PROBLEM SET
MTH 70200
REAL ANALYSIS

Problem 1. (Quals Fall 2016) Prove that an arbitrary collection of pairwise disjoint Lebesgue measurable subsets of $\mathbb{R}$, each of which has positive Lebesgue measure, is at most countable.

Problem 2. (Folland 2.3) Let $(f_n)_{n\geq 1}$ be a sequence of measurable functions on X. Show that $\{x\colon \lim_{n\to\infty} f_n(x) \text{ exists}\}$ is a measurable set.

Problem 3. (Folland 2.8) Show that if $f\colon \mathbb{R} \to \mathbb{R}$ is monotone, then f is Borel measurable.

Problem 4. (Folland 2.9, parts (a), (b), and (c) only) Let $f\colon [0,1] \to [0,1]$ be the Cantor function. You can use the properties proven about it at the end of Section 1.5 or in class, e.g., that it is increasing and locally constant on C^c, where C is the Cantor set. Let $g(x) = f(x)+x$.

a. Prove that g is a bijection from $[0,1]$ to $[0,2]$, and that $h = g^{-1}$ is continuous from $[0,2]$ to $[0,1]$.
b. Prove that $m(g(C)) = 1$, where m is Lebesgue measure.

Let $A \subseteq g(C)$ be a Lebesgue nonmeasurable set. You don't have to prove that A exists. To see that it does, just take the nonmeasurable set we constructed on the first day of class and intersect it with $g(C)$. If the resulting set were measurable, then $g(C)$ would be the union of countably many disjoint translates of it, a contradiction since $g(C)$ has strictly positive but noninfinite Lebesgue measure.

c. Let $B = g^{-1}(A)$. Prove that B is Lebesgue measurable but not Borel.

Commentary: The result of this problem is that h is a continuous function and $h^{-1}(B)$ is nonmeasurable. That is, h pulls back a Lebesgue measurable set to set that is not Lebesgue measurable. This illustrates why we take the σ-algebra for the codomain of real-valued functions to be the Borel σ-algebra rather than the larger set of Lebesgue measurable sets. Otherwise even continuous functions could be nonmeasurable.

Problem 5. Let $f\colon \mathbb{R} \to \mathbb{R}$ be bi-Lipschitz, meaning that there exists $C > 0$ such that for all $x, y \in \mathbb{R}$,

$$C^{-1}|x-y| \leq |f(x) - f(y)| \leq C|x-y|.$$

Show that if E is Lebesgue measurable, then $f^{-1}(E)$ is Lebesgue measurable. (You can take for granted that the Lebesgue measurable sets are the completion of the Borel sets with respect to Lebesgue measure—we never proved this in class but it's a consequence of some straightforward exercises from Chapter 1.4. Thus every Lebesgue measurable set can be represented as the union of a Borel measurable set and a set of Lebesgue measure zero.) *Commentary:* This problem shows that the concern of continuous functions pulling back measurable sets to nonmeasurable sets doesn't happen with a more stringent continuity condition.

Problem 6. Let $(X, \mathcal{M}, \mu)$ be a complete measure space. Show that if f is measurable and $f = g$ μ-a.e., then g is measurable.

Date: February 20, 2024.