PROBLEM SET
MTH 70200
REAL ANALYSIS

Problem 1. (Folland 2.23) Given a bounded $f\colon [a,b] \to \mathbb{R}$, let

$$H(x) = \lim_{\delta\to 0} \sup_{|y-x|\le\delta} f(y), \qquad h(x) = \lim_{\delta\to 0} \inf_{|y-x|\le\delta} f(y).$$

Prove that f is Riemann integrable if and only if its set of discontinuities has Lebesgue measure 0 by establishing the following lemmas.

a. $H(x) = h(x)$ iff f is continuous at x.
b. In the notation of the proof of Theorem 2.28a, $H = G$ a.e. and $h = g$ a.e. Hence H and h are Lebesgue measurable, and $\int_{[a,b]} H\,dm = \bar{I}_a^b(f)$ and $\int_{[a,b]} h\,dm = \underline{I}_a^b(f)$.

You should work out for yourself why this proves the statement, but you don't need to write it up.

Problem 2. (Folland 2.26) Prove that if $f \in L^1(m)$ and $F(x) = \int_{-\infty}^x f(t)\,dt$, then F is continuous on $\mathbb{R}$.

Problem 3. (Folland 2.33) Show that if $f_n \in L^+$ and $f_n \to f$ in measure, then

$$\int f \le \liminf_{n\to\infty} \int f_n.$$

Problem 4. (Folland 2.34) Suppose $|f_n| \le g \in L^1$ and $f_n \to f$ in measure.

a. Show that $\int f = \lim_{n\to\infty} \int f_n$.
b. Show that $f_n \to f$ in L^1.

Problem 5. (Quals 2022, #1) Let f be a measurable function on some measure space (X,μ). Prove that the limit (possibly infinite)

$$\lim_{n\to\infty} \int |f|^{1/n}\,d\mu$$

exists and find it.

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