PROBLEM SET
MTH 70200
REAL ANALYSIS

Problem 1. Let μ be counting measure on the integers. Let ν be a signed measure on the integers. What is the Radon-Nikodym derivative $\frac{d\nu}{d\mu}$?

Problem 2. Let $f, g\colon \mathbb{R} \to [0,\infty]$ be Borel-measurable. Let $d\mu = f\,dm$ and $d\nu = g\,dm$, where m is Lebesgue measure. Let $S_f = \{x \in \mathbb{R}\colon f(x) > 0\}$ and $S_g = \{x \in \mathbb{R}\colon g(x) > 0\}$.

(a) Show that $\nu \perp \mu$ if and only if $m(S_f \cap S_g) = 0$.
(b) Show that $\nu \ll \mu$ if and only if $m(S_g \cap S_f^c) = 0$.
(c) Show that the Lebesgue decomposition of ν with respect to μ is $\nu = \rho + \lambda$ where $d\rho = \frac{g}{f}\,d\mu$ and $d\lambda = \chi_{S_f^c} g\,dm$.

Problem 3. (Half of Folland 3.9) Suppose $\{\nu_j\}$ is a sequence of positive measures and μ is a positive measure, all on the same space. If $\nu_j \perp \mu$ for all j, then $\sum_{j=1}^{\infty} \nu_j \perp \mu$.

Problem 4. (Folland 3.11) Let μ be a positive measure. A collection of functions $\{f_\alpha\}_{\alpha\in A} \subseteq L^1(\mu)$ is called *uniformly integrable* if for every $\epsilon > 0$, there exists $\delta > 0$ such that for all $\alpha \in A$,

$$\mu(E) < \delta \Longrightarrow \left| \int_E f_\alpha \, d\mu \right| < \epsilon.$$

(a) Prove that any finite subset of $L^1(\mu)$ is uniformly integrable.
(b) Prove that if $\{f_n\}$ is a sequence in $L^1(\mu)$ that converges in the L^1-metric to some $f \in L^1(\mu)$, then $\{f_n\}$ is uniformly integrable.

Date: March 28, 2024.