Math 231, Midterm 2, version A Name: ______________________

November 18, 2019

I pledge that I have neither given nor received unauthorized assistance during this examination.

Signature:

- **DON'T PANIC!** If you get stuck, take a deep breath and go on to the next question.
- Unless the problem says otherwise **you must show your work** sufficiently much that it's clear to me how you arrived at your answer.
- You may use a scientific calculator on this exam, but you may not use a graphing calculator.
- You may bring a two-sided sheet of notes on letter-sized paper in your own handwriting.
- There are 8 problems on 8 pages.

Question	Points	Score
1	12	
2	10	
3	10	
4	6	
5	8	
6	8	
7	10	
8	8	
Total:	72	

Good luck!

[12 points] 1. Find the derivatives of the following functions. Do **not** simplify your solutions.

(a) $f(x) = \dfrac{\cos x}{x^2}$

Solution:

$$f'(x) = \frac{x^2(-\sin x) - (\cos x)(2x)}{x^4}$$

(b) $f(x) = e^{-\sqrt{2x+1}}$

Solution:

$$f'(x) = e^{-\sqrt{2x+1}}\left(-\frac{1}{2\sqrt{2x+1}}\right)(2)$$

(c) $f(x) = x^3 + \dfrac{1}{x^2}$

Solution:

$$f'(x) = 3x^2 - 2x^{-3}$$

[10 points] 2. A spherical balloon is being inflated. Suppose the radius expands at a constant rate of 2 cm/s. At the moment when the radius reaches 10 cm, how quickly is the volume of the balloon growing? (Note that the volume of a sphere with radius r is $\frac{4}{3}\pi r^3$.

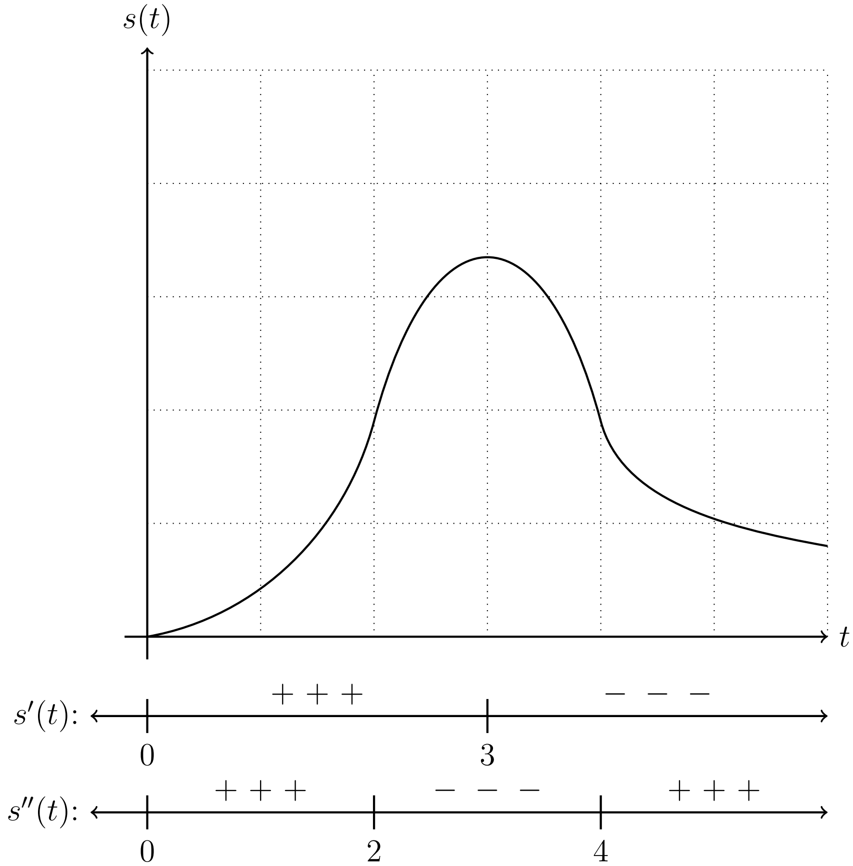
Solution: Let r be the radius and V the volume of the balloon. We're given that $\frac{dr}{dt} = 2$, and we want to find $\frac{dV}{dt}$ when $r = 10$. Differentiating both sides of the equation $V = \frac{4}{3}\pi r^3$, we get

$$\frac{dV}{dt} = 4\pi r^2 \frac{dr}{dt}.$$

Plugging in, we get

$$\frac{dV}{dt} = 4\pi(10)^2(2) = 800\pi.$$

[10 points] 3. Let $s(t)$ represent the number of subscribers of a streaming service at time t years (with $t = 0$ representing 2015, when the service started). The sign diagrams of $s'(t)$ and $s''(t)$ are given below.



(a) At time $t = 1$, which of the following is true about the streaming service:

√ **It's gaining subscribers, and the rate of gain is increasing.**

◯ It's gaining subscribers, but the rate of gain is decreasing.

◯ It's losing subscribers, and the rate of loss is increasing.

◯ It's losing subscribers, but the rate of loss is decreasing.

(b) At time $t = 5$, which of the following is true about the streaming service:

◯ It's gaining subscribers, and the rate of gain is increasing.

◯ It's gaining subscribers, but the rate of gain is decreasing.

◯ It's losing subscribers, and the rate of loss is increasing.

√ **It's losing subscribers, but the rate of loss is decreasing.**

(c) At what times does the number of subscribers achieve a local minimum? If the answer is never, say so.

Solution: I'd accept either $t = 0$ or never as correct answers (we've never really said whether or not an endpoint counts as a local minimum).

(d) At what times does the number of subscribers achieve a local maximum? If the answer is never, say so.

Solution: $t = 3$, since $s'(t)$ changes from positive to negative there.

(e) Give the t-coordinates of all inflection points of $s(t)$, or state that there are none.

Solution: $t = 2$ and $t = 4$, since $s''(t)$ changes sign there.

(f) Sketch the graph of $s(t)$ on the axes above the sign diagram. Assume that at time $t = 0$, the service has 0 users.

[6 points] 4. Compute

$$\lim_{x\to\infty} \frac{\ln x}{x}.$$

Solution: The top and bottom both approach infinity, so L'Hôpital's rule applies and gives

$$\lim_{x\to\infty} \frac{\ln x}{x} = \lim_{x\to\infty} \frac{1/x}{1} = 0.$$

[8 points] 5. Let $f(x) = \ln(2x+1)$. Give an estimate of $f(.1)$ using linearization.

Solution: The idea here is that we observe that $f(0) = \ln(1) = 0$, and we do linearization using this as our base point. We compute

$$f'(x) = \frac{2}{2x+1},$$

and $f'(0) = 2$. So,

$$f(.1) \approx f(0) + f'(0) \cdot (.1) = .2.$$

[8 points] 6. Let $g(x) = x^3 - 3x + 2$. Find the absolute maximum and minimum of $g(x)$ on the interval $[0, 3]$.

Solution: We find $g'(x) = 3x^2 - 3$ and set it equal to zero. This yields critical points $x = \pm 1$, of which only $x = 1$ is in the interval. Plugging in this point plus the endpoints into f gives:

$$g(0) = 2,$$
$$g(1) = 0,$$
$$g(3) = 20.$$

So, the minimum of $g(x)$ is 0 and the maximum is 4 on $[0, 3]$.

7. Let $f(x) = \frac{(x-1)^2}{x^2}$. The first two derivatives of this function can be computed to be

$$f'(x) = \frac{2(x-1)}{x^3},$$
$$f''(x) = -\frac{2(2x-3)}{x^4}.$$

[3 points] (a) Give the locations of any vertical asymptotes of the function, or state that it doesn't have any.

> **Solution:** There's a vertical asymptote at $x = 0$.

[3 points] (b) List all critical values of this function.

> **Solution:** There's a critical value at $x = 1$.

[4 points] (c) Give the intervals where this function is increasing and the intervals where it is decreasing.

> **Solution:** Increasing on $(-\infty, 0)$ and $(1, \infty)$, decreasing on $(0, 1)$.

[8 points] 8. Consider the curve defined by equation $x^2y^3 + x^2 = 2$. Find the equation for the tangent line to the curve at $(1, 1)$.

Solution: Differentiating with respect to x,

$$2xy^3 + 3x^2y^2\frac{dy}{dx} + 2x = 0.$$

Rearranging and isolating $\frac{dy}{dx}$, we get

$$\frac{dy}{dx} = \frac{-2x - 2xy^3}{3x^2y^2} = \frac{-2 - 2y^3}{3xy^2}.$$

At $(1, 1)$, the slope of the tangent line is therefore $(-2-2)/3 = -4/3$. In point-slope form, the equation for the tangent line is

$$y - 1 = -\frac{4}{3}(x - 1).$$