PROBLEM SET I
MTH 70200
REAL ANALYSIS

1. Show that the Borel σ-algebra $\mathcal{B}_{\mathbb{R}}$ of the real numbers has the following alternative generating sets. (You can assume part (a) of Proposition 1.2, which we proved in class.)
   (a) $\mathcal{B}_{\mathbb{R}} = \sigma(\{(a, b] \colon -\infty \leq a < b \leq \infty\})$
   (b) $\mathcal{B}_{\mathbb{R}} = \sigma(\{(-\infty, x] \colon x \in \mathbb{R})\})$
   (c) $\mathcal{B}_{\mathbb{R}} = \sigma(\{(-\infty, x] \colon x \in \mathbb{Q})\})$

2. A *monotone class* $\mathcal{M}$ of X is a collection of subsets of X that includes X and which is closed under unions of increasing sequences of subsets and also closed under intersections of decreasing sequences of subsets. That is, if $E_j \in \mathcal{M}$ and $E_1 \subseteq E_2 \subseteq \cdots$, then $\cup_{j=1}^{\infty} E_j \in \mathcal{M}$, and likewise for intersections and decreasing sequences. Prove that
   (a) An intersections of monotone classes is also a monotone class.
   So it makes sense to speak of the monotone class generated by a collection of subsets $\mathcal{A}$, denoted $\mathcal{M}(\mathcal{A})$, as the intersection of all monotone classes containing $\mathcal{A}$.
   (b) If $\mathcal{A}$ is an algebra of X, show that $\mathcal{M}(\mathcal{A}) = \sigma(\mathcal{A})$, that is the smallest monotone class containing $\mathcal{A}$ corresponds to the smallest σ-algebra containing $\mathcal{A}$.

   Hint: For 5b), the problem is to relate the notion of monotone class to complement and unions. To deal with this, consider for any $E \in \mathcal{M}(\mathcal{A})$

$$\mathcal{C}(E) = \{F \in \mathcal{M}(\mathcal{A}) \colon E \setminus F,\ F \setminus E, \text{ and } E \cap F \text{ are in } \mathcal{M}(\mathcal{A})\}.$$

   The point is to consider the subsets that behave well under union and complement with E.
   (a) Show that $\mathcal{C}(E)$ is a monotone class.
   (b) If $E \in \mathcal{A}$, show that $\mathcal{M}(\mathcal{A}) \subseteq \mathcal{C}(E)$.
   (c) Argue that $E \in \mathcal{C}(F)$ if and only if $F \in \mathcal{C}(E)$. Use this and the above to conclude that $\mathcal{M}(\mathcal{A}) \subseteq \mathcal{C}(E)$ for any $E \in \mathcal{M}(\mathcal{A})$.
   (d) Conclude from the above that $\mathcal{M}(A)$ is an algebra first, then prove it is a σ-algebra.

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