Regularity properties

By def., for $E \in \mathcal{M}_\mu$ (complete domain of $\mu = \mu_F$),

$$\mu(E) = \inf\left\{ \sum_{j=1}^{\infty} \left(F(b_j) - F(a_j)\right) : \bigcup_{j=1}^{\infty} (a_j, b_j] \supseteq E \right\}$$
$$= \inf\left\{ \sum_{j=1}^{\infty} \mu((a_j, b_j]) : \bigcup_{j=1}^{\infty} \cdots\cdots \right\}$$

Lemma. For $E \in \mathcal{M}_\mu$,

$$\mu(E) = \inf\left\{ \sum \mu((a_j, b_j)) : \bigcup_{j=1}^{\infty} (a_j, b_j) \supseteq E \right\} =: \nu(E)$$

Proof.

$\mu(E) \leq \nu(E)$: For any cover $\bigcup_j (a_j, b_j) \supseteq E$,

$\mu(E) \leq \sum \mu((a_j, b_j))$. Take infima

$\mu(E) \geq \nu(E)$:

Fix $\varepsilon > 0$, choose nearly optimal $\bigcup_{j=1}^{\infty} (a_j, b_j] \supseteq E$:

$$\sum_j \mu((a_j, b_j]) = \sum_j (F(b_j) - F(a_j)) \leq \mu(E) + \varepsilon$$

By right-continuity of F, increase b_j a tiny bit and only add a tiny bit of measure: choose ε_j

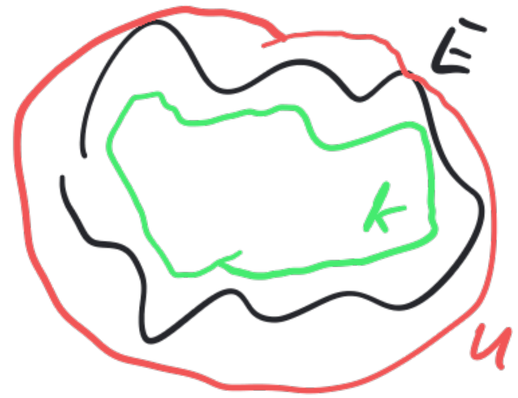
$$\mu((a_j, b_j + \delta_j]) \leq \mu((a_j, b_j]) + 2^{-j}\varepsilon$$

$$\nu(E) \leq \sum_j \mu((a_j, b_j + \varepsilon_j)) \leq \sum_j \mu((a_j, b_j]) + \varepsilon \leq \mu(E) + 2\varepsilon$$

Thm. $\mu = \mu_F$ is *regular*, meaning that for $E \in \mathcal{M}_\mu$,

$$\mu(E) = \inf\{\mu(U) : U \supseteq E,\ U \text{ open}\} \quad (1)$$

$$\mu(E) = \sup\{\mu(K) : K \subseteq E,\ K \text{ compact}\} \quad (2)$$



Proof.

For (1):

$\mu(E) \leq \inf\{ \cdots \}$ by monotonicity

For any $\varepsilon > 0$,

$\mu(E) \geq \inf\{\mu(U) : U \supseteq E,\ U \text{ open}\} - \varepsilon$ by choosing a nearly optimal cover of open intervals

For (2):

$\mu(E) \geq \sup\{\mu(K) : K \subseteq E,\ K \text{ compact}\}$ by monotonicity

Now, show $\exists$ compact $K \subseteq E$ s.t. $\mu(K)$ is almost $\mu(E)$.

First suppose E is bounded. Then $\bar{E} \setminus E$ has finite μ-measure.

Choose open $U \supseteq \bar{E} \setminus E$ w/ $\mu(U) \leq \mu(\bar{E} \setminus E) + \varepsilon$ by (1)

$$= \mu(\bar{E}) - \mu(E) + \varepsilon.$$

Let $K = \bar{E} \setminus U$, so $K \subseteq E$ and is compact.

$\mu(K) = \mu(\bar{E}) - \mu(U) \geq \mu(E) - \varepsilon$,

proving $\mu(E) \leq \sup\{\mu(K) : K \subseteq E,\ K \text{ compact}\} + \varepsilon$ when E is bounded. Take $\varepsilon \to 0$.

If E is unbounded, approximate $E \cap [-n, n]$ by compact $K_n \subseteq E \cap [-n, n]$

Take $n \to \infty$, then $\varepsilon \to 0$

Choose K_n s.t. $\mu(K_n) \geq \mu(E \cap [-n,n]) + \varepsilon$

Since holds for any n, and $\mu(E \cap [-n,n]) \to \mu(E)$,

shows $\mu(E) \leq \sup\{ \cdots \} + 2\varepsilon$

Want: $K \subseteq [0,1] \setminus \mathbb{Q}$ w/ $m(K) \geq 1-\varepsilon$

$\mathbb{Q} = \{q_1, q_2, \dots\}$

Cover $\mathbb{Q}$ by open U w/ $m(U) \leq \varepsilon$.

Let $K = [0,1] \setminus U$.

K is closed and bounded, $K \subseteq [0,1] \setminus \mathbb{Q}$ $m(K) \geq 1-\varepsilon$

Invariance of Lebesgue measure

Let $m = \mu_F$ for $F(x) = x$, let $\mathcal{L}$ be its complete domain, the Lebesgue-measurable sets.

For $s, r \in \mathbb{R}$, let

$$E + s = \{x + s : x \in E\}, \quad rE = \{rx : x \in E\}$$

Thm. If $E \in \mathcal{L}$, then $E + s \in \mathcal{L}$ and $rE \in \mathcal{L}$, and

$$m(E+s) = m(E) \text{ and } m(rS) = |r|\, m(S).$$

$0 \cdot \infty = 0$

Proof. Start w/ Borel sets. Let $E \in \mathcal{B}_{\mathbb{R}}$

$E+s \in \mathcal{B}_{\mathbb{R}}$:

Let $\mathcal{B}' = \{A \in \mathcal{B}_{\mathbb{R}} : A+s \in \mathcal{B}_{\mathbb{R}}\} \subseteq \mathcal{B}_{\mathbb{R}}$

$\mathcal{B}'$ is a σ-algebra (check!) and contains all open intervals. So, $\mathcal{B}_{\mathbb{R}} \subseteq \mathcal{B}'$. So $\mathcal{B}' = \mathcal{B}_{\mathbb{R}}$

Now, WTS that $m(E+s) = m(E)$.

Define $m_s(E) = m(E+s)$, which is a Borel measure (check!)

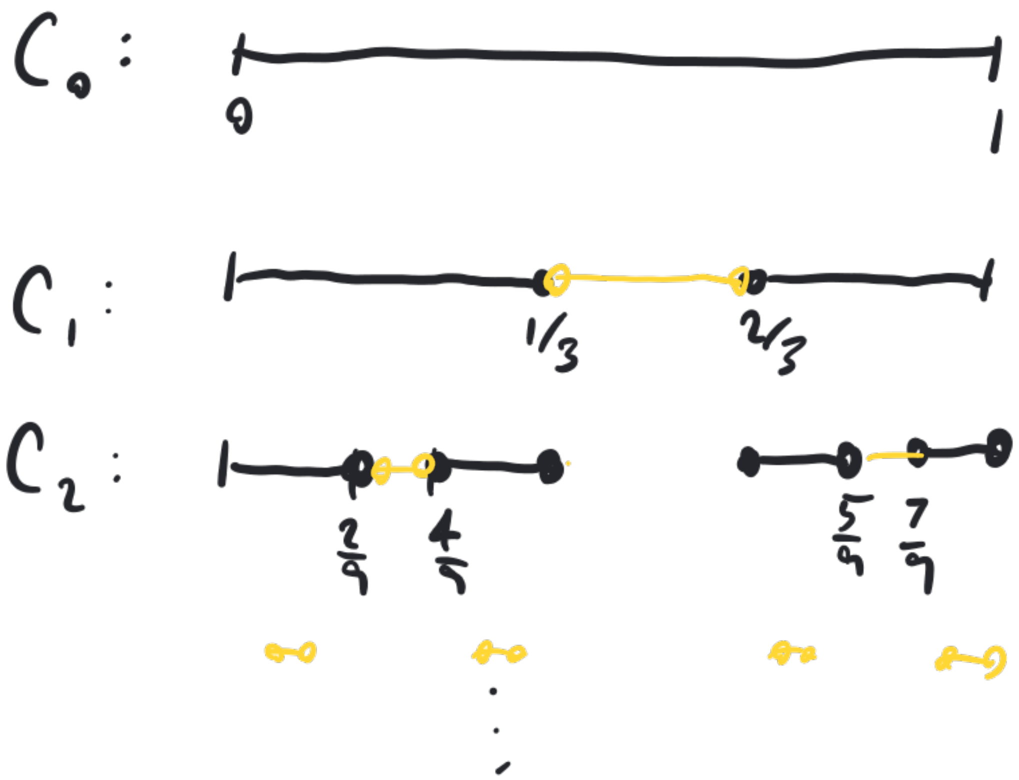
Since $m_s(E) = m(E)$ for all h-intervals E, m_s and m are same for all Borel E.

Completions of m_s and m same too.

Proof for dilations: same

Cantor sets and Function

Cantor set: start w/ $[0,1]$, remove middle third, remove middle thirds, etc...



$$C = \bigcap_{j=0}^{\infty} C_j$$

slightly more formally:

Take base-3 expansion

$$x = \sum_{j=1}^{\infty} a_j 3^{-j} \quad \text{for } x \in [0,1],$$

unique except when $x = .a_1 a_2 \cdots a_k 000\ldots$ (nonzero: a_k)

$$= .a_1 a_2 \cdots (a_k - 1) 222 \ldots$$

in which case we take expansion where $a_k \neq 1$

$$C_j = \{x \in [0,1] : a_1, \ldots, a_j \in \{0,2\}\}$$

$$C = \bigcap_{j=0}^{\infty} C_j = \{x \in [0,1] : a_1, a_2, \ldots \in \{0,2\}\}$$

Prop. C is uncountable and has $m(C)=0$

Proof. $m(C_j) = \left(\frac{2}{3}\right)^j$, so by continuity from above, $m(C) = \lim_{j\to\infty} \left(\frac{2}{3}\right)^j = 0$.

Define $f: C \to [0,1]$ by $\sum_{j=1}^{\infty} a_j 3^{-j} \longmapsto \sum_{j=1}^{\infty} \frac{a_j}{2} 2^{-j}$

f is surjective, so C is uncountable.

Def. The Cantor function $f:[0,1]\to[0,1]$ extends f above.

For $x<y$, $x,y\in C$, we have $f(x)\le f(y)$ with equality when:

$$\begin{aligned} x &= .a_1 \cdots a_k 0222\ldots \\ y &= .a_1 \cdots a_k 2000\ldots \end{aligned} \longmapsto \begin{aligned} &.\tfrac{a_1}{2} \cdots \tfrac{a_k}{2} 011111\ldots \\ &.\tfrac{a_1}{2} \cdots \tfrac{a_k}{2} 10000\ldots \end{aligned}$$

For $x \in [0,1]$, $x = .a_1 a_2 \cdots a_k \cdots$ w/ a_{k+1} the first 1, set

$$f(x) = f(.a_1 \cdots a_k 0222\ldots) = f(.a_1 \cdots a_k 2000).$$

We've extended f by making it locally constant off C.

(i.e., constant on each interval of C^c, which is a union of open intervals)

f is still increasing.

<u>Lemma</u>. f is continuous

By monotonicity, it has left and right limits at all x

Must be same or f wouldn't be surjective.