Large time asymptotics for the density of a branching Wiener process

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Abstract

Given an $\mathbb{R}^d$-valued supercritical branching Wiener process, let $\psi(A,T)$ be the number of particles in $A \subset \mathbb{R}^d$ at time T, $(T = 0,1,2,\ldots)$. We provide a complete asymptotic expansion of $\psi(A,T)$ as $T \to \infty$, generalizing the work of X. Chen ([2]).

1 Introduction

Consider the following model in $\mathbb{R}^d$ (with $d \geq 1$):

(i) a particle starts from the origin in $\mathbb{R}^d$ and executes a Wiener process $W(t) \in \mathbb{R}^d$,

(ii) arriving at time $t = 1$ at the new location $W(1)$, it dies,

(iii) at death it is replaced by Y offspring where

$$\mathbf{P}\{Y = \ell\} = p_\ell, \quad (\ell = 0,1,2,\ldots)$$
$$1 < \sum_{\ell=0}^{\infty} \ell p_\ell = m < \infty,$$
$$0 < \sum_{\ell=0}^{\infty} (\ell - m)^2 p_\ell = \sigma^2 < \infty,$$

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(iv) each offspring, starting from where its ancestor dies, executes a Wiener process (from its starting point) and repeats the above given steps and so on. All Wiener processes and offspring-numbers are assumed independent of each other.

Let

$$\lambda(x,t) = \begin{cases} 1 & \text{if } x \in I\!R^d \text{ is occupied by a particle at time } t, \\ 0 & \text{otherwise.} \end{cases}$$

We write

$$\psi(A,t) = \sum_{x \in A} \lambda(x,t),$$

which stands for the number of particles at time t located at $A \subset I\!R^d$. In particular, $\psi(I\!R^d, t)$ is the total number of particles alive at time t.

Since the branching is supercritical, it is well-known (Athreya and Ney [1], p. 9) that

$$N_0 := \lim_{T\to\infty} \frac{\psi(I\!R^d, T)}{m^T} \quad \text{a.s.,} \tag{1.1}$$

exists (and is finite), and that $\mathbf{P}(N_0 > 0) > 0$.

The limit properties of $\psi(A,T)$, $T \to \infty$, were studied by Chen ([2]) who proved

Theorem A. *There exist random variables N_1 and N_2 (N_1 being $I\!R^d$-valued) such that for any Borel set $A \subset I\!R^d$ with $\int_A \|x\|^2 dx < \infty$, we have, almost surely when $T \to \infty$,*

$$(2\pi T)^{d/2} \frac{\psi(A,T)}{m^T} = N_0 \int_A dx - \frac{1}{2T} \int_A (N_0 \|x\|^2 - 2N_1 \cdot x + N_2) dx + o(T^{-1}).$$

This result plays an important role in Révész ([5]) in the study of the concentration of particles in the branching process.

The goal of this paper is to provide a complete asymptotic expansion for $\psi(A,T)/m^T$ as $T \to \infty$. Let us first introduce some notation.

If $\alpha = (\alpha_1, \ldots, \alpha_d) \in Z_+^d$ and $x = (x_1, \ldots, x_d) \in R^d$ we use the notation $|\alpha| = \alpha_1 + \cdots + \alpha_d$, $\alpha! = \prod_{i=1}^d \alpha_i!$, $x^\alpha = \prod_{i=1}^d x_i^{\alpha_i}$ and

$$M_\alpha(A) = \int_A x^\alpha \, dx. \tag{1.2}$$

If also $\beta \in Z_+^d$ we will write $\beta \preceq \alpha$ to mean that $\beta_i \le \alpha_i$ for all i, and if $\beta \preceq \alpha$ we set

$$\binom{\alpha}{\beta} = \prod_{i=1}^d \binom{\alpha_i}{\beta_i}. \tag{1.3}$$

Here is the main result of the paper:

Theorem 1.1 *There exist random variables* $(N_\alpha,\ \alpha \in Z_+^d)$ *such that for any* $k \geq 1$ *and any bounded Borel set* $A \subset \mathbb{R}^d$*, when* $T \to \infty$*,*

$$(1.4)\qquad (2\pi T)^{d/2}\frac{\psi(A,T)}{m^T}$$
$$= \sum_{n=0}^{k} \frac{(-T)^{-n}}{2^n} \sum_{|\alpha|=n} \frac{1}{\alpha!} \sum_{\beta \preceq 2\alpha} \binom{2\alpha}{\beta} (-1)^{|\beta|} M_\beta(A) N_{2\alpha-\beta} + o(T^{-k}), \qquad \text{a.s.}$$

Remark 1.2 The random variables $(N_\alpha,\ \alpha \in Z_+^d)$ are described in the proof of Theorem 1.1. They are limits of explicit martingales related to the branching Wiener process.

Although the distributions of the random variables $(N_\alpha,\ \alpha \in Z_+^d)$ are not known, Theorem 1.1 can nevertheless be used to make predictions to any degree of accuracy.

To see this, choose an integer k and disjoint sets $(A_\alpha \subseteq \mathbb{R}^d,\ |\alpha| \leq k)$. Consider (1.4) for each A_α. Then we have a linear system of equations with the unknowns $N_{2\alpha-\beta}$. One can solve this system of equations if the corresponding determinant is not equal to 0. It is easy to see that we can choose the sets A_α such that the determinant is not 0 for any T $(T = 1, 2, \ldots)$. Observe the number of particles of a branching Wiener process which are located in the above given sets $(A_\alpha,\ |\alpha| \leq k)$ at time T_0. Having these observations one can evaluate the actual values of the random variables $(N_\alpha,\ |\alpha| \leq k)$ with an error term $o(T_0^{-k})$. Having these values one can use Theorem 1.1 to get the values of the process $(2\pi T)^{d/2}\psi(A,T)/m^T$ for any $A \subseteq \mathbb{R}^d,\ T \geq T_0$ with an error term $o(T_0^{-k})$.

The proof of Theorem 1.1 is presented in Section 2. In Section 3 we show that if the offspring distribution Y has p moments for some even integer p then the martingales described in Remark 1.2 converge to the random variables $(N_\alpha,\ \alpha \in Z_+^d)$ in L^p.

2 The proof

We start with a preliminary result concerning the transition kernel of the Wiener process. Let

$$p_t^{(d)}(x) = \frac{1}{(2\pi t)^{d/2}} \exp\left(-\frac{\|x\|^2}{2t}\right).$$

Define the Hermite polynomials by

$$(2.1)\qquad H_n(x,t) = \sum_{j=0}^{[n/2]} \frac{n!}{j!(n-2j)!}\left(\frac{-t}{2}\right)^j x^{n-2j}.$$

Lemma 2.1 *For any $0 < t < T$ and any $x \in I\!R^1$,*

$$p^{(1)}_{T-t}(x) = \frac{1}{(2\pi T)^{1/2}} \sum_{n=0}^{\infty} \frac{(-T)^{-n}}{2^n n!} H_{2n}(x,t). \tag{2.2}$$

Proof. Let us recall the Hermite polynomials:

$$\begin{aligned} H_n(x) &= (-1)^n e^{x^2} \frac{d^n}{dx^n}(e^{x^2}) \\ &= n! \sum_{j=0}^{\lfloor n/2 \rfloor} \frac{(-1)^j}{j!(n-2j)!}(2x)^{n-2j}, \qquad x \in I\!R, \end{aligned} \tag{2.3}$$

so that

$$H_n(x,t) = (t/2)^{n/2} H_n\left(\frac{x}{\sqrt{2t}}\right), \qquad x \in I\!R, \;\; t > 0. \tag{2.4}$$

We use the following identity, see for example Lebedev ([3], p. 75): for any $a > 0$ and $y \in I\!R$,

$$e^{-a^2y^2} = \sum_{n=0}^{\infty} \frac{(-1)^n a^{2n}}{2^{2n} n!(1+a^2)^{n+(1/2)}} H_{2n}(y).$$

Taking $y = x/\sqrt{2t} \in I\!R^1$ and $a = \sqrt{t/(T-t)}$, and multiplying both sides by $(2\pi(T-t))^{-1/2}$, we readily get (2.2). $\diamondsuit$

If $\alpha = (\alpha_1, \ldots, \alpha_d) \in Z^d_+$ and $x = (x_1, \ldots, x_d) \in R^d$ we use the notation

$$H_\alpha(x,t) = \prod_{i=1}^{d} H_{\alpha_i}(x_i, t). \tag{2.5}$$

Lemma 2.2 *For any $0 < t < T$ and any $(x,y) \in I\!R^d \times I\!R^d$,*

$$p^{(d)}_{T-t}(x) = \frac{1}{(2\pi T)^{d/2}} \sum_{n=0}^{\infty} \frac{(-T)^{-n}}{2^n} \sum_{|\alpha|=n} \frac{1}{\alpha!} H_{2\alpha}(x,t), \tag{2.6}$$

and

$$\begin{aligned} &p^{(d)}_{T-t}(x-y) \\ &= \frac{1}{(2\pi T)^{d/2}} \sum_{n=0}^{\infty} \frac{(-T)^{-n}}{2^n} \sum_{|\alpha|=n} \frac{1}{\alpha!} \sum_{\beta \preceq 2\alpha} \binom{2\alpha}{\beta} (-x)^\beta H_{2\alpha-\beta}(y,t). \end{aligned} \tag{2.7}$$

Proof. Since for $x = (x_1, \ldots, x_d) \in I\!R^d$

$$p_t^{(d)}(x) = \prod_{i=1}^{d} p_t^{(1)}(x_i), \tag{2.8}$$

(2.6) follows from (2.2). To obtain (2.7) we use the fact that

$$H_n(x + y, t) = \sum_{j=0}^{n} \binom{n}{j} x^{n-j} H_j(y, t). \tag{2.9}$$

For this we recall that (Lebedev [3], p. 60)

$$\sum_{n=0}^{\infty} \frac{s^n}{n!} H_n(x) = e^{2sx - s^2} \tag{2.10}$$

so that

$$\sum_{n=0}^{\infty} \frac{s^n}{n!} H_n(x, t) = e^{sx - ts^2/2}. \tag{2.11}$$

Then

$$\sum_{n=0}^{\infty} \frac{s^n}{n!} H_n(x + y, t) = e^{s(x+y) - ts^2/2} \tag{2.12}$$

$$= e^{sx} e^{sy - ts^2/2} = \sum_{k=0}^{\infty} \frac{s^k x^k}{k!} \sum_{j=0}^{\infty} \frac{s^j}{j!} H_j(y, t),$$

and comparing powers of s^n proves (2.9). ◇

Now we turn to the study of the branching Wiener process. Clearly, for any $T \geq 1$ and $A \subset I\!R^d$,

$$\mathbf{E}\left(\psi(A, T) \,|\, \mathcal{F}(T-1)\right) = m \int_A \sum_y p_1^{(d)}(y - x) \lambda(y, T-1) dx,$$

(as usual, $\mathcal{F}(t)$ denoting the σ-algebra induced by the branching process until time t). A simple argument by induction yields that for all $0 < t < T$,

$$\mathbf{E}\left(\psi(A, T) \,|\, \mathcal{F}(t)\right) = m^{T-t} \int_A \sum_y p_{T-t}^{(d)}(y - x) \lambda(y, t) dx. \tag{2.13}$$

It turns out that $\psi(A, T)$ is quite close to its conditional expectation, as is confirmed by the following results.

Fact 2.3 (Révész [4], (6.16)) *Fix $\gamma \in (0,1)$ and let $t = \lfloor T^\gamma \rfloor$. Let $A \subset I\!R^d$ be a bounded Borel set. Let $\varepsilon > 0$. We have, almost surely for $T \to \infty$,*

$$\frac{\psi(A,T)}{m^T} - \frac{1}{m^t} \int_A \sum_y p^{(d)}_{T-t}(y-x)\lambda(y,t)dx = o\left(m^{-t/(2+\varepsilon)}\right). \tag{2.14}$$

Fact 2.4 (Révész [4], (6.11)) *There exists a constant $C = C(m,d) > 0$ such that for all $1 \le t < T$,*

$$\mathbf{E}\left(\sum_{y \in I\!R^d} \{\lambda(y,T) - \mathbf{E}[\lambda(y,T) \,|\, \mathcal{F}(t)]\}^2 \right) \le C \, \frac{m^{2T-t}}{(T-t)^{d/2}}. \tag{2.15}$$

Lemma 2.5 *Let $\varepsilon > 0$. Almost surely for all large t, we have $\lambda(y,t) = 0$ whenever $\|y\| > t^{1+\varepsilon}$.*

Proof. This follows from the usual estimate for the tail of the Wiener process, the Borel–Cantelli lemma, and (1.1). ♢

Lemma 2.6 *Let $\alpha \in Z^d_+$, and let*

$$V_\alpha(t) = \sum_y H_\alpha(y,t)\lambda(y,t). \tag{2.16}$$

Then, $(\frac{1}{m^t} V_\alpha(t),\ t \ge 0)$ is a martingale and

$$N_\alpha := \lim_{t\to\infty} \frac{V_\alpha(t)}{m^t}$$

exists and is finite almost surely.

Proof. We start by proving the martingale property. Recall that $\psi(I\!R^d, t)$ stands for the total number of particles at time t. Thus, by numbering these particles and considering them all starting from time $t = 0$ (many of them share common paths, at least partially), we can write $\sum_y H_\alpha(y,t)\lambda(y,t) = \sum_{i=1}^{\psi(I\!R^d,t)} H_\alpha(W^{(i)}(t), t)$, where $(W^{(i)},\ i \ge 1)$ is a sequence of $I\!R^d$-valued Wiener processes (they are *not* independent). Conditioning on $\mathcal{F}(t-1)$ and on $\psi(I\!R^d, t)$, we have

$$\begin{aligned} & \mathbf{E}\left(\sum_{i=1}^{\psi(I\!R^d,t)} H_\alpha(W^{(i)}(t), t) \,\Big|\, \mathcal{F}(t-1),\ \psi(I\!R^d, t) \right) \\ = & \sum_{i=1}^{\psi(I\!R^d,t)} H_\alpha(W^{(i)}(t-1), t-1) \\ = & \sum_{i=1}^{\psi(I\!R^d,t-1)} Y_{i,t-1} H_\alpha(W^{(i)}(t-1), t-1), \end{aligned}$$

the last identity following from the fact that many particles at time t come from the same ancestor at time $t-1$, with $Y_{i,t-1}$ denoting the number of offspring from the i-th particle at time $t-1$.

Integrating on both sides gives that

$$\begin{aligned}
(2.17)\qquad & \mathbf{E}\left(\sum_{i=1}^{\psi(I\!R^d,t)} H_\alpha(W^{(i)}(t),t)\,\Big|\,\mathcal{F}(t-1)\right) \\
&= \sum_{i=1}^{\psi(I\!R^d,t-1)} \mathbf{E}(Y) H_\alpha(W^{(i)}(t-1),t-1) \\
&= m \sum_{i=1}^{\psi(I\!R^d,t-1)} H_\alpha(W^{(i)}(t-1),t-1),
\end{aligned}$$

proving that $t \mapsto \frac{1}{m^t}V_\alpha(t)$ is a martingale.

We now show that $(\frac{1}{m^t}\,V_\alpha(t),\ t \geq 0)$ converges to a finite limit almost surely. With the above notation we first write

$$(2.18)\quad V_\alpha(t) = \sum_{l=1}^{\psi(I\!R^d,t)} H_\alpha(W^{(l)}(t),t) = \sum_{l=1}^{\psi(I\!R^d,t-1)} \sum_{m=1}^{Y_{l,t-1}} H_\alpha(W^{(l,m)}(t),t)$$

where $W^{(l,m)}(t)$ is the m-th child of the l-th particle which dies at time $t-1$. Then we can write

$$\begin{aligned}
(2.19)\quad V_\alpha(t)^2 \;=\;& \sum_{l=1}^{\psi(I\!R^d,t-1)} \sum_{m=1}^{Y_{l,t-1}} H^2_\alpha(W^{(l,m)}(t),t) \\
&+ \sum_{l=1}^{\psi(I\!R^d,t-1)} \sum_{m\neq n,\, m,n=1}^{Y_{l,t-1}} H_\alpha(W^{(l,m)}(t),t) H_\alpha(W^{(l,n)}(t),t) \\
&+ \sum_{i\neq j,\, i,j=1}^{\psi(I\!R^d,t-1)} \sum_{m=1}^{Y_{i,t-1}} \sum_{n=1}^{Y_{j,t-1}} H_\alpha(W^{(i,m)}(t),t) H_\alpha(W^{(j,n)}(t),t)
\end{aligned}$$

Therefore

$$\begin{aligned}
(2.20)\qquad & \mathbf{E}\left(V_\alpha(t)^2\,\Big|\,\mathcal{F}(t-1),\ \psi(I\!R^d,t)\right) \\
=\;& \sum_{i=1}^{\psi(I\!R^d,t-1)} Y_{i,t-1}\mathbf{E}\left(H^2_\alpha(W^{(i,1)}(t),t)\,\Big|\,\mathcal{F}(t-1)\right) \\
+\;& \sum_{i=1}^{\psi(I\!R^d,t-1)} (Y^2_{i,t-1} - Y_{i,t-1}) H^2_\alpha(W^{(i)}(t-1),t-1) \\
+\;& \sum_{i\neq j,\, i,j=1}^{\psi(I\!R^d,t-1)} Y_{i,t-1}Y_{j,t-1} H_\alpha(W^{(i)}(t-1),t-1) H_\alpha(W^{(j)}(t-1),t-1).
\end{aligned}$$

Thus

$$\begin{aligned}
(2.21)\quad & \mathbf{E}\left(V_\alpha(t)^2 \,\Big|\, \mathcal{F}(t-1)\right) \\
= & \sum_{i=1}^{\psi(I\!R^d,t-1)} m\mathbf{E}\left(H_\alpha^2(W^{(i)}(t),t) \,\Big|\, \mathcal{F}(t-1)\right) \\
+ & \sum_{i=1}^{\psi(I\!R^d,t-1)} (\sigma^2+m^2-m)H_\alpha^2(W^{(i)}(t-1),t-1) \\
+ & \sum_{i\neq j,\, i,j=1}^{\psi(I\!R^d,t-1)} m^2 H_\alpha(W^{(i)}(t-1),t-1)H_\alpha(W^{(j)}(t-1),t-1). \\
= & \sum_{i=1}^{\psi(I\!R^d,t-1)} \Big[m\mathbf{E}\left(H_\alpha^2(W^{(i)}(t),t) \,\Big|\, \mathcal{F}(t-1)\right) \\
& \qquad +(\sigma^2-m)H_\alpha^2(W^{(i)}(t-1),t-1)\Big] + m^2V_\alpha(t-1)^2.
\end{aligned}$$

Recall that $\mathbf{E}(\psi(I\!R^d,t-1)) = m^{t-1}$ (Athreya and Ney [1], p. 9). It is easy to see using (2.11) that $\mathbf{E}\left(H_\alpha^2(W^{(1)}(t),t)\right) = \alpha! t^{|\alpha|}$. Hence

$$\begin{aligned}
(2.22)\quad & \mathbf{E}\left(V_\alpha(t)^2\right) \\
& = m^{t-1}\alpha!(mt^{|\alpha|} + (\sigma^2-m)(t-1)^{|\alpha|}) + m^2\mathbf{E}\left(V_\alpha(t-1)^2\right) \\
& = m^{t-1}\alpha!(m(t^{|\alpha|}-(t-1)^{|\alpha|}) + \sigma^2(t-1)^{|\alpha|}) + m^2\mathbf{E}\left(V_\alpha(t-1)^2\right).
\end{aligned}$$

This gives us that

$$0 < \mathbf{E}\left(\frac{V_\alpha(t)^2}{m^{2t}} - \frac{V_\alpha(t-1)^2}{m^{2(t-1)}}\right) \le c\frac{t^{|\alpha|}}{m^t}. \tag{2.23}$$

Hence, using the fact that $V_\alpha(t)/m^t$ is a martingale we have that

$$\begin{aligned}
\mathbf{E}\left(\sum_{t=1}^{\infty}\left|\frac{V_\alpha(t)}{m^t} - \frac{V_\alpha(t-1)}{m^{t-1}}\right|\right) &\le \sum_{t=1}^{\infty}\left\{\mathbf{E}\left(\left(\frac{V_\alpha(t)}{m^t} - \frac{V_\alpha(t-1)}{m^{t-1}}\right)^2\right)\right\}^{1/2} \\
(2.24)\qquad &= \sum_{t=1}^{\infty}\left\{\mathbf{E}\left(\frac{V_\alpha(t)^2}{m^{2t}} - \frac{V_\alpha(t-1)^2}{m^{2(t-1)}}\right)\right\}^{1/2} \\
&\le c\sum_{t=1}^{\infty}\frac{t^{|\alpha|/2}}{m^{t/2}} < \infty,
\end{aligned}$$

so that

$$\sum_{t=1}^{\infty}\left|\frac{V_\alpha(t)}{m^t} - \frac{V_\alpha(t-1)}{m^{t-1}}\right| < \infty, \qquad \text{a.s.} \tag{2.25}$$

This shows that $(\frac{1}{m^t}V_\alpha(t),\ t\ge 0)$ converges to a finite limit almost surely.$\diamondsuit$

Remark 2.7 Note that by induction from (2.22)

$$\mathbf{E}\left(V_\alpha(t)^2\right) = m^{t-1}\alpha!(\sigma^2 \sum_{j=1}^{t-1} m^{t-j} j^{|\alpha|} + m \sum_{j=1}^{t} m^{t-j}(j^{|\alpha|} - (j-1)^{|\alpha|})) \tag{2.26}$$

and therefore

$$\mathbf{E}\left(N_\alpha^2\right) = m^{-1}\alpha!(\sigma^2 \sum_{j=1}^{\infty} m^{-j} j^{|\alpha|} + m \sum_{j=1}^{\infty} m^{-j}(j^{|\alpha|} - (j-1)^{|\alpha|})). \tag{2.27}$$

Lemma 2.8 *Let $\alpha \in Z_+^d$, and let V_α, N_α be as in Lemma 2.6. Then for any $\varepsilon > 0$, we have that almost surely as $t \to \infty$,*

$$\frac{V_\alpha(t)}{m^t} = N_\alpha + o\left(m^{-t/(2+\varepsilon)}\right). \tag{2.28}$$

Proof. We claim that

$$\frac{V_\alpha(t^2)}{m^{t^2}} = \mathbf{E}\left(\frac{V_\alpha(t^2)}{m^{t^2}} \,\middle|\, \mathcal{F}(t)\right) + o\left(m^{-t/(2+2\varepsilon)}\right), \qquad \text{a.s.} \tag{2.29}$$

To see this, we first observe that by Fact 2.4, Chebyshev's inequality and the Borel–Cantelli lemma that almost surely for $t \to \infty$,

$$\max_{y \in \mathbb{R}^d} \left|\lambda(y,t^2) - \mathbf{E}\left(\lambda(y,t^2) \,|\, \mathcal{F}(t)\right)\right| = o\left(m^{t^2 - t/(2+\varepsilon)}\right).$$

Assembling this estimate with (2.16) and Lemma 2.5, together with the fact that $\sup_{\|y\| \le t^{2(1+\varepsilon)}} H_\alpha(y,t^2) \le ct^{2(1+\varepsilon)|\alpha|}$, we get (2.29).

Since $\mathbf{E}(\frac{V_\alpha(t^2)}{m^{t^2}} \,|\, \mathcal{F}(t)) = \frac{V_\alpha(t)}{m^t}$ (by Lemma 2.6), it follows from (2.29) that

$$\frac{V_\alpha(t^2)}{m^{t^2}} - \frac{V_\alpha(t)}{m^t} = o\left(m^{-t/(2+2\varepsilon)}\right), \qquad \text{a.s.}$$

As a consequence,

$$N_\alpha - \frac{V_\alpha(t)}{m^t} = \sum_{j=0}^{\infty} \left(\frac{V_\alpha(t^{2^{j+1}})}{m^{t^{2^{j+1}}}} - \frac{V_\alpha(t^{2^j})}{m^{t^{2^j}}}\right) = o\left(m^{-t/(2+2\varepsilon)}\right), \qquad \text{a.s.} \tag{2.30}$$

This proves our lemma, since $\varepsilon > 0$ is arbitrary. ♢

We have now all the ingredients to prove Theorem 1.1.

Proof of Theorem 1.1. Fix $k \geq 1$. Fix $0 < \gamma < \frac{1}{2(k+1)}$, and let $t = \lfloor T^\gamma \rfloor$. Let $\varepsilon > 0$ be such that $(1+\varepsilon)\gamma < \frac{1}{2(k+1)}$. We will show that, almost surely for $T \to \infty$,

$$\frac{\psi(A,T)}{m^T} = \frac{1}{(2\pi T)^{d/2}} \sum_{n=0}^{k} \frac{(-T)^{-n}}{2^n} \sum_{|\alpha|=n} \frac{1}{\alpha!} \sum_{\beta \preceq 2\alpha} \binom{2\alpha}{\beta} (-1)^{|\beta|} M_\beta(A) \frac{V_{2\alpha-\beta}(t)}{m^t}$$
$$+ o\left(T^{-(k+d/2)}\right) + O\left(m^{-t/(2+\varepsilon)}\right), \tag{2.31}$$

where $V_{2\alpha-\beta}$ is defined in (2.16). Our Theorem will then follow from Lemma 2.8.

By Fact 2.3, we have, almost surely for $T \to \infty$,

$$\frac{\psi(A,T)}{m^T} = \frac{1}{m^t} \int_A \sum_y p^{(d)}_{T-t}(y-x)\lambda(y,t)dx + o\left(m^{-t/(2+\varepsilon)}\right).$$

On the other hand we can write

$$(2\pi T)^{d/2} p^{(d)}_{T-t}(y-x) = \frac{1}{(1-t/T)^{d/2}} \exp\left(-\frac{\|y-x\|^2}{2(T-t)}\right) = f(z,t,x,y), \tag{2.32}$$

where $z = 1/T$ and

$$f(z,t,x,y) = \frac{1}{(1-tz)^{d/2}} \exp\left(-\frac{\|y-x\|^2 z}{2(1-tz)}\right) \tag{2.33}$$

is a C^∞ function of z near $z=0$ as long as $tz \ll 1$. If we expand $f(z,t,x,y)$ in a finite Taylor series in z around $z=0$, it is clear that we can bound the remainder $R_{k+1}(z,t,x,y)$ of order $k+1$ by a polynomial in $\|y-x\|$ of order at most $2(k+1)$.

According to Lemma 2.5, almost surely for all large T, $\lambda(y,t) = 0$ as long as $\|y\| > T^{(1+\varepsilon)\gamma}$. Together with (1.1) which implies that the number of points y with $\lambda(y,t) \neq 0$ is bounded by cm^t and the fact that A is bounded we have

$$\frac{1}{m^t} \int_A \sum_y R_{k+1}(T^{-1},t,x,y)\lambda(y,t)dx \leq cT^{2(1+\varepsilon)\gamma(k+1)} = o(T). \tag{2.34}$$

By inspection of Lemma 2.2, the first k terms in the Taylor series for $f(z,t,x,y)$ give rise to the the first line of (2.31), completing the proof of that formula and hence of our Theorem. ♢

3 L^p convergence

In this section we show that if the offspring distribution Y has p moments for some even integer p then $\frac{V_\alpha(t)}{m^t}$ converges in L^p.

Introduce the notation

$$\widetilde{\sum}_{i_1,\ldots,i_j=1}^{n} =: \sum_{\substack{i_1,\ldots,i_j=1\\ i_l\neq i_m,\,\forall l\neq m}}^{n}$$

for summation over non-repeated indices. Let $Z_t = \psi(I\!R^d, t)$, $F_{\alpha;i}(t) = H_\alpha(W^{(i)}(t), t)$ and

$$U_{\alpha^{(1)},\ldots,\alpha^{(p)}}(t) = \widetilde{\sum}_{i_1,\ldots,i_p=1}^{Z_t} \prod_{h=1}^{p} F_{\alpha^{(h)};i_h}(t). \tag{3.1}$$

The following Lemma will play an important role in showing that $\frac{V_\alpha(t)}{m^t}$ converges in L^p.

Lemma 3.1 *Let k be an integer with $\mathbf{E}(|Y|^k) < \infty$. Then for any $\alpha^{(1)},\ldots,\alpha^{(k)}$ we can find $c, \beta < \infty$ independent of t such that*

$$\left|\mathbf{E}\left(U_{\alpha^{(1)},\ldots,\alpha^{(k)}}(t)\right)\right| \leq ct^\beta m^{kt}. \tag{3.2}$$

Proof of Lemma 3.1. We will prove this Lemma by induction on k. The case of $k=1$ is trivial. Assume that we have proven this Lemma for all $k \leq p-1$.

We can write

$$\begin{aligned} U_{\alpha^{(1)},\ldots,\alpha^{(p)}}(t) &= \widetilde{\sum}_{i_1,\ldots,i_p=1}^{Z_t} \prod_{h=1}^{p} F_{\alpha^{(h)};i_h}(t) \\ &= \sum_{k=1}^{p} \widetilde{\sum}_{i_1,\ldots,i_k=1}^{Z_{t-1}} \sum_{A_1\cup\cdots\cup A_k=[1,p]} \prod_{h=1}^{k} \left(\widetilde{\sum}_{j_s=1,\forall s\in A_h}^{Y_{i_h,t-1}} \prod_{m\in A_h} F_{\alpha^{(m)};i_h,j_m}(t) \right) \end{aligned} \tag{3.3}$$

where the sum $\sum_{A_1\cup\cdots\cup A_k=[1,p]}$ runs over all partitions of $[1,p] = \{1,\ldots,p\}$ by k non-empty sets $A_1,\ldots,A_k$ and $F_{\alpha;l,m}(t) = H_\alpha(W^{(l,m)}(t), t)$. Introducing the falling factorial notation $(x)_k = x(x-1)\cdots(x-k+1))$ we have that

$$\begin{aligned} &\mathbf{E}\left(\prod_{h=1}^{k} \left(\widetilde{\sum}_{j_s=1,\forall s\in A_h}^{Y_{i_h,t-1}} \prod_{m\in A_h} F_{\alpha^{(m)};i_h,j_m}(t) \right) \Big| \mathcal{F}(t-1) \right) \\ &= \prod_{h=1}^{k} \mathbf{E}\left((Y)_{|A_h|}\right) \prod_{m\in A_h} F_{\alpha^{(m)};i_h}(t-1). \end{aligned} \tag{3.4}$$

Hence

$$(3.5)\quad \mathbf{E}\left(U_{\alpha^{(1)},\dots,\alpha^{(p)}}(t)\,\Big|\,\mathcal{F}(t-1)\right)$$
$$= \sum_{k=1}^{p} \widetilde{\sum}_{i_1,\dots,i_k=1}^{Z_{t-1}} \sum_{A_1\cup\dots\cup A_k=[1,p]} \prod_{h=1}^{k} \mathbf{E}\left((Y)_{|A_h|}\right) \prod_{m\in A_h} F_{\alpha^{(m)};\,i_h}(t-1)$$
$$= m^p U_{\alpha^{(1)},\dots,\alpha^{(p)}}(t-1)$$
$$+\sum_{k=1}^{p-1} \widetilde{\sum}_{i_1,\dots,i_k=1}^{Z_{t-1}} \sum_{A_1\cup\dots\cup A_k=[1,p]} \prod_{h=1}^{k} \mathbf{E}\left((Y)_{|A_h|}\right) \prod_{m\in A_h} F_{\alpha^{(m)};\,i_h}(t-1).$$

Note that by (2.11)

$$(3.6)\qquad \sum_{n=0}^{\infty} \frac{r^n}{n!} H_n(x,t) \sum_{m=0}^{\infty} \frac{s^m}{m!} H_m(x,t) = e^{rx-tr^2/2} e^{sx-ts^2/2}.$$
$$= e^{(r+s)x-t(r+s)^2/2} e^{trs}$$
$$= \sum_{j=0}^{\infty} \frac{(r+s)^j}{j!} H_j(x,t) \sum_{k=0}^{\infty} \frac{(trs)^k}{k!}$$
$$= \sum_{j=0}^{\infty} \sum_{i=0}^{j} \frac{r^i s^{j-i}}{i!(j-i)!} H_j(x,t) \sum_{k=0}^{\infty} \frac{(trs)^k}{k!}.$$

Equating coefficients of $r^n s^m$ we find that

$$(3.7)\quad H_n(x,t)H_m(x,t) = n!m! \sum_{k=0}^{m\wedge n} \frac{t^k}{k!} \frac{1}{(n-k)!(m-k)!} H_{n+m-2k}(x,t).$$

Using this to reduce products of Hermite functions to sums we find that

$$(3.8)\quad \mathbf{E}\left(U_{\alpha^{(1)},\dots,\alpha^{(p)}}(t)\,\Big|\,\mathcal{F}(t-1)\right) = m^p U_{\alpha^{(1)},\dots,\alpha^{(p)}}(t-1)$$
$$+\sum_{j=1}^{p-1} \sum_{\beta^{(1)},\dots,\beta^{(j)}} c(\alpha;\,p;\beta^{(1)},\dots,\beta^{(j)};t) U_{\beta^{(1)},\dots,\beta^{(j)}}(t-1)$$

where $\sum_{\beta^{(1)},\dots,\beta^{(j)}}$ is a finite sum over $\beta^{(1)},\dots,\beta^{(j)}$ such that $\sum_{l=1}^{j} |\beta^{(l)}| \leq \sum_{l=1}^{p} |\alpha^{(l)}|$ and the $c(\alpha;\,p;\beta^{(1)},\dots,\beta^{(j)};t)$ are polynomials in t. Hence by our induction hypothesis

$$(3.9)\qquad \mathbf{E}\left(U_{\alpha^{(1)},\dots,\alpha^{(p)}}(t)\right) = m^p \mathbf{E}\left(U_{\alpha^{(1)},\dots,\alpha^{(p)}}(t-1)\right) + \mathcal{R}_{\alpha^{(1)},\dots,\alpha^{(p)}}(t)$$

with $|\mathcal{R}_{\alpha^{(1)},\dots,\alpha^{(p)}}(t)| \leq ct^\beta m^{(p-1)(t-1)}$ for some $\beta, c < \infty$ independent of t. Iterating this completes the proof of our Lemma for $k = p$. ♢

Proposition 3.2 *Let p be an even integer with $\mathbf{E}(|Y|^p) < \infty$. Then $\frac{V_\alpha(t)}{m^t}$ converges in L^p.*

Proof of Proposition 3.2. Note that because of the presence of the polynomial factor t^β in (3.2) we cannot simply use Lemma 3.1 to show that $\frac{V_\alpha(t)}{m^t}$ is bounded uniformly in L^p. Rather, we will show that for some $c, \beta < \infty$ independent of t

$$|\mathbf{E}\left(\{V_\alpha(t) - mV_\alpha(t-1)\}^p\right)| \le ct^\beta m^{t(p-1)}. \tag{3.10}$$

Then

$$\left|\mathbf{E}\left(\left\{\frac{V_\alpha(t)}{m^t} - \frac{V_\alpha(t-1)}{m^{t-1}}\right\}^p\right)\right| \le ct^\beta m^{-t} \tag{3.11}$$

and therefore (it is here that we need p even)

$$\sum_{t=1}^{\infty}\left\|\frac{V_\alpha(t)}{m^t} - \frac{V_\alpha(t-1)}{m^{t-1}}\right\|_p \le c\sum_{t=1}^{\infty} t^{\beta/p}m^{-t/p} < \infty \tag{3.12}$$

which will complete the proof of the proposition.

The basic idea of the proof of (3.10) is that the subtraction eliminates the highest order term in the expectation leaving only sums of terms of the form $U_{\alpha^{(1)},\dots,\alpha^{(k)}}(t)$ with $k \le p-1$.

We now prove (3.10). We have that

$$\begin{aligned} &\mathbf{E}\left(\{V_\alpha(t) - mV_\alpha(t-1)\}^p\right) \\ &= \sum_{k=0}^{p}\binom{p}{k}(-1)^k m^k \mathbf{E}\left(V_\alpha^{p-k}(t)V_\alpha^k(t-1)\right) \\ &= \sum_{k=0}^{p}\binom{p}{k}(-1)^k m^k \mathbf{E}\left(\mathbf{E}\left(V_\alpha^{p-k}(t)\,\middle|\,\mathcal{F}(t-1)\right)V_\alpha^k(t-1)\right). \end{aligned} \tag{3.13}$$

By (2.18) we have

$$V_\alpha(t) = \sum_{l=1}^{Z_{t-1}}\sum_{m=1}^{Y_{l,t-1}} F_{\alpha;\,l,m}(t) \tag{3.14}$$

where $F_{\alpha;\,l,m}(t) = H_\alpha(W^{(l,m)}(t), t)$. Thus

$$\begin{aligned} &V_\alpha^n(t) \\ &= \sum_{j=1}^{n}\widetilde{\sum}_{i_1,\dots,i_j=1}^{Z_{t-1}}\sum_{l_1+\cdots+l_j=n}\binom{n}{l_1,\dots,l_j}\prod_{h=1}^{j}\left(\sum_{r=1}^{Y_{i_h,t-1}} F_{\alpha;\,i_h,r}(t)\right)^{l_h} \end{aligned} \tag{3.15}$$

$$= \widetilde{\sum}_{i_1,\dots,i_n=1}^{Z_{t-1}} \prod_{h=1}^{n} \left(\sum_{r=1}^{Y_{i_h,t-1}} F_{\alpha;\, i_h,r}(t) \right)$$

$$+ \sum_{j=1}^{n-1} \widetilde{\sum}_{i_1,\dots,i_j=1}^{Z_{t-1}} \sum_{l_1+\dots+l_j=n} \binom{n}{l_1,\dots,l_j} \prod_{h=1}^{j} \left(\sum_{r=1}^{Y_{i_h,t-1}} F_{\alpha;\, i_h,r}(t) \right)^{l_h}$$

and

$$\left(\sum_{r=1}^{Y_{i_h,t-1}} F_{\alpha;\, i_h,r}(t) \right)^{l_h} = \sum_{s=1}^{l_h} \widetilde{\sum}_{r_1,\dots,r_s=1}^{Y_{i_h,t-1}} \sum_{q_1+\dots+q_s=l_h} \binom{l_h}{q_1,\dots,q_s} \prod_{f=1}^{s} F^{q_f}_{\alpha;\, i_h,r_f}(t). \tag{3.16}$$

Thus

$$\mathbf{E}\left(\left(\sum_{r=1}^{Y_{i_h,t-1}} F_{\alpha;\, i_h,r}(t) \right)^{l_h} \Big| \mathcal{F}(t-1) \right) = \sum_{s=1}^{l_h} \mathbf{E}\left((Y)_s \right) \sum_{q_1+\dots+q_s=l_h} \binom{l_h}{q_1,\dots,q_s} \prod_{f=1}^{s} \mathbf{E}\left(F^{q_f}_{\alpha;\, i_h}(t) \Big| \mathcal{F}(t-1) \right). \tag{3.17}$$

Using (3.7) to reduce products of Hermite functions to sums we find that by (3.15)-(3.17) we can write, with $\alpha^{(i)} = \alpha$, $i = 1, \dots, n$

$$\mathbf{E}\left(V^n_\alpha(t) \Big| \mathcal{F}(t-1) \right) = m^n U_{\alpha^{(1)},\dots,\alpha^{(n)}}(t-1) + \sum_{j=1}^{n-1} \sum_{\beta^{(1)},\dots,\beta^{(j)}} c(\alpha;\, n; \beta^{(1)},\dots,\beta^{(j)}; t) U_{\beta^{(1)},\dots,\beta^{(j)}}(t-1) \tag{3.18}$$

where $\sum_{\beta^{(1)},\dots,\beta^{(j)}}$ is a finite sum and the $c(\alpha;\, n; \beta^{(1)},\dots,\beta^{(j)}; t)$ are polynomials in t.

We next observe that

$$\begin{aligned} V^n_\alpha(t-1) &= \left(\sum_{l=1}^{Z_{t-1}} F_{\alpha;\, l}(t) \right)^n \\ &= \sum_{j=1}^{n} \widetilde{\sum}_{i_1,\dots,i_j=1}^{Z_{t-1}} \sum_{l_1+\dots+l_j=n} \binom{n}{l_1,\dots,l_j} \prod_{h=1}^{j} F^{l_h}_{\alpha;\, i_h}(t-1) \\ &= U_{\alpha^{(1)},\dots,\alpha^{(n)}}(t-1) \\ &\quad + \sum_{j=1}^{n-1} \widetilde{\sum}_{i_1,\dots,i_j=1}^{Z_{t-1}} \sum_{l_1+\dots+l_j=n} \binom{n}{l_1,\dots,l_j} \prod_{h=1}^{j} F^{l_h}_{\alpha;\, i_h}(t-1) \end{aligned} \tag{3.19}$$

$$= U_{\alpha^{(1)},\dots,\alpha^{(n)}}(t-1) + \sum_{j=1}^{n-1} \sum_{\gamma^{(1)},\dots,\gamma^{(j)}} d(\alpha;\, n; \gamma^{(1)},\dots,\gamma^{(j)}; t) U_{\gamma^{(1)},\dots,\gamma^{(j)}}(t-1)$$

where we have again used (3.7) to reduce products of Hermite functions to sums, and the $d(\alpha;\, n; \gamma^{(1)},\dots,\gamma^{(j)}; t)$ are polynomials in t.

Similarly

$$(3.20)\quad \begin{aligned} &U_{\beta^{(1)},\dots,\beta^{(j)}}(t-1) U_{\gamma^{(1)},\dots,\gamma^{(k)}}(t-1) \\ &= \left(\widetilde{\sum}_{i_1,\dots,i_j=1}^{Z_{t-1}} \prod_{h=1}^{j} F_{\beta^{(h)};\, i_h}(t-1) \right) \left(\widetilde{\sum}_{j_1,\dots,j_k=1}^{Z_{t-1}} \prod_{l=1}^{k} F_{\gamma^{(l)};\, j_l}(t-1) \right) \\ &= U_{\beta^{(1)},\dots,\beta^{(j)},\gamma^{(1)},\dots,\gamma^{(k)}}(t-1) \\ &\quad + \sum_{m=1}^{j+k-1} \sum_{\zeta^{(1)},\dots,\zeta^{(m)}} f(\beta, \gamma; \zeta^{(1)},\dots,\zeta^{(m)}; t) U_{\zeta^{(1)},\dots,\zeta^{(m)}}(t-1) \end{aligned}$$

where we have abreviated $\beta = (\beta^{(1)},\dots,\beta^{(j)})$, $\gamma = (\gamma^{(1)},\dots,\gamma^{(k)})$.

Combining (3.18)-(3.20) we have that for each $k \le p$

$$(3.21)\quad \begin{aligned} &m^k \mathbf{E}\left(V_\alpha^{p-k}(t) \,\middle|\, \mathcal{F}(t-1) \right) V_\alpha^k(t-1) \\ &= m^p U_{\alpha^{(1)},\dots,\alpha^{(p)}}(t-1) \\ &\quad + \sum_{j=1}^{p-1} \sum_{\gamma^{(1)},\dots,\gamma^{(j)}} f(\alpha;\, n; \gamma^{(1)},\dots,\gamma^{(j)}; t) U_{\gamma^{(1)},\dots,\gamma^{(j)}}(t-1) \end{aligned}$$

where the $f(\alpha;\, n; \gamma^{(1)},\dots,\gamma^{(j)}; t)$ are polynomials in t. Substituting back into (3.13) and using the fact that $\sum_{k=0}^{p} \binom{p}{k}(-1)^k = 0$ we find that the $m^p U_{\alpha^{(1)},\dots,\alpha^{(p)}}(t-1)$'s cancel, and we can write

$$(3.22)\quad \begin{aligned} &\mathbf{E}\left(\{V_\alpha(t) - m V_\alpha(t-1)\}^p \right) \\ &= \sum_{j=1}^{p-1} \sum_{\gamma^{(1)},\dots,\gamma^{(j)}} g(\alpha;\, n; \gamma^{(1)},\dots,\gamma^{(j)}; t) \mathbf{E}\left(U_{\gamma^{(1)},\dots,\gamma^{(j)}}(t-1) \right) \end{aligned}$$

where the $g(\alpha;\, n; \gamma^{(1)},\dots,\gamma^{(j)}; t)$ are polynomials in t. (3.2) then completes the proof of (3.10) and hence of our Proposition. ◊

Remark 3.3 Note that by Proposition 3.2 we have that $\|\frac{V_\alpha(t)}{m^t}\|_p$ is bounded uniformly in t, so that

$$(3.23)\qquad \|V_\alpha(t)\|_p \le c m^t.$$

Arguing as before, any $U_{\alpha^{(1)},\dots,\alpha^{(p)}}(t)$, where $\alpha^{(1)},\dots,\alpha^{(k)}$ are now arbitrary, can be written as

$$U_{\alpha^{(1)},\dots,\alpha^{(p)}}(t) = \prod_{i=1}^{p} V_{\alpha^{(i)}}(t) + \text{ terms of `lower order'} \tag{3.24}$$

and thus using (3.23), Hölder's inequality and (3.2) for $k \le p-1$ we can refine (3.2) and find $c, \beta < \infty$ independent of t such that

$$\left|\mathbf{E}\left(U_{\alpha^{(1)},\dots,\alpha^{(p)}}(t)\right)\right| \le cm^{pt}. \tag{3.25}$$

(Here we require that Y have r momnets for some even $r \ge p$).

References

[1] Athreya, K.B. and Ney, P.E. (1972). *Branching Processes*. Springer, New York.

[2] Chen, X. (2001). Exact convergence rates for the distribution of particles in branching random walks. *Ann. Appl. Probab.* **11**, 1242–1262.

[3] Lebedev, N.N. (1972). *Special Functions and their Applications*. Dover, New York.

[4] Révész, P. (1994). *Random Walks of Infinitely Many Particles*. World Scientific, Singapore.

[5] Révész, P. (2004). A prediction problem of the branching random walk. *J. Appl. Probab.* **41A**, 25–31.

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