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Cover times for Brownian motion and random walks in two dimensions

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Abstract

Let $\mathcal{T}(x,\varepsilon)$ denote the first hitting time of the disc of radius ε centered at x for Brownian motion on the two dimensional torus $\mathbb{T}^2$. We prove that $\sup_{x\in\mathbb{T}^2}\mathcal{T}(x,\varepsilon)/|\log\varepsilon|^2 \to 2/\pi$ as $\varepsilon\to 0$. The same applies to Brownian motion on any smooth, compact connected, two-dimensional, Riemannian manifold with unit area and no boundary. As a consequence, we prove a conjecture, due to Aldous (1989), that the number of steps it takes a simple random walk to cover all points of the lattice torus $\mathbb{Z}_n^2$ is asymptotic to $4n^2(\log n)^2/\pi$. Determining these asymptotics is an essential step toward analyzing the fractal structure of the set of uncovered sites before coverage is complete; so far, this structure was only studied nonrigorously in the physics literature. We also establish a conjecture, due to Kesten and Révész, that describes the asymptotics for the number of steps needed by simple random walk in $\mathbb{Z}^2$ to cover the disc of radius n.

1. Introduction

In this paper, we introduce a unified method for analyzing cover times for random walks and Brownian motion in two dimensions, and resolve several open problems in this area.

1.1. *Covering the discrete torus.* The time it takes a random walk to cover a finite graph is a parameter that has been studied intensively by probabilists, combinatorialists and computer scientists, due to its intrinsic appeal and its applications to designing universal traversal sequences [5], [10], [11], testing graph connectivity [5], [19], and protocol testing [24]; see [2] for an introduction

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to cover times. Aldous and Fill [4, Chap. 7] consider the cover time for random walk on the discrete d-dimensional torus $\mathbb{Z}_n^d = \mathbb{Z}^d/n\mathbb{Z}^d$, and write:

> "*Perhaps surprisingly, the case $d = 2$ turns out to be the hardest of all explicit graphs for the purpose of estimating cover times.*"

The problem of determining the expected cover time $\mathcal{T}_n$ for $\mathbb{Z}_n^2$ was posed informally by Wilf [29] who called it "the white screen problem" and wrote

> "*Any mathematician will want to know how long, on the average, it takes until each pixel is visited.*"

(see also [4, p. 1]).

In 1989, Aldous [1] conjectured that $\mathcal{T}_n/(n\log n)^2 \to 4/\pi$. He noted that the upper bound $\mathcal{T}_n/(n\log n)^2 \le 4/\pi + o(1)$ was easy, and pointed out the difficulty of obtaining a corresponding lower bound. A lower bound of the correct order of magnitude was obtained by Zuckerman [30], and in 1991, Aldous [3] showed that $\mathcal{T}_n/\mathbb{E}(\mathcal{T}_n) \to 1$ in probability. The best lower bound prior to the present work is due to Lawler [21], who showed that $\liminf \mathbb{E}(\mathcal{T}_n)/(n\log n)^2 \ge 2/\pi$.

Our main result in the discrete setting, is the proof of Aldous's conjecture:

THEOREM 1.1. *If $\mathcal{T}_n$ denotes the time it takes for the simple random walk in $\mathbb{Z}_n^2$ to cover $\mathbb{Z}_n^2$ completely, then*

$$\lim_{n\to\infty} \frac{\mathcal{T}_n}{(n\log n)^2} = \frac{4}{\pi} \quad \textit{in probability.} \tag{1.1}$$

The main interest in this result is not the value of the constant, but rather that establishing a limit theorem, with matching upper and lower bounds, forces one to develop insight into the delicate process of coverage, and to understand the fractal structure, and spatial correlations, of the configuration of uncovered sites in $\mathbb{Z}_n^2$ before coverage is complete.

The fractal structure of the uncovered set in $\mathbb{Z}_n^2$ has attracted the interest of physicists, (see [25], [12] and the references therein), who used simulations and nonrigorous heuristic arguments to study it. One cannot begin the rigorous study of this fractal structure without knowing precise asymptotics for the cover time; an estimate of cover time up to a bounded factor will not do. See [14] for quantitative results on the uncovered set, based on the ideas of the present paper.

Our proof of Theorem 1.1 is based on strong approximation of random walks by Brownian paths, which reduces that theorem to a question about Brownian motion on the 2-torus.

1.2. *Brownian motion on surfaces.* For x in the two-dimensional torus $\mathbb{T}^2$, denote by $D_{\mathbb{T}^2}(x,\varepsilon)$ the disk of radius ε centered at x, and consider the hitting time

$$\mathcal{T}(x,\varepsilon) = \inf\{t > 0 \mid X_t \in D_{\mathbb{T}^2}(x,\varepsilon)\}.$$

Then

$$\mathcal{C}_\varepsilon = \sup_{x\in\mathbb{T}^2} \mathcal{T}(x,\varepsilon)$$

is the ε-covering time of the torus $\mathbb{T}^2$, i.e. the amount of time needed for the Brownian motion X_t to come within ε of each point in $\mathbb{T}^2$. Equivalently, $\mathcal{C}_\varepsilon$ is the amount of time needed for the Wiener sausage of radius ε to completely cover $\mathbb{T}^2$. We can now state the continuous analog of Theorem 1.1, which is the key to its proof.

THEOREM 1.2. *For Brownian motion in* $\mathbb{T}^2$, *almost surely* (a.s.),

$$\lim_{\varepsilon\to 0} \frac{\mathcal{C}_\varepsilon}{(\log\varepsilon)^2} = \frac{2}{\pi}. \tag{1.2}$$

Matthews [23] studied the ε-cover time for Brownian motion on a d dimensional sphere (embedded in $\mathbb{R}^{d+1}$) and on a d-dimensional projective space (that can be viewed as the quotient of the sphere by reflection). He calls these questions the "one-cap problem" and "two-cap problem", respectively. Part of the motivation for this study is a technique for viewing multidimensional data developed by Asimov [7]. Matthews obtained sharp asymptotics for all dimensions $d \geq 3$, but for the more delicate two dimensional case, his upper and lower bounds had a ratio of 4 between them; he conjectured the upper bound was sharp. We can now resolve this conjecture; rather than handling each surface separately, we establish the following extension of Theorem 1.2. See Section 8 for definitions and references concerning Brownian motion on manifolds.

THEOREM 1.3. *Let M be a smooth, compact, connected, two-dimensional, Riemannian manifold without boundary. Denote by $\mathcal{C}_\varepsilon$ the ε-covering time of M, i.e., the amount of time needed for the Brownian motion to come within* (*Riemannian*) *distance ε of each point in M. Then*

$$\lim_{\varepsilon\to 0} \frac{\mathcal{C}_\varepsilon}{(\log\varepsilon)^2} = \frac{2}{\pi}A \quad \text{a.s.}, \tag{1.3}$$

where A denotes the Riemannian area of M.

(When M is a sphere, this indeed corresponds to the upper bound in [23], once a computational error in [23] is corrected; the hitting time in (4.3) there is twice what it should be. This error led to doubling the upper and the lower bounds for cover time in [23, Theorem 5.7].)

1.3. *Covering a large disk by random walk in* $\mathbb{Z}^2$. Over ten years ago, Kesten (as quoted by Aldous [1] and Lawler [21]) and Révész [26] independently considered a problem about simple random walks in $\mathbb{Z}^2$: *How long does it take for the walk to completely cover the disc of radius* n? Denote this time by T_n. Kesten and Révész proved that

(1.4)
$$e^{-b/t} \le \liminf_{n\to\infty} \mathbf{P}(\log T_n \le t(\log n)^2) \le \limsup_{n\to\infty} \mathbf{P}(\log T_n \le t(\log n)^2) \le e^{-a/t}$$

for certain $0 < a < b < \infty$. Révész [26] conjectured that the limit exists and has the form $e^{-\lambda/t}$ for some (unspecified) λ. Lawler [21] obtained (1.4) with the constants $a = 2$, $b = 4$ and quoted a conjecture of Kesten that the limit equals $e^{-4/t}$. We can now prove this:

Theorem 1.4. *If* T_n *denotes the time it takes for the simple random walk in* $\mathbb{Z}^2$ *to completely cover the disc of radius* n, *then*

$$\lim_{n\to\infty} \mathbf{P}(\log T_n \le t(\log n)^2) = e^{-4/t}. \tag{1.5}$$

1.4. *A birds-eye view.* The basic approach of this paper, as in [13], is to control ε-hitting times using excursions between concentric circles. The number of excursions between two fixed concentric circles before ε-coverage is so large, that the ε-hitting times will necessarily be concentrated near their conditional means given the excursion counts (see Lemma 3.2).

The key idea in the proof of the lower bound in Theorem 1.2, is to control excursions on many scales simultaneously, leading to a ‘multi-scale refinement’ of the classical second moment method. This is inspired by techniques from probability on trees, in particular the analysis of first-passage percolation by Lyons and Pemantle [22]. The approximate tree structure that we (implicitly) use arises by consideration of circles of varying radii around different centers; for fixed centers x, y, and “most” radii r (on a logarithmic scale) the discs $D_{\mathbb{T}^2}(x, r)$ and $D_{\mathbb{T}^2}(y, r)$ are either well-separated (if $r \ll d(x, y)$) or almost coincide (if $r \gg d(x, y)$). This tree structure was also the key to our work in [13], but the dependence problems encountered in the present work are more severe. While in [13] the number of macroscopic excursions was bounded, here it is large; In the language of trees, one can say that while in [13] we studied the maximal number of visits to a leaf until visiting the root, here we study the number of visits to the root until every leaf has been visited. For the analogies between trees and Brownian excursions to be valid, the effect of the initial and terminal points of individual excursions must be controlled. To prevent conditioning on the endpoints of the numerous macroscopic excursions to affect the estimates, the ratios between radii of even the largest pair of concentric circles where excursions are counted, must grow to infinity as ε decreases to zero.

Section 2 provides simple lemmas which will be useful in exploiting the link between excursions and ε-hitting times. These lemmas are then used to obtain the upper bound in Theorem 1.2. In Section 3 we explain how to obtain the analogous lower bound, leaving some technical details to lemmas which are proven in Sections 6 and 7. In Section 4 we prove the lattice torus covering time conjecture, Theorem 1.1, and in Section 5 we prove the Kesten-Révész conjecture, Theorem 1.4. In Section 8 we consider Brownian motion on manifolds and prove Theorem 1.3. Complements and open problems are collected in the final section.

2. Hitting time estimates and upper bounds

We start with some definitions. Let $\{W_t\}_{t\geq 0}$ denote planar Brownian motion started at the origin. We use $\mathbb{T}^2$ to denote the two dimensional torus, which we identify with the set $(-1/2, 1/2]^2$. The distance between $x, y \in \mathbb{T}^2$, in the natural metric, is denoted $d(x, y)$. Let $X_t = W_t \,\overline{\text{mod}}\, \mathbb{Z}^2$ denote the Brownian motion on $\mathbb{T}^2$, where $a \,\overline{\text{mod}}\, \mathbb{Z}^2 = [a+(1/2, 1/2)] \bmod \mathbb{Z}^2 - (1/2, 1/2)$. Throughout, $D(x, r)$ and $D_{\mathbb{T}^2}(x, r)$ denote the open discs of radius r centered at x, in $\mathbb{R}^2$ and in $\mathbb{T}^2$, respectively.

Fixing $x \in \mathbb{T}^2$ let $\tau_\xi = \inf\{t \geq 0 : X_t \in \partial D_{\mathbb{T}^2}(x, \xi)\}$ for $\xi > 0$. Also let $\widetilde{\tau}_\xi = \inf\{t \geq 0 : B_t \in \partial D(0, \xi)\}$, for a standard Brownian motion B_t on $\mathbb{R}^2$. For any $x \in \mathbb{T}^2$, the natural bijection $i = i_x : D_{\mathbb{T}^2}(x, 1/2) \mapsto D(0, 1/2)$ with $i_x(x) = 0$ is an isometry, and for any $z \in D_{\mathbb{T}^2}(x, 1/2)$ and Brownian motion X_t on $\mathbb{T}^2$ with $X_0 = z$, we can find a Brownian motion B_t starting at $i_x(z)$ such that $\tau_{1/2} = \widetilde{\tau}_{1/2}$ and $\{i_x(X_t), t \leq \tau_{1/2}\} = \{B_t, t \leq \widetilde{\tau}_{1/2}\}$. We shall hereafter use i to denote i_x, whenever the precise value of x is understood from the context, or does not matter.

We start with some uniform estimates on the hitting times $\mathbb{E}^y(\tau_r)$.

LEMMA 2.1. *For some $c < \infty$ and all $r > 0$ small enough*,

$$\|\tau_r\| := \sup_y \mathbb{E}^y(\tau_r) \leq c|\log r|\,. \tag{2.1}$$

Further, there exists $\eta(R) \to 0$ as $R \to 0$, such that for all $0 < 2r \leq R$, $x \in \mathbb{T}^2$,

$$\begin{aligned} \frac{(1-\eta)}{\pi}\log\left(\frac{R}{r}\right) &\leq \inf_{y\in\partial D_{\mathbb{T}^2}(x,R)} \mathbb{E}^y(\tau_r) \\ &\leq \sup_{y\in\partial D_{\mathbb{T}^2}(x,R)} \mathbb{E}^y(\tau_r) \leq \frac{(1+\eta)}{\pi}\log\left(\frac{R}{r}\right)\,. \end{aligned} \tag{2.2}$$

Proof of Lemma 2.1. Let Δ denote the Laplacian, which on $\mathbb{T}^2$ is just the Euclidean Laplacian with periodic boundary conditions. It is well known that for any $x \in \mathbb{T}^2$ there exists a Green's function $G_x(y)$, defined for $y \in \mathbb{T}^2 \setminus \{x\}$, such that $\Delta G_x = 1$ and $F(x,y) = G_x(y) + \frac{1}{2\pi}\log d(x,y)$ is continuous on $\mathbb{T}^2 \times \mathbb{T}^2$ (c.f. [8, p. 106] or [16] where this is shown in the more general context of smooth, compact two-dimensional Riemannian manifolds without boundary). For completeness, we explicitly construct such $G_x(\cdot)$ at the end of the proof.

Let $e(y) = E^y(\tau_r)$. We have Poisson's equation $\frac{1}{2}\Delta e = -1$ on $\mathbb{T}^2 \setminus D_{\mathbb{T}^2}(x,r)$ and $e = 0$ on $\partial D_{\mathbb{T}^2}(x,r)$. Hence, with x fixed,

$$\Delta\left(G_x + \frac{1}{2}e\right) = 0 \quad \text{on } \mathbb{T}^2 \setminus D_{\mathbb{T}^2}(x,r). \tag{2.3}$$

Applying the maximum principle for the harmonic function $G_x + \frac{1}{2}e$ on $\mathbb{T}^2 \setminus D_{\mathbb{T}^2}(x,r)$, we see that for all $y \in \mathbb{T}^2 \setminus D_{\mathbb{T}^2}(x,r)$,

$$\inf_{z \in \partial D_{\mathbb{T}^2}(x,r)} G_x(z) \le G_x(y) + \frac{1}{2}e(y) \le \sup_{z \in \partial D_{\mathbb{T}^2}(x,r)} G_x(z). \tag{2.4}$$

Our lemma follows then, with

$$\eta(R) = \frac{2\pi}{\log 2} \sup_{x \in \mathbb{T}^2} \sup_{y,z \in D_{\mathbb{T}^2}(x,R)} |F(x,z) - F(x,y)|$$
$$c = (1/\pi) + [(1/\pi)\log \operatorname{diam}(\mathbb{T}^2) + 4 \sup_{x,y \in \mathbb{T}^2} |F(x,y)|]/\log 4 < \infty\ ,$$

except that we have proved (2.1) so far only for $y \notin D_{\mathbb{T}^2}(x,r)$. To complete the proof, fix $x' \in \mathbb{T}^2$ with $d(x,x') = 3\rho > 0$. For $r < \rho$, starting at $X_0 = y \in D_{\mathbb{T}^2}(x,r)$, the process X_t hits $\partial D_{\mathbb{T}^2}(x,r)$ before it hits $\partial D_{\mathbb{T}^2}(x',r)$. Consequently, $E^y(\tau_r) \le c|\log r|$ also for such y and r, establishing (2.1).

Turning to constructing $G_x(y)$, we use the representation $\mathbb{T}^2 = (-1/2, 1/2]^2$. Let $\phi \in C^\infty(\mathbb{R})$ be such that $\phi = 1$ in a small neighborhood of 0, and $\phi = 0$ outside a slightly larger neighborhood of 0. With $r = |z|$ for $z = (z_1, z_2)$, let

$$h(z) = -\frac{1}{2\pi}\phi(r)\log r$$

and note that by Green's theorem

$$\int_{\mathbb{T}^2} \Delta h(z)\, dz = 1. \tag{2.5}$$

Recall that for any function f which depends only on $r = |z|$,

$$\Delta f = f'' + \frac{1}{r}f',$$

and therefore, for $r > 0$

$$\Delta h(z) = -\frac{1}{2\pi}(\phi''(r)\log r + \frac{2 + \log r}{r}\phi'(r)).$$

Because of the support properties of $\phi(r)$ we see that $H(z) = \Delta h(z) - 1$ is a C^∞ function on $\mathbb{T}^2$, and consequently has an expansion in Fourier series

$$H(z) = \sum_{j,k=0}^{\infty} a_{j,k} \cos(2\pi j z_1)\cos(2\pi k z_2)$$

with $a_{j,k}$ rapidly decreasing. Note that as a consequence of (2.5) we have $a_{0,0} = 0$. Set

$$F(z) = \sum_{\substack{j,k=0 \\ (j,k)\neq(0,0)}}^{\infty} \frac{a_{j,k}}{4\pi^2(j^2+k^2)} \cos(2\pi j z_1)\cos(2\pi k z_2).$$

The function $F(z)$ is then a C^∞ function on $\mathbb{T}^2$ and it satisfies $\Delta F = -H$. Hence, if we set $g(z) = h(z) + F(z)$ we have $\Delta g(z) = 1$ for $|z| > 0$ and $g(z) + \frac{1}{2\pi}\log|z|$ has a continuous extension to all of $\mathbb{T}^2$. The Green's function for $\mathbb{T}^2$ is then $G_x(y) = g((x-y)_{\mathbb{T}^2})$. $\square$

Fixing $x \in \mathbb{T}^2$ and constants $0 < 2r \leq R < 1/2$ let

$$\tau^{(0)} = \inf\{t \geq 0 \mid X_t \in \partial D_{\mathbb{T}^2}(x,R)\}, \tag{2.6}$$

$$\sigma^{(1)} = \inf\{t \geq 0 \mid X_{t+\tau^{(0)}} \in \partial D_{\mathbb{T}^2}(x,r)\} \tag{2.7}$$

and define inductively for $j = 1, 2, \ldots$

$$\tau^{(j)} = \inf\{t \geq \sigma^{(j)} \mid X_{t+\mathfrak{T}_{j-1}} \in \partial D_{\mathbb{T}^2}(x,R)\}, \tag{2.8}$$

$$\sigma^{(j+1)} = \inf\{t \geq 0 \mid X_{t+\mathfrak{T}_j} \in \partial D_{\mathbb{T}^2}(x,r)\}, \tag{2.9}$$

where $\mathfrak{T}_j = \sum_{i=0}^{j} \tau^{(i)}$ for $j = 0, 1, 2, \ldots$. Thus, $\tau^{(j)}$ is the length of the j-th excursion $\mathcal{E}_j$ from $\partial D_{\mathbb{T}^2}(x,R)$ to itself via $\partial D_{\mathbb{T}^2}(x,r)$, and $\sigma^{(j)}$ is the amount of time it takes to hit $\partial D_{\mathbb{T}^2}(x,r)$ during the j-th excursion $\mathcal{E}_j$.

The next lemma, which shows that excursion times are concentrated around their mean, will be used to relate excursions to hitting times.

LEMMA 2.2. *With the above notation, for any* $N \geq N_0$, $\delta_0 > 0$ *small enough*, $0 < \delta < \delta_0$, $0 < 2r \leq R < R_1(\delta)$, *and* $x, x_0 \in \mathbb{T}^2$,

$$\mathbf{P}^{x_0}\left(\sum_{j=0}^{N} \tau^{(j)} \leq (1-\delta)N\frac{1}{\pi}\log(R/r)\right) \leq e^{-C\delta^2 N} \tag{2.10}$$

and

$$\mathbf{P}^{x_0}\left(\sum_{j=0}^{N} \tau^{(j)} \geq (1+\delta)N\frac{1}{\pi}\log(R/r)\right) \leq e^{-C\delta^2 N}. \tag{2.11}$$

Moreover, $C = C(R,r) > 0$ *depends only upon* δ_0 *as soon as* $R > r^{1-\delta_0}$.

Proof of Lemma 2.2. Applying Kac's moment formula for the first hitting time τ_r of the strong Markov process X_t (see [17, equation (6)]), we see that for any $\theta < 1/\|\tau_r\|$,

$$\sup_y \mathbb{E}^y(e^{\theta\tau_r}) \leq \frac{1}{1-\theta\|\tau_r\|}\,. \tag{2.12}$$

Consequently, by (2.1) we have that for some $\lambda > 0$,

$$\sup_{0<r\leq r_0}\sup_{x,y}\mathbb{E}^y(e^{\lambda\tau_r/|\log r|}) < \infty. \tag{2.13}$$

By the strong Markov property of X_t at $\tau^{(0)}$ and at $\tau^{(0)}+\sigma^{(1)}$ we then deduce that

$$\sup_{0<2r\leq R<r_0}\sup_{x,y}\mathbb{E}^y(e^{\lambda\mathfrak{T}_1/|\log r|}) < \infty. \tag{2.14}$$

Fixing $x \in \mathbb{T}^2$ and $0 < 2r \leq R < 1/2$ let $\tau = \tau^{(1)}$ and $v = \frac{1}{\pi}\log(R/r)$. Recall that $\{X_t : t \leq \tau_R\}$ starting at $X_0 = z$ for some $z \in \partial D_{\mathbb{T}^2}(x,r)$, has the same law as $\{B_t : t \leq \widetilde{\tau}_R\}$ starting at $B_0 = i(z) \in \partial D(0,r)$. Consequently,

$$\|\tau_R\|_R := \sup_x \sup_{z\in D_{\mathbb{T}^2}(x,R)} \mathbb{E}^z(\tau_R) \leq \mathbb{E}^0(\widetilde{\tau}_R) = \frac{R^2}{2} \to_{R\to 0} 0\,, \tag{2.15}$$

by the radial symmetry of the Brownian motion B_t.

By the strong Markov property of X_t at $\tau^{(0)}+\sigma^{(1)}$ we thus have that

$$\mathbb{E}^y(\tau_r) \leq \mathbb{E}^y(\tau) \leq \mathbb{E}^y(\tau_r) + \|\tau_R\|_R \qquad \text{for all } y \in \partial D_{\mathbb{T}^2}(x,R)\,.$$

Consequently, with $\eta = \delta/6$, let $R_1(\delta) \leq r_0$ be small enough so that (2.2) and (2.15) imply

$$\begin{aligned}(1-\eta)v &\leq \inf_x \inf_{y\in\partial D_{\mathbb{T}^2}(x,R)} \mathbb{E}^y(\tau)\\ &\leq \sup_x \sup_{y\in\partial D_{\mathbb{T}^2}(x,R)} \mathbb{E}^y(\tau) \leq (1+2\eta)v\,,\end{aligned} \tag{2.16}$$

whenever $R \leq R_1$. It follows from (2.14) and (2.16) that there exists a universal constant $c_4 < \infty$ such that for $\rho = c_4|\log r|^2$ and all $\theta \geq 0$,

$$\begin{aligned}&\sup_x \sup_{y\in\partial D_{\mathbb{T}^2}(x,R)} \mathbb{E}^y(e^{-\theta\tau})\\ &\quad\leq 1-\theta\inf_x\inf_{y\in\partial D_{\mathbb{T}^2}(x,R)}\mathbb{E}^y(\tau) + \frac{\theta^2}{2}\sup_x\sup_{y\in\partial D_{\mathbb{T}^2}(x,R)}\mathbb{E}^y(\tau^2)\\ &\quad\leq 1-\theta(1-\eta)v + \rho\theta^2 \leq \exp(\rho\theta^2-\theta(1-\eta)v).\end{aligned} \tag{2.17}$$

Since $\tau^{(0)} \geq 0$, using Chebyshev's inequality we bound the left-hand side of (2.10) by

(2.18)
$$\begin{aligned}\mathbf{P}^{x_0}\Big(\sum_{j=1}^{N}\tau^{(j)} \leq (1-6\eta)vN\Big) &\leq e^{\theta(1-3\eta)vN}\mathbb{E}^{x_0}\Big(e^{-\theta\sum_{j=1}^{N}\tau^{(j)}}\Big)\\ &\leq e^{-\theta vN\delta/3}\Big[e^{\theta(1-\eta)v}\sup_{y\in\partial D_{\mathbb{T}^2}(x,R)}\mathbb{E}^{y}(e^{-\theta\tau})\Big]^{N},\end{aligned}$$

where the last inequality follows by the strong Markov property of X_t at $\{\mathfrak{T}_j\}$. Combining (2.17) and (2.18) for $\theta = \delta v/(6\rho)$, results in (2.10), where $C = v^2/36\rho > 0$ is bounded below by $\delta_0^2/(36c_4\pi^2)$ if $r^{1-\delta_0} < R$.

To prove (2.11) we first note that for $\theta = \lambda/|\log r| > 0$ and $\lambda > 0$ as in (2.14), it follows that

$$\mathbf{P}^{x_0}\left(\tau^{(0)} \geq \frac{\delta}{3}vN\right) \leq e^{-\theta v(\delta/3)N}\mathbb{E}^{x_0}(e^{\lambda\tau^{(0)}/|\log r|}) \leq c_5 e^{-c_6\delta N},$$

where $c_5 < \infty$ is a universal constant and $c_6 = c_6(r,R) > 0$ does not depend upon N, δ or x_0 and is bounded below by some $c_7(\delta_0) > 0$ when $r^{1-\delta_0} < R$. Thus, the proof of (2.11), in analogy to that of (2.10), comes down to bounding

(2.19)
$$\mathbf{P}^{x_0}\Big(\sum_{j=1}^{N}\tau^{(j)} \geq (1+4\eta)vN\Big) \leq e^{-\theta\delta vN/3}\Big(e^{-\theta(1+2\eta)v}\sup_{y\in\partial D_{\mathbb{T}^2}(x,R)}\mathbb{E}^{y}(e^{\theta\tau})\Big)^{N}.$$

By (2.14) and (2.16), there exists a universal constant $c_8 < \infty$ such that for $\rho = c_8|\log r|^2$ and all $0 < \theta < \lambda/(2|\log r|)$,

$$\begin{aligned}\sup_{x}\sup_{y\in\partial D_{\mathbb{T}^2}(x,R)}\mathbb{E}^{y}(e^{\theta\tau}) &\leq 1+\theta(1+2\eta)v+\sup_{x}\sup_{y\in\partial D_{\mathbb{T}^2}(x,R)}\sum_{n=2}^{\infty}\frac{\theta^n}{n!}\mathbb{E}^{y}(\tau^n)\\ &\leq 1+\theta(1+2\eta)v+\rho\theta^2 \leq \exp(\theta(1+2\eta)v+\rho\theta^2);\end{aligned}$$

the proof of (2.11) now follows as in the proof of (2.10). $\square$

LEMMA 2.3. *For any $\delta > 0$ there exist $c < \infty$ and $\varepsilon_0 > 0$ so that for all $\varepsilon \leq \varepsilon_0$ and $y \geq 0$*

$$\mathbf{P}^{x_0}\left(\mathcal{T}(x,\varepsilon) \geq y(\log\varepsilon)^2\right) \leq c\varepsilon^{(1-\delta)\pi y} \tag{2.20}$$

for all $x, x_0 \in \mathbb{T}^2$.

Proof of Lemma 2.3. We use the notation of the last lemma and its proof, with $R < R_1(\delta)$ and $r = R/e$ chosen for convenience so that $\log(R/r) = 1$. Let

$n_\varepsilon := (1-\delta)\pi y(\log\varepsilon)^2$. Then,

$$\begin{aligned}(2.21)\quad &\mathbf{P}^{x_0}\left(\mathcal{T}(x,\varepsilon)\geq y(\log\varepsilon)^2\right)\\ &\leq \mathbf{P}^{x_0}\left(\mathcal{T}(x,\varepsilon)\geq \sum_{j=0}^{n_\varepsilon}\tau^{(j)}\right)+\mathbf{P}^{x_0}\left(\sum_{j=0}^{n_\varepsilon}\tau^{(j)}\geq y(\log\varepsilon)^2\right).\end{aligned}$$

It follows from Lemma 2.2 that

$$\mathbf{P}^{x_0}\left(\sum_{j=0}^{n_\varepsilon}\tau^{(j)}\geq y(\log\varepsilon)^2\right)\leq e^{-C'y(\log\varepsilon)^2} \tag{2.22}$$

for some $C'=C'(\delta)>0$. On the other hand, the first probability in the second line of (2.21) is bounded above by the probability of B_t not hitting $i(D_{\mathbb{T}^2}(x,\varepsilon))=D(0,\varepsilon)$ during n_ε excursions, each starting at $i(\partial D_{\mathbb{T}^2}(x,r))=\partial D(0,r)$ and ending at $i(\partial D_{\mathbb{T}^2}(x,R))=\partial D(0,R)$, so that

$$\mathbf{P}^{x_0}\left(\mathcal{T}(x,\varepsilon)\geq \sum_{j=0}^{n_\varepsilon}\tau^{(j)}\right)\leq\left(1-\frac{1}{\log\frac{R}{\varepsilon}}\right)^{n_\varepsilon}\leq e^{-(1-\delta)\pi y|\log\varepsilon|} \tag{2.23}$$

and (2.20) follows. □

We next show that

$$\limsup_{\varepsilon\to 0}\sup_{x\in\mathbb{T}^2}\frac{\mathcal{T}(x,\varepsilon)}{(\log\varepsilon)^2}\leq\frac{2}{\pi}\,,\quad \text{a.s.} \tag{2.24}$$

from which the upper bound for (1.2) follows.

Set $h(\varepsilon)=|\log\varepsilon|^2$. Fix $\delta>0$, and set $\tilde\varepsilon_n=e^{-n}$ so that

$$h(\tilde\varepsilon_{n+1})=(1+\frac{1}{n})^2h(\tilde\varepsilon_n). \tag{2.25}$$

For $\tilde\varepsilon_{n+1}\leq\varepsilon\leq\tilde\varepsilon_n$,

$$\frac{\mathcal{T}(x,\tilde\varepsilon_{n+1})}{h(\tilde\varepsilon_{n+1})}=\frac{h(\tilde\varepsilon_n)}{h(\tilde\varepsilon_{n+1})}\frac{\mathcal{T}(x,\tilde\varepsilon_{n+1})}{h(\tilde\varepsilon_n)}\geq(1+\frac{1}{n})^{-2}\frac{\mathcal{T}(x,\varepsilon)}{h(\varepsilon)}\,. \tag{2.26}$$

Fix $x_0\in\mathbb{T}^2$ and let $\{x_j: j=1,\dots,\bar K_n\}$, denote a maximal collection of points in $\mathbb{T}^2$, such that $\inf_{\ell\neq j} d(x_\ell,x_j)\geq\delta\tilde\varepsilon_n$. Let $a=(2+\delta)/(1-10\delta)$ and $\mathcal{A}_n$ be the set of $1\leq j\leq\bar K_n$, such that

$$\mathcal{T}(x_j,(1-\delta)\tilde\varepsilon_n)\geq(1-2\delta)ah(\tilde\varepsilon_n)/\pi.$$

It follows by Lemma 2.3 that

$$\mathbf{P}^{x_0}(\mathcal{T}(x,(1-\delta)\tilde\varepsilon_n)\geq(1-2\delta)ah(\tilde\varepsilon_n)/\pi)\leq c\,\tilde\varepsilon_n^{(1-10\delta)a}\,,$$

for some $c=c(\delta)<\infty$, all sufficiently large n and any $x\in\mathbb{T}^2$. Thus, for all sufficiently large n, any j and $a>0$,

$$\mathbf{P}^{x_0}(j\in\mathcal{A}_n)\leq c\,\tilde\varepsilon_n^{(1-10\delta)a}\,. \tag{2.27}$$

This implies

$$\sum_{n=1}^{\infty} \mathbf{P}^{x_0}(|\mathcal{A}_n| \geq 1) \leq \sum_{n=1}^{\infty} \mathbb{E}^{x_0}|\mathcal{A}_n| \leq c' \sum_{n=1}^{\infty} \tilde{\varepsilon}_n^{\,\delta} < \infty\,.$$

By Borel-Cantelli, it follows that $\mathcal{A}_n$ is empty a.s. for all $n > n_0(\omega)$ and some $n_0(\omega) < \infty$. By (2.26) we then have for some $n_1(\delta, \omega) < \infty$ and all $n > n_1(\omega)$

$$\sup_{\varepsilon \leq \tilde{\varepsilon}_{n_1}} \sup_{x \in \mathbb{T}^2} \frac{\mathcal{T}(x, \varepsilon)}{(\log \varepsilon)^2} \leq \frac{a}{\pi}\,,$$

and (2.24) follows by taking $\delta \downarrow 0$. $\square$

3. Lower bound for covering times

Fixing $\delta > 0$ and $a < 2$, we prove in this section that

$$\liminf_{\varepsilon \to 0} \frac{\mathcal{C}_\varepsilon}{(\log \varepsilon)^2} \geq (1 - \delta)\frac{a}{\pi} \qquad \text{a.s.} \tag{3.1}$$

In view of (2.24), we then obtain Theorem 1.2.

We start by constructing an almost sure lower bound on $\mathcal{C}_\varepsilon$ for a specific deterministic sequence $\varepsilon_{n,1}$. To this end, fix $\varepsilon_1 \leq R_1(\delta)$ as in Lemma 2.2 and the square $S = [\varepsilon_1, 2\varepsilon_1]^2$. Let $\varepsilon_k = \varepsilon_1 (k!)^{-3}$ and $n_k = 3ak^2 \log k$. For fixed $n \geq 3$, let $\varepsilon_{n,k} = \rho_n \varepsilon_n (k!)^3$ for $\rho_n = n^{-25}$ and $k = 1, \ldots, n$. Observe that $\varepsilon_{n,1} = \rho_n \varepsilon_n$, $\varepsilon_{n,n} = \rho_n \varepsilon_1$, and $\varepsilon_{n,k} \leq \rho_n \varepsilon_{n+1-k} \leq \varepsilon_{n+1-k}$ for all $1 \leq k \leq n$. Recall the natural bijection $i : D_{\mathbb{T}^2}(0, 1/2) \mapsto D(0, 1/2)$. For any $x \in S$, let $\mathcal{R}_n^x$ denote the time until X_t completes n_n excursions from $i^{-1}(\partial D(x, \varepsilon_{n,n-1}))$ to $i^{-1}(\partial D(x, \varepsilon_{n,n}))$. (In the notation of Section 2, if we set $R = \varepsilon_{n,n}$ and $r = \varepsilon_{n,n-1}$, then $\mathcal{R}_n^x = \sum_{j=0}^{n_n} \tau^{(j)}$.) Note that $i^{-1}(\partial D(x, \varepsilon_{n,k}))$ is just $\partial D_{\mathbb{T}^2}(i^{-1}(x), \varepsilon_{n,k})$, but the former notation will allow easy generalization to the case of general manifolds treated in Section 8.

For $x \in S$, $2 \leq k \leq n$ let $N_{n,k}^x$ denote the number of excursions of X_t from $i^{-1}(\partial D(x, \varepsilon_{n,k-1}))$ to $i^{-1}(\partial D(x, \varepsilon_{n,k}))$ until time $\mathcal{R}_n^x$. Thus, $N_{n,n}^x = n_n = 3an^2 \log n$. A point $x \in S$ is called *n-successful* if

$$N_{n,2}^x = 0, \qquad n_k - k \leq N_{n,k}^x \leq n_k + k \quad \forall k = 3, \ldots, n-1\,. \tag{3.2}$$

In particular, if x is n-successful, then $\mathcal{T}(i^{-1}(x), \varepsilon_{n,1}) > \mathcal{R}_n^x$.

For $n \geq 3$ we partition S into $M_n = \varepsilon_1^2/(2\varepsilon_n)^2 = (1/4)\prod_{l=1}^n l^6$ nonoverlapping squares of edge length $2\varepsilon_n = 2\varepsilon_1/(n!)^3$, with $x_{n,j}$, $j = 1, \ldots, M_n$ denoting the centers of these squares. Let $Y(n, j)$, $j = 1, \ldots, M_n$, be the sequence of random variables defined by

$$Y(n, j) = 1 \ \text{ if } x_{n,j} \text{ is } n\text{-successful}$$

and $Y(n, j) = 0$ otherwise. Set $\bar{q}_n = \mathbf{P}(Y(n, j) = 1) = \mathbb{E}(Y(n, j))$, noting that this probability is independent of j (and of the value of ρ_n).

The next lemma, which is a direct consequence of Lemmas 6.2 and 7.1, provides bounds on the first and second moments of $Y(n,j)$, that are used in order to show the existence of at least one n-successful point $x_{n,j}$ for large enough n.

LEMMA 3.1. *There exists $\delta_n \to 0$ such that for all $n \geq 1$,*

$$\bar{q}_n = \mathcal{P}\ (x \text{ is } n\text{-successful}) \geq \varepsilon_n^{a+\delta_n}\ . \tag{3.3}$$

For some $C_0 < \infty$ and all n, if $|x_{n,i} - x_{n,j}| \geq 2\varepsilon_{n,n}$, then

$$\mathbb{E}(Y(n,i)Y(n,j)) \leq (1 + C_0 n^{-1} \log n)\bar{q}_n^2\ . \tag{3.4}$$

Further, for any $\gamma > 0$ there exists $C = C(\gamma) < \infty$ so that for all n and $l = l(i,j) = \max\{k \leq n\ :\ |x_{n,i} - x_{n,j}| \geq 2\varepsilon_{n,k}\} \vee 1$,

$$\mathbb{E}(Y(n,i)Y(n,j)) \leq \bar{q}_n^2 C^{n-l} n^{39} \left(\frac{\varepsilon_{n,n}}{\varepsilon_{n,l+1}}\right)^{a+\gamma}\ . \tag{3.5}$$

Fix $\gamma > 0$ such that $2 - a - \gamma > 0$. By (3.3) for all n large enough,

$$\mathbb{E}\left(\sum_{j=1}^{M_n} Y(n,j)\right) = M_n \bar{q}_n \geq \varepsilon_n^{-(2-a-\gamma)}. \tag{3.6}$$

In the sequel, we let C_m denote generic finite constants that are independent of n, l, i and j. Recall that there are at most $C_1 \varepsilon_{n,l+1}^2 \varepsilon_n^{-2}$ points $x_{n,j}$, $j \neq i$, in $D(x_{n,i}, 2\varepsilon_{n,l+1})$. Further, our choice of ρ_n guarantees that $(\varepsilon_{n,n}/\varepsilon_n)^2 \leq C_2 M_n n^{-50}$. Hence, it follows from (3.5) that for $n - 1 \geq l \geq 1$,

$$\begin{aligned} V_l &:= (M_n \bar{q}_n)^{-2} \sum_{\substack{i \neq j = 1 \\ l(i,j) = l}}^{M_n} \mathbb{E}\Big(Y(n,i)Y(n,j)\Big) \\ &\leq C_1 M_n^{-1} \varepsilon_{n,l+1}^2 \varepsilon_n^{-2} C^{n-l} n^{39} \left(\frac{\varepsilon_{n,l+1}}{\varepsilon_{n,n}}\right)^{-a-\gamma} \\ &\leq C_1 C_2 n^{-3} C^{n-l} \left(\frac{\varepsilon_{n,l+1}}{\varepsilon_{n,n}}\right)^{2-a-\gamma}, \end{aligned} \tag{3.7}$$

and since $(\varepsilon_{n,l+1}/\varepsilon_{n,n}) \leq (\varepsilon_{n-l}/\varepsilon_1)$ for all $1 \leq l \leq n-1$, we deduce that

$$\sum_{l=1}^{n-1} V_l \leq C_3 n^{-3} \sum_{j=1}^{\infty} C^j \varepsilon_j^{2-a-\gamma} \leq C_4 n^{-3}\,. \tag{3.8}$$

We have, by Chebyshev's inequality (see [6, Theorem 4.3.1]) and (3.4), that

$$\begin{aligned} \mathbf{P}(\sum_{j=1}^{M_n} Y(n,j) = 0) &\leq (M_n \bar{q}_n)^{-2} \mathbb{E}\left\{\Big(\sum_{i=1}^{M_n} Y(n,i)\Big)^2\right\} - 1 \\ &\leq (M_n \bar{q}_n)^{-1} + C_0 n^{-1} \log n + \sum_{l=1}^{n-1} V_l\,. \end{aligned}$$

Combining this with (3.6) and (3.8), we see that

$$\mathbf{P}(\sum_{j=1}^{M_n} Y(n,j)=0) \le C_5 n^{-1}\log n\,. \tag{3.9}$$

The next lemma relates the notion of n-successful to the $\varepsilon_{n,1}$-hitting time.

LEMMA 3.2. *For each n let $\mathcal{V}_n$ be a finite subset of S with cardinality bounded by $e^{o(n^2)}$. There exists $m(\omega)<\infty$ a.s. such that for all $n \ge m$ and all $x \in \mathcal{V}_n$, if x is n-successful then*

$$\mathcal{T}(i^{-1}(x),\varepsilon_{n,1}) \ge (\log \varepsilon_{n,1})^2\Big(\frac{a}{\pi}-\frac{2}{\sqrt{\log n}}\Big). \tag{3.10}$$

Proof of Lemma 3.2. Recall that if x is n-successful then $\mathcal{T}(i^{-1}(x),\varepsilon_{n,1}) > \sum_{j=0}^{n_n}\tau^{(j)}$. Hence, using (2.10) with $N=n_n=3an^2\log n$, $\delta_n=\pi/(a\sqrt{\log n})$, $R=\varepsilon_{n,n}$, and $r=\varepsilon_{n,n-1}$ so that $\log(R/r)=3\log n$ and $R>r^{0.8}$, we see that for some $C>0$ that is independent of n,

$$\begin{aligned}
P_x &:= \mathbf{P}^{x_0}\left(\mathcal{T}(i^{-1}(x),\varepsilon_{n,1}) \le (\frac{a}{\pi}-\frac{2}{\sqrt{\log n}})(\log\varepsilon_{n,1})^2\,,\, x \text{ is } n\text{-successful}\right)\\
&\le \mathbf{P}^{x_0}\left(\sum_{j=0}^{N}\tau^{(j)} \le (\frac{a}{\pi}-\frac{1}{\sqrt{\log n}})(3n\log n)^2\right)\\
&\le \mathbf{P}^{x_0}\left(\frac{1}{N}\sum_{j=0}^{N}\tau^{(j)} \le (1-\delta_n)\frac{\log(R/r)}{\pi}\right) \le e^{-Cn^2}.
\end{aligned}$$

Consequently, the sum of P_x over all $x \in \mathcal{V}_n$ and then over all n is finite, and the Borel-Cantelli lemma then completes the proof of Lemma 3.2. $\square$

Taking $\mathcal{V}_n=\{x_{n,k} : k=1,\dots,M_n\}$, and the subsequence $n(j)=j(\log j)^3$, it follows from (3.9), (3.10) and the Borel-Cantelli lemma that a.s.

$$\mathcal{C}_{\varepsilon_{n(j),1}} \ge (\log\varepsilon_{n(j),1})^2\Big(\frac{a}{\pi}-\frac{2}{\sqrt{\log n(j)}}\Big)\,, \tag{3.11}$$

for all j large enough. Since $\varepsilon \mapsto \mathcal{C}_\varepsilon$ is monotone nondecreasing, it follows that for any $\varepsilon_{n(j+1),1} \le \varepsilon \le \varepsilon_{n(j),1}$

$$\frac{\mathcal{C}_\varepsilon}{(\log\varepsilon)^2} \ge \frac{\mathcal{C}_{\varepsilon_{n(j+1),1}}}{(\log\varepsilon_{n(j),1})^2}\,.$$

Observing that $(\log\varepsilon_{n(j+1),1})/(\log\varepsilon_{n(j),1}) \to 1$ as $j\to\infty$, we thus see that (3.1) is an immediate consequence of (3.11). $\square$

Remark. We note for use in Section 5 that essentially the same proof shows that for any $\widehat{a} < 2$, almost surely,

$$\sup_{x \in n(j)^{-4}S} \mathcal{T}(x, \varepsilon_{n(j),1}) \geq (\log \varepsilon_{n(j),1})^2 \Big(\frac{\widehat{a}}{\pi} - \frac{2}{\sqrt{\log n(j)}} \Big), \tag{3.12}$$

for all j large enough. To see this we need only prove (3.9) with the sum now going over j' such that $x_{n,j'} \in n^{-4}S$. This has the effect of replacing M_n by n^{-4} times its previous value. Clearly (3.6) still holds, with perhaps a different $\gamma > 0$. Also, we now have only $(\varepsilon_{n,n}/\varepsilon_n)^2 \leq C_2 M_n n^{-42}$, but this is enough to establish (3.7). The rest of the proof follows as before.

4. Proof of the lattice torus covering time conjecture

To establish Theorem 1.1 it suffices to prove that for any $\delta > 0$

$$\lim_{n\to\infty} \mathbf{P} \left(\frac{\mathcal{T}_n}{(n \log n)^2} \geq \frac{4}{\pi} - \delta \right) = 1 \tag{4.1}$$

since the complementary upper bound on $\mathcal{T}_n$ is contained in [4, Cor. 25, Chap. 7] (see also the references therein). Our approach is to use Theorem 1.2 together with the strong approximation results of [15] and [20].

Fix $\gamma > 0$ and let $\varepsilon_n = 2n^{\gamma-1}$. Then by Theorem 1.2 for all $n \geq N_0$ with some $N_0 = N_0(\gamma, \delta) < \infty$,

$$\mathbf{P} \left(\mathcal{C}_{\varepsilon_n} > \frac{2(1-\gamma-\delta)^2}{\pi} (\log n)^2 \right) \geq 1 - \delta. \tag{4.2}$$

By Einmahl's [15, Theorem 1] multidimensional extension of the Komlós-Major-Tusnády [20] strong approximation theorem, we may, for each n, construct $\{S_k\}$ and $\{W_t\}$ on the same probability space so that a.s. for some $n_0 = n_0(\omega) < \infty$,

$$\max_{k \leq 4n^2(\log n)^2} |W_k - \sqrt{2} S_k| \leq n^\gamma/6, \qquad \forall n \geq n_0.$$

Hence, dividing by $\sqrt{2}n$ we have

$$\max_{k \leq 4n^2(\log n)^2} |\frac{W_k}{\sqrt{2}n} - \frac{S_k}{n}| \leq \varepsilon_n/2, \qquad \forall n \geq n_0$$

or, using Brownian scaling, we have

$$\mathbf{P} \left(\max_{k \leq 4n^2(\log n)^2} |W_{k/2n^2} - \frac{S_k}{n}| \geq \varepsilon_n/2 \right) \leq \delta \tag{4.3}$$

for all $n \geq N_0'$ with some $N_0' = N_0'(\gamma, \delta) < \infty$.

Now, by (4.2) we see that with probability at least $1-\delta$ some disc $D_{\mathbb{T}^2}(x, \varepsilon_n) \subseteq \mathbb{T}^2$ is completely missed by

$$\left\{ W_{k/2n^2} \,\overline{\mathrm{mod}}\, \mathbb{Z}^2 \,;\, k \leq \frac{4(1-\gamma-\delta)^2}{\pi} n^2 (\log n)^2 \right\};$$

hence by (4.3) with probability at least $1-2\delta$ we have that

$$\left\{\frac{S_k}{n} \ \overline{\mathrm{mod}}\ \mathbb{Z}^2\,;\ k \leq \frac{4(1-\gamma-\delta)^2}{\pi} n^2(\log n)^2\right\}$$

avoids some disc of radius $\varepsilon_n/2 = n^{\gamma-1}$. Thus, the probability that

$$\left\{S_k \ \mathrm{mod}\ n\mathbb{Z}^2\,;\ k \leq \frac{4(1-\gamma-\delta)^2}{\pi} n^2(\log n)^2\right\}$$

avoids some disc of radius n^γ is at least $1-2\delta$, which implies (4.1).

5. Proof of the Kesten-Révész conjecture

Let $D_r = D(0,r) \cap \mathbb{Z}^2$ denote the disc of radius r in $\mathbb{Z}^2$ and define its boundary

$$\partial D_r = \{z \notin D_r \,\big|\, |z-y| = 1 \text{ for some } y \in D_r\}.$$

Let $\phi_n = (\log n)^2/\log\log n$ and let $\mathcal{N}_n$ denote the number of excursions in $\mathbb{Z}^2$ from ∂D_{2n} to $\partial D_{n(\log n)^3}$ after first hitting $\partial D_{n(\log n)^3}$, that is needed to cover D_n. By [21, Theorem 1.1], it suffices to show that

$$\limsup_{n\to\infty} \mathbf{P}(\log T_n \leq t(\log n)^2) \leq e^{-4/t},$$

and by [21, equation (7), p. 196], this is a direct consequence of the next lemma.

Lemma 5.1.

$$\liminf_{n\to\infty} \frac{\mathcal{N}_n}{\phi_n} \geq \frac{2}{3} \quad \textit{in probability.} \tag{5.1}$$

Remark. Though not needed for our proof of Theorem 1.4, it is not hard to modify the proof of Lemma 5.1 so as to show that $\mathcal{N}_n/\phi_n \to \frac{2}{3}$ in probability.

Let $K(z,u)$ denote the Poisson kernel for the annular region

$$\mathcal{A}_r := \{z : r < |z| < 1/2\},$$

such that for any continuous function $g \geq 0$ on $\partial\mathcal{A}_r$, we have

$$\mathbb{E}^z(g(W_\theta)) = \int_{\partial\mathcal{A}_r} g(u)K(z,u)du\,,$$

where $\theta := \inf\{t \geq 0 : W_t \in \partial\mathcal{A}_r\}$, and W_t is a planar Brownian motion, starting at $W_0 = z \in \mathcal{A}_r$. A preliminary step in proving Lemma 5.1 is the following estimate about $K(z,u)$ when $|z| \gg r = |u|$.

Lemma 5.2. *There exists finite $c > 2$ such that if $cr \leq |z| < 1/(2c)$, then*

$$\sup_{\{u:|u|=r\}} K(z,u) \leq \Big(1 + \frac{40r\log(2r)}{|z|\log(2|z|)}\Big) \inf_{\{u:|u|=r\}} K(z,u). \tag{5.2}$$

Proof of Lemma 5.2. The series expansion

$$P_A(x,u) = c_0(x) + \sum_{m=1}^{\infty} c_m(x) Z_m(x, \frac{u}{|u|})$$

is provided in [9, 10.11–10.13, p. 191] for the Poisson kernel $P_A(\cdot,\cdot)$ in the region $A = \{x : r_0 < |x| < 1\}$, at its inner boundary $|u| = r_0$, where

$$c_m(x) = |x|^{-m} \left\{ \frac{r_0}{|x|} \right\}^m \frac{1-|x|^{2m}}{1-(r_0)^{2m}}, \qquad m \geq 1,$$

and the "zonal harmonic" functions

$$Z_m(x, e^{i\phi}) = 2|x|^m \cos(m(\mathrm{Arg}(x) - \phi))$$

are as given in [9, 5.9 and 5.18]. Note that for any $x \in A$

$$\begin{aligned} |P_A(x,u) - c_0(x)| &\leq \sum_{m=1}^{\infty} c_m(x) |Z_m(x, \frac{u}{|u|})| \\ &\leq 2 \sum_{m=1}^{\infty} \left(\frac{r_0}{|x|} \right)^m = \frac{2r_0}{|x| - r_0} \, . \end{aligned} \tag{5.3}$$

The function $c_0(x) = \log(1/|x|)/\log(1/r_0)$ is the harmonic function in A corresponding to the boundary condition $\mathbf{1}_{|x|=r_0}$. By Brownian scaling $K(z,u) = P_A(2z, 2u)$ for $r_0 = 2r$. Hence, it follows from (5.3) and the value of $c_0(\cdot)$, that for all $2r \leq |z| < 1/2$,

$$\sup_{\{u:|u|=r\}} K(z,u) \leq \left(1 + \frac{8f(r)}{f(|z|) - 4f(r)} \right) \inf_{\{u:|u|=r\}} K(z,u) \, ,$$

where $f(t) := t \log(1/(2t))$. The proof is completed when we note that $f(t) \geq 5f(r)$ for all $cr \leq t \leq 1/(2c)$ provided c is large enough ($c = 10$ suffices). □

With $\mathbb{T}^2 = (-1/2, 1/2]^2$, our application of Lemma 5.2 is via the following estimate.

LEMMA 5.3. *Assume $W_0 = X_0 = \beta$ with $|\beta| = R \in (r, 1/2)$, and let $\tau_r := \inf\{t \geq 0 : |W_t| = r\}$. There exists finite $c > 2$, such that if $cr \leq R < 1/(2c)$, then the law of W_{τ_r} is absolutely continuous with respect to the law of X_{τ_r}, with Radon-Nikodym derivative $h_r(\beta, \cdot)$ such that*

$$\sup_{|\beta|=R, |\alpha|=r} h_r(\beta, \alpha) \leq 1 + \frac{40 r \log(2r)}{R \log(2R)} \, . \tag{5.4}$$

Proof of Lemma 5.3. Recall that the exit time θ from the annular region $\mathcal{A}_r$ is such that $\theta \leq \tau_r$, with equality if and only if the path exits $\mathcal{A}_r$ via its inner boundary $\partial D(0,r)$. Moreover, with $X_0 = W_0 = z \in \mathcal{A}_r$, the law of the path

$\{X_t : 0 \leq t \leq \theta\}$ is identical to that of the path $\{W_t : 0 \leq t \leq \theta\}$. Let L denote the number of excursions of ω_t between $\partial D(0,R)$ and $\partial D(0,1/2)$ completed by time τ_r. For each $k \geq 0$, let $\mu_k(\beta,\cdot)$ denote the hitting (probability) measure of $\partial D(0,R)$ induced by W_t upon completing k such excursions, conditional upon $L \geq k$. Let $\nu_k(\beta,\cdot)$ denote the corresponding hitting measure induced by the process X_t. Note that L has a geometric(p) law, where $p < 1$ is the same for both processes X_t and W_t and is independent of the initial condition $z \in \partial D(0,R)$. Consequently, for any Borel set $B \subset \partial D(0,r)$,

$$
\begin{aligned}
\mathbf{P}^\beta(W_{\tau_r} \in B) &= \sum_{k=0}^{\infty} \mathbf{P}^\beta(W_{\tau_r} \in B,\, L = k) \\
&= \sum_{k=0}^{\infty} p^k \int_{\partial D(0,R)} \mu_k(\beta, dz) \int_B K(z,u)du \leq \frac{1}{1-p} \int_B [\sup_{|z|=R} K(z,u)]du\,,
\end{aligned}
$$

where $K(z,u)$ is the Poisson kernel for W_t and the region $\mathcal{A}_r$. Similarly,

$$
\begin{aligned}
\mathbf{P}^\beta(X_{\tau_r} \in B) &= \sum_{k=0}^{\infty} p^k \int_{\partial D(0,R)} \nu_k(\beta, dz) \int_B K(z,u)du \\
&\geq \frac{1}{1-p} \int_B [\inf_{|z|=R} K(z,u)]du\,.
\end{aligned}
$$

Hence, for any $B \subset \partial D(0,r)$,

$$
\mathbf{P}^\beta(W_{\tau_r} \in B) \leq P^\beta(X_{\tau_r} \in B) \frac{\sup_{|z|=R,|u|=r} K(z,u)}{\inf_{|z|=R,|u|=r} K(z,u)}\,,
$$

implying that W_{τ_r} is absolutely continuous with respect to X_{τ_r}, and by (5.2) the Radon-Nikodym derivative $h_r(\beta,\cdot)$ clearly satisfies (5.4). $\square$

Proof of Lemma 5.1. For any $K \subseteq \mathbb{T}^2$ let

$$
\mathcal{C}_\varepsilon(K) = \sup_{x \in K} \mathcal{T}(x,\varepsilon)
$$

be the ε-covering time of K. Fix $a > 0$ and $b \in (0,1)$. Set $r_\varepsilon = a/|\log\varepsilon|^3$. Taking the isometry $i : D_{\mathbb{T}^2}(0,1/2) \mapsto D(0,1/2)$ to be the identity, omitting i^{-1} throughout the proof, we can find sequences $n(j) \uparrow \infty$ and $\varepsilon_{n(j),1} \downarrow 0$ with $(\log\varepsilon_{n(j+1),1})/(\log\varepsilon_{n(j),1}) \to 1$ such that for any $\widehat{a} < 2$, almost surely

$$
\frac{\mathcal{C}_{\varepsilon_{n(j),1}}(D(0, br_{\varepsilon_{n(j+1),1}}))}{\left(\log\varepsilon_{n(j),1}\right)^2} \geq \left(\frac{\widehat{a}}{\pi} - \frac{2}{\sqrt{\log n(j)}}\right),
$$

for all j large enough. Indeed, this follows from (3.12) after noting that $n(j)^{-4}S \subseteq D(0, br_{\varepsilon_{n(j+1),1}})$. By first interpolating for $\varepsilon_{n(j+1),1} \leq \varepsilon \leq \varepsilon_{n(j),1}$ using monotonicity and then letting $\widehat{a} \uparrow 2$ we thus have that almost surely,

$$
\lim_{\varepsilon\to 0} \frac{\mathcal{C}_\varepsilon(D(0,br_\varepsilon))}{\left(\log\varepsilon\right)^2} = \frac{2}{\pi}\,. \tag{5.5}
$$

Fix $1 > \gamma > 0$. For the remainder of this section only we set $\varepsilon_n = n^{\gamma-1}$ and $r_n = r_{\varepsilon_n}$. Using the notation of Section 2, for $x = 0$, $r = r_n$ and any $R \in (0, 1/2)$, let

$$\mathcal{N}'_n(a, R, b) = \max\{j : \mathfrak{T}_j \leq \mathcal{C}_{\varepsilon_n}(D(0, br_n))\}$$

denote the number of excursions of the Brownian motion X_t in the torus $\mathbb{T}^2$ from $\partial D_{\mathbb{T}^2}(0, r_n) = \partial D(0, r_n)$ to $\partial D_{\mathbb{T}^2}(0, R) = \partial D(0, R)$ up to time $\mathcal{C}_{\varepsilon_n}(D(0, br_n))$. Fixing $\delta > 0$, let $N_n = (2/3)(1-\gamma)^2(1-2\delta)\phi_n$, noting that

$$\frac{2}{\pi}(1-\delta)(\log \varepsilon_n)^2 \geq (1+\delta)\frac{N_n}{\pi}\log(R/r)\,,$$

for all $n \geq n_0(a, R, \delta, \gamma)$, implying that,

$$\begin{aligned}\mathbf{P}(\mathcal{N}'_n(a, R, b) \leq N_n) \leq & \,\mathbf{P}\left(\mathcal{C}_{\varepsilon_n}(D(0, br_n)) \leq \frac{2}{\pi}(1-\delta)(\log \varepsilon_n)^2\right) \\ & + \mathbf{P}\left(\sum_{j=0}^{N_n} \tau^{(j)} \geq (1+\delta)\frac{\log(R/r)}{\pi} N_n\right).\end{aligned}$$

Hence, by (2.11) and (5.5) it follows that for any $R < R_1(\delta)$, $a > 0$ and $b \in (0, 1)$,

$$\lim_{n\to\infty} \mathbf{P}\left(\mathcal{N}'_n(a, R, b) \leq N_n\right) = 0. \tag{5.6}$$

Our next task is to show that (5.6) applies for the excursion counts $\mathcal{N}_n(a, R, b)$ that correspond to $\mathcal{N}'_n(a, R, b)$, when X_t is replaced by the planar Brownian motion W_t. To this end, consider the random vectors

$$\mathbf{W}_k := (W_{\mathfrak{T}_{j-1}+\sigma^{(j)}}, j = 1, \ldots, k)$$

and $\mathbf{X}_k := (X_{\mathfrak{T}_{j-1}+\sigma^{(j)}}, j = 1, \ldots, k)$. Recall that the j-th excursion of X_t from $\partial D_{\mathbb{T}^2}(0, r)$ to $\partial D_{\mathbb{T}^2}(0, R)$, starting at $\alpha_j = X_{\mathfrak{T}_{j-1}+\sigma^{(j)}}$ is precisely the isomorphic image of a planar Brownian motion started at α_j, and run until first hitting $\partial D(0, R)$ (and the same applies in case $\alpha_0 = X_0 = 0$). Thus, by the strong Markov property of both X_t and W_t at the stopping times $\mathfrak{T}_0, \mathfrak{T}_0 + \sigma^{(1)}, \mathfrak{T}_1, \mathfrak{T}_1 + \sigma^{(2)}, \ldots$ we see that for every Borel set $B \subset (\partial D(0, r))^k$

$$\mathbf{P}^0(\mathbf{W}_k \in B) = \mathbb{E}^0(\prod_{j=0}^{k-1} h_r(X_{\mathfrak{T}_j}, X_{\mathfrak{T}_j+\sigma^{(j+1)}}); \mathbf{X}_k \in B)\,.$$

Recall that $|X_{\mathfrak{T}_j}| = R$ and $|X_{\mathfrak{T}_j+\sigma^{(j+1)}}| = r$ for all $j \geq 0$. Consequently, the law of $\mathbf{W}_k$ is absolutely continuous with respect to the law of $\mathbf{X}_k$, with Radon-Nikodym derivative $h_{k,r}$ such that

$$\|h_{k,r}\|_\infty \leq \left(\sup_{|\beta|=R, |\alpha|=r} h_r(\beta, \alpha)\right)^k\,.$$

With $r = r_n \to 0$, we thus have by (5.4) that for small enough $R > 0$ and all n large enough,

$$\|h_{N_n,r_n}\|_\infty \le \left(1 + \frac{40 r_n \log(2r_n)}{R\log(2R)}\right)^{N_n} . \tag{5.7}$$

Since $N_n r_n |\log(2r_n)| \to 0$, we see that $\|h_{N_n,r_n}\|_\infty \to 1$ as $n \to \infty$. Since $b < 1$, and with the j-th excursion of X_t from $\partial D_{\mathbb{T}^2}(0, r)$ to $\partial D_{\mathbb{T}^2}(0, R)$, starting at some $\alpha_j = X_{\mathfrak{T}_{j-1}+\sigma^{(j)}}$ being the isomorphic image of a planar Brownian motion started at α_j, and run until first hitting $\partial D(0, R)$, we get by the strong Markov property of both X_t and W_t that for any k,

$$\mathbb{E}\left(\mathbf{1}_{\mathcal{N}_n(a,R,b)\le k} \,|\, \sigma(\mathbf{W}_k)\right) = \mathbb{E}\left(\mathbf{1}_{\mathcal{N}'_n(a,R,b)\le k} \,|\, \sigma(\mathbf{X}_k)\right) ,$$

implying that

$$\mathbf{P}\left(\mathcal{N}_n(a, R, b) \le k\right) = \mathbb{E}\left(h_{k,r_n}(\mathbf{X}_k) \,,\, \mathcal{N}'_n(a, R, b) \le k\right) . \tag{5.8}$$

It thus follows from (5.6), (5.7) and (5.8) that

$$\begin{aligned}\mathbf{P}\left(\mathcal{N}_n(a, R, b) \le N_n\right) &= \mathbb{E}\left(h_{N_n,r_n}(\mathbf{X}_{N_n}) \,,\, \mathcal{N}'_n(a, R, b) \le N_n\right) \\ &\le \|h_{N_n,r_n}\|_\infty \mathbf{P}\left(\mathcal{N}'_n(a, R, b) \le N_n\right) \to 0 \,.\end{aligned} \tag{5.9}$$

Setting $R < R_0(\delta)$ small enough for (5.9) to apply, with $a := 2R(1-\gamma)^5$ and $b := 1/(2(1-\gamma))$, we next use strong approximation, as in Section 4, to show how (5.1) follows from this. Indeed, with $t_n := \exp((\log n)^3)$, we may and shall, for each n, construct $\{S_k\}$ and $\{W_t\}$ on the same probability space so that for some $n_0 = n_0(\omega) < \infty$

$$\max_{k\le t_n} |W_k - \sqrt{2}S_k| \le n^{\gamma/2}, \qquad \forall n \ge n_0 \qquad \text{a.s.}$$

Hence, multiplying by $\rho_n := br_n/(\sqrt{2}n)$ we have

$$\max_{k\le t_n} |\rho_n W_k - \rho_n\sqrt{2}S_k| \le \varepsilon_n/3, \qquad \forall n \ge n_0 \qquad \text{a.s.}$$

or, using Brownian scaling, we have

$$\mathbf{P}\left(\max_{k\le t_n} |W_{k\rho_n^2} - \rho_n\sqrt{2}S_k| \le \varepsilon_n/3\right) \ge 1 - \delta \tag{5.10}$$

for all $n \ge N'_0$ with some $N'_0 = N'_0(\gamma, \delta) < \infty$.

Recall that $\mathbf{P}(T_n > t_n) \to 0$, see [21, Theorem 1.1], hence by (5.9), we see that for all n sufficiently large,

$$\mathbf{P}\left(\mathcal{N}_n(a, R, b) > N_n,\, T_n \le t_n\right) \ge 1 - \delta \,. \tag{5.11}$$

Now, by (5.11) we have that with probability at least $1-\delta$ some disc $D(x, \varepsilon_n) \subseteq D(0, br_n)$ is completely missed by $\{W_{k\rho_n^2}\}$ during the first N_n excursions from $\partial D(0, r_n)$ to $\partial D(0, R)$. Moreover, by (5.11), also the fact that $\{\sqrt{2}\rho_n S_k : k \le t_n\}$ covers $\sqrt{2}\rho_n D_n$, with probability at least $1 - 2\delta$, we also have by (5.10), that the sequence $\{W_{k\rho_n^2} : k \le t_n\}$ provides a $(2\varepsilon_n/3)$-cover of the set $D(0, \sqrt{2}\rho_n n)$. Our choice of ρ_n guarantees that the latter set is exactly $D(0, br_n)$. Consequently, in this case we know that the N_n excursions mentioned above are completed by time $\rho_n^{-2} t_n$. Observe that $b > 1/2$ and $br_n(\log n)^3 = R(1-\gamma)$, hence $(r_n + \varepsilon_n/3) < \sqrt{2}\rho_n(2n)$ and $(R - \varepsilon_n/3) > \sqrt{2}\rho_n n(\log n)^3$, for all n large. Appealing again to (5.10) we thus further have that $\{\sqrt{2}\rho_n S_k\}$ avoids some disc of radius $\varepsilon_n/3 = \frac{1}{3}n^{\gamma-1}$ in $D(0, \sqrt{2}\rho_n n)$ during its first N_n excursions from $\sqrt{2}\rho_n \partial D_{2n}$ to $\sqrt{2}\rho_n \partial D_{n(\log n)^3}$. Thus, the probability that $\{S_k\}$ avoids some lattice point in D_n during its first $N_n = \frac{2}{3}(1-\gamma)^2(1-2\delta)\phi_n$ excursions from ∂D_{2n} to $\partial D_{n(\log n)^3}$ is at least $1-2\delta$. Considering $\delta \to 0$, followed by $\gamma \to 0$, we get (5.1). □

6. First moment estimates

We start by analyzing the birth-death Markov chain $\{Y_l\}$ on the state space $\{-n, -(n-1), \dots, -1\}$, starting at $Y_0 = -n$, having both $-n$ and -1 as reflecting boundaries (so that $\mathbf{P}(Y_l = -(n-1)|Y_{l-1} = -n) = 1$, $\mathbf{P}(Y_l = -2|Y_{l-1} = -1) = 1$) and the transition probabilities

$$\overline{p}_k := \mathbf{P}(Y_l = -(k-1)|Y_{l-1} = -k) = 1 - \mathbf{P}(Y_l = -(k+1)|Y_l = -k) = \frac{\log(k+1)}{\log k + \log(k+1)} \,. \tag{6.1}$$

for $k = 2, \dots, n-1$. Let $\zeta = 3a > 0$ and

$$\mathcal{S} := \inf\{m : \sum_{j=1}^{m} \mathbf{1}_{\{-n\}}(Y_j) = \zeta n^2 \log n\},$$

denote the number of steps it takes this birth-death Markov chain to complete $\zeta n^2 \log n$ excursions from $-(n-1)$ to $-n$. For each $-n \le k \le -2$,

$$\overline{L}_k = \sum_{l=1}^{\mathcal{S}} \mathbf{1}_{\{Y_{l-1}=k, Y_l=k+1\}} \,,$$

denote the number of transitions of $\{Y_l\}$ from state k to state $k+1$ up to time $\mathcal{S}$. (Thus, $\overline{L}_{-n} = \zeta n^2 \log n$.) As we show below, fixing $x \in S$, the law of $\{N_{n,k}^x\}_{k=2}^n$ relevant for the n-successful property, is exactly that of $\{\overline{L}_{-k}\}_{k=2}^n$. To get a hold on the latter, note that conditional on $\overline{L}_{-(k+1)} = \ell_{k+1} \ge 0$ we have the representation

$$\overline{L}_{-k} = \sum_{i=1}^{\ell_{k+1}} Z_i \,, \tag{6.2}$$

where the Z_i are independent, identically distributed (geometric) random variables with

$$\mathbf{P}(Z_i = j) = (1-\overline{p}_k)\overline{p}_k^{\,j}\,. \qquad j = 0,1,2,\dots\,. \tag{6.3}$$

Consequently, $\{\overline{L}_k\}_{k=-n}^{-2}$ is a Markov chain on $\mathbb{Z}_+$ with initial condition $\overline{L}_{-n} = \zeta n^2 \log n$, and transition probabilities $\mathbf{P}(\overline{L}_{-k} = 0 | \overline{L}_{-(k+1)} = 0) = 1$,

$$\mathbf{P}\left(\overline{L}_{-k} = \ell \,\middle|\, \overline{L}_{-(k+1)} = \widetilde{m}\right) = \binom{\widetilde{m}-1+\ell}{\widetilde{m}-1}\overline{p}_k^{\,\ell}(1-\overline{p}_k)^{\widetilde{m}}, \tag{6.4}$$

for $\widetilde{m} \geq 1$, $\ell \geq 0$ and $k = n-1,\dots,2$.

Let $n_k = \zeta k^2 \log k$ for $k = 3,\dots,n-1$ and define for $2 \leq i < j \leq n$,

$$h_{i,j}(\ell_j) := \sum_{\substack{\ell_i,\dots,\ell_{j-1} \\ |\ell_k - n_k| \leq k}} \prod_{k=i}^{j-1} \mathbf{P}\left(\overline{L}_{-k} = \ell_k \,\middle|\, \overline{L}_{-(k+1)} = \ell_{k+1}\right), \tag{6.5}$$

where $\ell_n = \zeta n^2 \log n$ and $\ell_2 = 0$. The next lemma is key to estimating the growth of $h_{i,n}(\ell_n)$ in n.

LEMMA 6.1. *For some $C = C(\zeta) < \infty$ and all $3 \leq k \leq n-1$, $|\ell - n_k| \leq k$, $|\widetilde{m} - n_{k+1}| \leq k+1$, $\widetilde{m} \geq 1$,*

$$C^{-1}\frac{k^{-(\zeta+1)}}{\sqrt{\log k}} \leq \mathbf{P}\left(\overline{L}_{-k} = \ell \,\middle|\, \overline{L}_{-(k+1)} = \widetilde{m}\right) \leq C\frac{k^{-(\zeta+1)}}{\sqrt{\log k}}\,. \tag{6.6}$$

Proof of Lemma 6.1. With $p_k = 1 - \overline{p}_k$ and $m = \widetilde{m} - 1 \geq 0$, we see that

$$\frac{1-p_k}{p_k}\mathbf{P}\left(\overline{L}_{-k} = \ell \,\middle|\, \overline{L}_{-(k+1)} = \widetilde{m}\right) = \binom{m+\ell}{m} p_k^m (1-p_k)^{\ell+1}\,. \tag{6.7}$$

The right-hand side of (6.7) is from [13, (7.6)] for which the bounds of (6.6) are derived in [13, Lemma 7.2]. To complete the proof, note that $p_k = 1 - \overline{p}_k$ is bounded away from 0 and 1 (see (6.1)). ∎

Note that

$$\inf_{\widetilde{m} \leq n_3+3} \mathbf{P}\left(\overline{L}_{-2} = 0 \,\middle|\, \overline{L}_{-3} = \widetilde{m}\right) \geq (1-\overline{p}_2)^{n_3+3} > 0.$$

Hence, setting $h_{n,n}(\ell_n) = 1$, it follows from (6.5) and (6.6) that for some $C_1 < \infty$,

$$C_1^{-1}\frac{k^{-\zeta}}{\sqrt{\log k}} \leq \frac{h_{k,n}(\ell_n)}{h_{k+1,n}(\ell_n)} \leq C_1\frac{k^{-\zeta}}{\sqrt{\log k}} \qquad \forall\ 2 \leq k \leq n-1\,. \tag{6.8}$$

Applying (6.8) we conclude also that for any $\gamma > 0$ there exists $C_2 = C_2(\gamma) > 0$ such that for all $2 \leq l \leq n-1$,

$$h_{l,n}(\ell_n) \geq \prod_{k=l}^{n-1} C_1^{-1}\frac{k^{-\zeta}}{\sqrt{\log k}} \geq C_2^{n-l}\left\{\frac{n!}{l!}\right\}^{-\zeta-\gamma}. \tag{6.9}$$

Recall that $\varepsilon_k = \varepsilon_1(k!)^{-3}$ and $\varepsilon_{n,k} = \rho_n\varepsilon_n(k!)^3$ for $\rho_n = n^{-21}$ and $k = 1,\dots,n$. For $n \geq 3$ and $x \in S = [\varepsilon_1, 2\varepsilon_1]^2$, $\mathcal{R}_n^x$ denotes the time up to X_t completes $\zeta n^2 \log n$ excursions from $i^{-1}(\partial D(x,\varepsilon_{n,n-1}))$ to $i^{-1}(\partial D(x,\varepsilon_{n,n}))$ and $N_{n,k}^x$, $k = 2,\dots,n$, denote the number of excursions from $i^{-1}(\partial D(x,\varepsilon_{n,k-1}))$ to $i^{-1}(\partial D(x,\varepsilon_{n,k}))$ until $\mathcal{R}_n^x$. A point $x \in S$ is *n-successful* if

$$N_{n,2}^x = 0, \qquad n_k - k \leq N_{n,k}^x \leq n_k + k \quad \forall k = 3,\dots,n-1\,.$$

The next lemma applies (6.8) to estimate the first moment of the n-successful property.

LEMMA 6.2. *For all $n \geq 3, x \in S$ and some $\delta_n \to 0$, independent of ρ_n,*

$$\bar{q}_n := \mathbf{P}(x \text{ is } n\text{-successful}) = (n!)^{-\zeta-\delta_n}\,. \tag{6.10}$$

Proof of Lemma 6.2. Observe that

$$\overline{p}_k = \frac{\log(\varepsilon_{n,k+1}/\varepsilon_{n,k})}{\log(\varepsilon_{n,k+1}/\varepsilon_{n,k-1})}\,,$$

is exactly the probability that the planar Brownian motion B_t starting at any $z \in \partial D(x,\varepsilon_{n,k})$ will hit $\partial D(x,\varepsilon_{n,k-1})$ prior to hitting $\partial D(x,\varepsilon_{n,k+1})$, with (Y_{l-1}, Y_l) recording the order of excursions the Brownian path makes between the sets $\{\partial D(x,\varepsilon_{n,k}), n \geq k \geq 1\}$. Note that $0 \notin D(x,\varepsilon_1)$ for $x \in S$, the above mentioned probabilities are independent of the starting points of the excursions, and $\partial D(x,\varepsilon_{n,k}) \subset D(x,\varepsilon_1) \subset D(0,1/2)$, for all $k = 1,\dots,n$. Hence, by the strong Markov property of the Brownian motion X_t on $\mathbb{T}^2$ with respect to the starting times of its first n_n excursions from $i^{-1}(\partial D(x,\varepsilon_{n,n-1}))$ to $i^{-1}(\partial D(x,\varepsilon_{n,n}))$, it follows that in computing $\bar{q}_n$ of (6.10) we may and shall replace X_t by the planar Brownian motion $B_t = i(X_t)$. It follows from radial symmetry and the strong Markov property of Brownian motion that $\bar{q}_n$ is independent of $x \in S$. By Brownian scaling, $\bar{q}_n$ is also independent of the value of $\rho_n \leq 1$. Moreover, as already mentioned, fixing $x \in S$, the law of $\{N_{n,k}^x\}_{k=2}^n$ is exactly that of $\{\overline{L}_{-k}\}_{k=2}^n$. We thus deduce that

$$\bar{q}_n = \mathbf{P}\left(|\overline{L}_{-k} - n_k| \leq k\,;\, 3 \leq k \leq n-1\,;\, \overline{L}_{-2} = 0\right) = h_{2,n}(\ell_n). \tag{6.11}$$

Since $n^{-1}\log n! \to \infty$ and for some $\eta_n \to 0$

$$\prod_{k=2}^n \log(k) = (n!)^{\eta_n}\,,$$

we see that the estimate (6.10) on $\bar{q}_n$ is a direct consequence of the bound (6.8). □

In Section 7 we control the second moment of the n-successful property. To do this, we need to consider excursions between disks centered at $x \in S$ as well as those between disks centered at $y \in S$, $y \neq x$. The radial symmetry we used in proving Lemma 6.2 is hence lost. The next lemma shows that, in terms of the number of excursions, not much is lost when we focus on a certain σ-algebra $\mathcal{G}_l^y$ which contains more information than just the number of excursions in the previous level. To define $\mathcal{G}_l^y$, let $\tau_0 = 0$ and for $i = 1, 2, \dots$ let

$$\begin{aligned} \tau_{2i-1} &= \inf\{t \geq \tau_{2i-2} : \ X_t \in i^{-1}(\partial D(y, \varepsilon_{n,l-1}))\}, \\ \tau_{2i} &= \inf\{t \geq \tau_{2i-1} : \ X_t \in i^{-1}(\partial D(y, \varepsilon_{n,l}))\}. \end{aligned}$$

Thus, $N_{n,l}^y = \max\{i : \tau_{2i} \leq \mathcal{R}_n^y\}$. For each $j = 1, 2, \dots, N_{n,l}^y$ let

$$e^{(j)} = \{X_{\tau_{2j-2}+t} : 0 \leq t \leq \tau_{2j-1} - \tau_{2j-2}\}$$

be the j-th excursion from $i^{-1}(\partial D(y, \varepsilon_{n,l}))$ to $i^{-1}(\partial D(y, \varepsilon_{n,l-1}))$ (but note that for $j = 1$ we do begin at $t = 0$). Finally, let

$$e^{(N_{n,l}^y + 1)} = \{X_{\tau_{2N_{n,l}^y}+t} : t \geq 0\}.$$

We let $J_l := \{l-1, \dots, 2\}$ and take $\mathcal{G}_l^y$ to be the σ-algebra generated by the excursions $e^{(1)}, \dots, e^{(N_{n,l}^y)}$, $e^{(N_{n,l}^y+1)}$.

LEMMA 6.3. *For some $C_0 < \infty$, any $3 \leq l \leq n$, $|m_l - n_l| \leq l$ and all $y \in S$,*

$$\begin{aligned} &\mathbf{P}(N_{n,k}^y = m_k; k \in J_l \,|\, N_{n,l}^y = m_l, \mathcal{G}_l^y) \\ &\qquad \leq (1 + C_0 l^{-1} \log l) \prod_{k=2}^{l-1} \mathbf{P}\left(\overline{L}_{-k} = m_k \,|\, \overline{L}_{-(k+1)} = m_{k+1}\right). \end{aligned} \tag{6.12}$$

The key to the proof of Lemma 6.3 is to demonstrate that the number of Brownian excursions involving concentric disks of radii $\varepsilon_{n,k}$, $k \in J_l$ prior to first exiting the disk of radius $\varepsilon_{n,l}$ is almost independent of the initial and final points of the overall excursion between the $\varepsilon_{n,l-1}$ and $\varepsilon_{n,l}$ disks. The next lemma provides uniform estimates sufficient for this task.

LEMMA 6.4. *Consider a Brownian path B. starting at $z \in \partial D(y, \varepsilon_{n,l-1})$, for some $3 \leq l \leq n$. Let $\bar{\tau} = \inf\{t > 0 : B_t \notin D(y, \varepsilon_{n,l})\}$ and Z_k, $k \in J_l$, denote the number of excursions of the path from $\partial D(y, \varepsilon_{n,k-1})$ to $\partial D(y, \varepsilon_{n,k})$, prior to $\bar{\tau}$. Then, there exists a universal constant $c < \infty$, such that for all $\{m_k : k \in J_l\}$, uniformly in $v \in \partial D(y, \varepsilon_{n,l})$ and y,*

$$\mathbf{P}^z(Z_k = m_k, k \in J_l \,\big|\, B_{\bar{\tau}} = v) \leq (1 + c l^{-3}) \mathbf{P}^z(Z_k = m_k, \, k \in J_l)\,. \tag{6.13}$$

Proof of Lemma 6.4. This is essentially [13, Lemma 7.4]. The only difference is that here we use the sequence of radii $\varepsilon_{n,k}$, for $k = l, l-1, l-2, \dots, 2$, whereas [13] uses the radii ε_k, for $k = l-1, l, l+1, \dots, n$. The proof of [13, Lemma 7.4] involves only the ratio $\varepsilon_l/\varepsilon_{l-1} = l^{-3}$ between the two exterior disks and the fact that the probability p_l of reaching the next disk (of radius ε_{l+1} there), is uniformly bounded away from 1. The ratio of the two exterior disks here is $\varepsilon_{n,l-1}/\varepsilon_{n,l} = l^{-3}$ which is the same as in [13], whereas p_l is replaced here by $\overline{p}_{l-1}$, which is also uniformly bounded away from 1. $\square$

Proof of Lemma 6.3. Fixing $3 \le l \le n$ and $y \in S$, let $Z_k^{(j)}$, $k \in J_l$ denote the number of excursions from $i^{-1}(\partial D(y, \varepsilon_{n,k-1}))$ to $i^{-1}(\partial D(y, \varepsilon_{n,k}))$ during the j-th excursion of the path X_t from $i^{-1}(\partial D(y, \varepsilon_{n,l-1}))$ to $i^{-1}(\partial D(y, \varepsilon_{n,l}))$. If $m_l = 0$ then the probabilities on both sides of (6.12) are zero unless $m_k = 0$ for all $k \in J_l$, in which case they are both one; so the lemma trivially applies when $m_l = 0$. Considering hereafter $m_l > 0$, since $0 \notin i^{-1}(D(y, \varepsilon_1))$ we have that conditioned upon $\{N_{n,l}^y = m_l\}$,

$$N_{n,k}^y = \sum_{j=1}^{m_l} Z_k^{(j)} \,, \qquad k \in J_l \,. \tag{6.14}$$

Conditioned upon $\mathcal{G}_l^y$, the random vectors $\{Z_k^{(j)}, k \in J_l\}$ are independent for $j = 1, 2, \dots, m_l$. Moreover, as X_t is the isomorphic image of a planar Brownian motion B_t within $D(y, \varepsilon_{n,l})$, we see that $\{Z_k^{(j)}, k \in J_l\}$ then has the conditional law of $\{Z_k, k \in J_l\}$ of Lemma 6.4 for some random $z_j \in \partial D(y, \varepsilon_{n,l-1})$ and $v_j \in \partial D(y, \varepsilon_{n,l})$, both measurable on $\mathcal{G}_l^y$ (as z_j corresponds to the final point of $e^{(j)}$, the j-th excursion from $i^{-1}(\partial D(y, \varepsilon_{n,l}))$ to $i^{-1}(\partial D(y, \varepsilon_{n,l-1}))$ and v_j corresponds to the initial point of the $(j+1)$-st such excursion $e^{(j+1)}$). Let $\mathcal{P}_l$ denote the finite set of all partitions $\{m_k^{(j)}, k \in J_l, j = 1, \dots, m_l : m_k = \sum_{j=1}^{m_l} m_k^{(j)}, k \in J_l\}$. Then, by the uniform upper bound of (6.13) and radial symmetry,

$$\begin{aligned}
\mathbf{P}(N_{n,k}^y = m_k,\, k \in J_l \big| \, N_{n,l}^y = m_l, \mathcal{G}_l^y)& \\
&= \sum_{\mathcal{P}_l} \prod_{j=1}^{m_l} \mathbf{P}^{z_j}(Z_k = m_k^{(j)},\, k \in J_l \,\big|\, B_{\bar{\tau}} = v_j) \\
&\le \sum_{\mathcal{P}_l} \prod_{j=1}^{m_l} (1 + cl^{-3}) \mathbf{P}^{z_j}(Z_k = m_k^{(j)},\, k \in J_l) \\
&= (1 + cl^{-3})^{m_l} \mathbf{P}\left(N_{n,k}^y = m_k,\, k \in J_l \big| \, N_{n,l}^y = m_l \right) .
\end{aligned}$$

Since $m_l \le c_1 l^2 \log l$ we thus get the bound (6.12) by the representation used in the proof of Lemma 6.2. $\square$

7. Second moment estimates

Recall that $N^x_{n,k}$ for $x \in S$, $2 \le k \le n$, denotes the number of excursions from $i^{-1}(\partial D(x,\varepsilon_{n,k-1}))$ to $i^{-1}(\partial D(x,\varepsilon_{n,k}))$ prior to $\mathcal{R}^x_n$. With $n_k = \zeta k^2 \log k$ we shall write $N \overset{k}{\sim} n_k$ if $|N - n_k| \le k$ for $3 \le k \le n-1$ and $N = 0$ when $k = 2$. Relying upon the first moment estimates of Lemmas 6.2 and 6.3, we next bound the second moment of the n-successful property.

Lemma 7.1. *For any $\gamma > 0$ there exists $C = C(\gamma) < \infty$ such that for all $x, y \in S$,*

$$\mathbf{P}\,(x \text{ and } y \text{ are } n\text{-successful}) \le \bar{q}_n^2\, n^{3+5\zeta} C^{n-l} \left(\frac{n!}{l!}\right)^{\zeta+\gamma}, \tag{7.1}$$

where $l = \max\{k \le n : |x-y| \ge 2\varepsilon_{n,k}\} \vee 1$ and $\bar{q}_n := \mathbf{P}\ (x \text{ is } n\text{-successful})$. Furthermore, if $|x-y| \ge 2\varepsilon_{n,n}$ then for some $C_0 < \infty$,

$$\mathbf{P}\,(x \text{ and } y \text{ are } n\text{-successful}) \le (1 + C_0 n^{-1}\log n)\bar{q}_n^2\,. \tag{7.2}$$

Proof of Lemma 7.1. Fixing $x, y \in S$, suppose $2\varepsilon_{n,l+1} > |x-y| \ge 2\varepsilon_{n,l}$ for some $n-1 \ge l \ge 3$. Since $\varepsilon_{n,l+2} - \varepsilon_{n,l} \ge 2\varepsilon_{n,l+1}$, it is easy to see that $i^{-1}(D(y,\varepsilon_{n,l})) \cap i^{-1}(\partial D(x,\varepsilon_{n,k})) = \emptyset$ for all $k \ne l+1$. Replacing hereafter l by $l \wedge (n-3)$, it is easy to see that for $k \ne l+1$, $k \ne l+2$, the events $\{N^x_{n,k} \overset{k}{\sim} n_k\}$ are measurable on the σ-algebra $\mathcal{G}^y_l$ defined above Lemma 6.3. With $J_l := \{l-1, \dots, 2\}$ and $I_l := \{2, \dots, l, l+3, \dots, n-1\}$, we note that

$$\{x, y \text{ are } n\text{-successful}\} \subset \{N^x_{n,k} \overset{k}{\sim} n_k,\ k \in I_l\} \bigcap \{N^y_{n,k} \overset{k}{\sim} n_k,\ k \in J_{l+1}\}\,.$$

Let $\mathcal{M}(I_l) := \{m_2, \dots, m_{n-1} : m_k \overset{k}{\sim} n_k, k \in I_l\}$ (note that the range of m_{l+1}, m_{l+2} is unrestricted), and $\mathcal{M}(J_l) := \{m_2, \dots, m_{l-1} : m_k \overset{k}{\sim} n_k, k \in J_l\}$. Applying (6.12), we have that for some universal constant $C_3 < \infty$,

$$\begin{aligned}
&\mathbf{P}\,(x \text{ and } y \text{ are } n\text{-successful}) \\
&\quad\le \sum_{\mathcal{M}(J_{l+1})} \mathbb{E}\left[\mathbf{P}(N^y_{n,k} = m_k,\ k \in J_l \,\middle|\, N^y_{n,l} = m_l, \mathcal{G}^y_l)\,; N^x_{n,k} \overset{k}{\sim} n_k,\ k \in I_l\right] \\
&\quad\le C_3 \mathbf{P}\left(N^x_{n,k} \overset{k}{\sim} n_k,\ k \in I_l\right) \sum_{|m_l - n_l| \le l} h_{2,l}(m_l).
\end{aligned} \tag{7.3}$$

Since,

$$\begin{aligned}
\sum_{m_{l+1},\, m_{l+2}} \prod_{k=l}^{l+2} \mathbf{P}\left(\overline{L}_{-k} = m_k \,\middle|\, \overline{L}_{-(k+1)} = m_{k+1}\right) \\
= \mathbf{P}\left(\overline{L}_{-l} = m_l \,\middle|\, \overline{L}_{-(l+3)} = m_{l+3}\right) \le 1,
\end{aligned}$$

taking $m_n = \zeta n^2 \log n$, we have by the representation (6.11) of Lemma 6.2, that

$$\mathbf{P}\left(N^x_{n,k} \stackrel{k}{\sim} n_k,\ k \in I_l\right) = \sum_{\mathcal{M}(I_l)} \prod_{k=2}^{n-1} \mathbf{P}\left(\overline{L}_{-k} = m_k \,\middle|\, \overline{L}_{-(k+1)} = m_{k+1}\right) \le h_{l+3,n}(m_n) \sum_{|m_l - n_l| \le l} h_{2,l}(m_l) \tag{7.4}$$

(as mentioned, the sum over $\mathcal{M}(I_l)$ involves the unrestricted m_{l+1} and m_{l+2}). Combining (7.3) and (7.4), we have

$$\mathbf{P}\left(x \text{ and } y \text{ are } n\text{-successful}\right) \le C_3 h_{l+3,n}(m_n) \Big[\sum_{|m_l - n_l| \le l} h_{2,l}(m_l) \Big]^2. \tag{7.5}$$

By (6.11) and the bounds of Lemma 6.1 we have the inequalities,

$$\begin{aligned} \bar{q}_n = h_{2,n}(m_n) &\ge h_{l,n}(m_n) \inf_{|m_l - n_l| \le l} h_{2,l}(m_l) \\ &\ge h_{l,n}(m_n) C^{-2} \sup_{|m_l - n_l| \le l} h_{2,l}(m_l) \\ &\ge h_{l,n}(m_n) C^{-2} (2l+1)^{-1} \sum_{|m_l - n_l| \le l} h_{2,l}(m_l). \end{aligned} \tag{7.6}$$

Combining (7.5) and (7.6), we see that for some universal constant $C_4 < \infty$,

$$\mathbf{P}\left(x \text{ and } y \text{ are } n\text{-successful}\right) \le C_4 n^2 \bar{q}_n^2 \frac{h_{l+3,n}(m_n)}{h_{l,n}(m_n)^2}.$$

By (6.8), $h_{l+3,n}(m_n)/h_{l,n}(m_n) \le C_5 n^{3\zeta+1}$ for some $C_5 < \infty$ and all $l \le n-3$. Thus, we get (7.1) via the bound (6.9) on $h_{l,n}(m_n)$, with the extra $n^{2\zeta}$ factor coming from the use of $l \wedge (n-3)$ throughout the above proof. It also follows from (6.9) and (6.11) that when $2\varepsilon_{n,3} > |x-y|$, the trivial bound $\mathbf{P}\left(x \text{ and } y \text{ are } n\text{-successful}\right) \le \bar{q}_n$ already implies (7.1).

Suppose next that $|x-y| \ge 2\varepsilon_{n,n}$, in which case (7.1) is contained in the sharper bound (7.2). To prove the latter, note that if $|x-y| \ge 2\varepsilon_{n,n}$, then the event $\{x \text{ is } n\text{-successful }\}$ is $\mathcal{G}^y_n$ measurable; hence

$$\begin{aligned} &\mathbf{P}\left(x \text{ and } y \text{ are } n\text{-successful }\right) \\ &\quad = \mathbb{E}\left(\left\{\mathbf{P}(y \text{ is } n\text{-successful } \,\middle|\, \mathcal{G}^y_n)\right\},\ x \text{ is } n\text{-successful }\right) \\ &\quad = \mathbb{E}\left(\left\{\mathbf{P}(\, N^y_{n,k} \stackrel{k}{\sim} n_k,\, k \in J_n \,\middle|\, N^y_{n,n} = m_n,\, \mathcal{G}^y_n)\right\},\ x \text{ is } n\text{-successful }\right), \end{aligned}$$

and (7.2) follows from Lemma 6.3. □

8. The ε-covering time of a compact Riemannian manifold

Let M be a smooth, compact, connected two-dimensional, Riemannian manifold without boundary. Let $\{X_t\}_{t\geq 0}$ denote Brownian motion on M starting at some nonrandom $x_0 \in M$. The process $\{X_t\}_{t\geq 0}$ is a symmetric, strong Markov process with reference measure given by the Riemannian measure dA and infinitesimal generator 1/2 the Laplace-Beltrami operator Δ_M. We use $d(x,y)$ to denote the Riemannian distance between $x, y \in M$. With this notion of distance we can take over the definitions used for the plane and the flat torus: $D_M(x,r)$ denotes the open disc in M of radius r centered at x. For x in M we have the ε-hitting time

$$\mathcal{T}(x,\varepsilon) = \inf\{t > 0 \,|\, X_t \in D_M(x,\varepsilon)\}.$$

Then

$$\mathcal{C}_\varepsilon = \sup_{x\in M} \mathcal{T}(x,\varepsilon)$$

is the ε-covering time of M.

Proof of Theorem 1.3. If g denotes the Riemannian metric for M, let M' denote the Riemannian manifold obtained by changing the Riemannian metric for M to $g' = g/A$, so that the area of M' is 1. Since $\Delta_{M'} = \frac{1}{A}\Delta_M$, it follows that $X'_t = X_{t/A}$ is the Brownian motion on M'. With $\mathcal{C}'_{\varepsilon'}$ denoting the ε'-covering time of M', we see that $\mathcal{C}_\varepsilon$ has the same law as $A\mathcal{C}'_{\varepsilon/\sqrt{A}}$. Consequently, it suffices to prove the theorem only for manifolds of area $A = 1$, which we assume hereafter. Then, the statement and proof of Lemma 2.1 applies for any fixed $x \in M$, upon replacing $D_{\mathbb{T}^2}(x,\cdot)$ by $D_M(x,\cdot)$.

Our assumptions about M imply the existence for some $\xi > 0$ of a smooth isothermal coordinate system in each disc $D_M(u,\xi)$, $u \in M$ (cf. for example [28, p. 386 and Addendum 1]). This implies that with respect to such coordinates, the Laplace-Beltrami operator Δ_M is given on $D_M(u,\xi)$ by $a(z)(\partial_1^2 + \partial_2^2)$ for some smooth, scalar function $a : M \to (0,\infty)$, with $a(z) = a_u(z)$ possibly depending on u. Moreover, for each $u \in M$ and $\delta > 0$, upon choosing $\xi = \xi(u,\delta) > 0$ small enough, we may after translation and dilation, assume that for the above mentioned coordinate system $i : D_M(u,\xi) \mapsto \mathbb{R}^2$, we have $i(u) = 0$, $D(0,\rho) \subset i(D_M(u,\xi/2))$ for some $\rho = \rho(u,\delta)$ with $0 < \rho < \xi$ and if $x, x' \in D_M(u,\xi)$, then

$$(1-\delta)|i(x) - i(x')| \leq d(x,x') \leq (1+\delta)|i(x) - i(x')|\,. \tag{8.1}$$

For any open $G \subseteq D_M(u,\xi)$, let $\tau_G = \inf\{t \geq 0 : X_t \notin G\}$. It follows that for any $z \in D_M(u,\xi)$ we can find a Brownian motion B_t starting at $i(z)$ such that $\{i(X_t),\, t \leq \tau_G\} = \{B_{T_t},\, t \leq \tau_G\}$ where $T_t = \int_0^t a(X_s)\,ds$; see [27, §V.1]. Thus, $T_{\tau_G} = \widetilde{\tau}_{i(G)}$, where for any set $D \subseteq \mathbb{R}^2$ we write

$$\widetilde{\tau}_D = \inf\{t \geq 0 : B_t \notin D\}.$$

Consequently

$$\left(\inf_{v\in G} a(v)\right)\tau_G \leq \int_0^{\tau_G} a(X_s)\,ds = \widetilde{\tau}_{i(G)}. \tag{8.2}$$

The upper bound in (1.3) is obtained by adapting the proof provided in Section 2. To this end, fixing $1/2 > \delta > 0$, extract a finite open subcover $\cup_j D_M(u_j, \xi_j/4)$ of the compact manifold M out of $\cup_{u\in M} D_M(u, \xi(u,\delta)/4)$. Since $\underline{a} = \min_j \inf_{z\in D_M(u_j,\xi_j)}\ a_{u_j}(z) > 0$, we have by (8.1), (8.2) and (2.15) that for any $R \leq \min_j \xi_j/4$

$$\|\tau_R\|_R := \sup_{x\in M}\sup_{z\in D_M(x,R)} \mathbb{E}^z(\tau_{D_M(x,R)}) \leq \frac{R^2}{2\underline{a}(1-\delta)^2} \to_{R\to 0} 0.$$

With its proof otherwise unchanged, Lemma 2.2 applies for M. Moreover, fixing j, we have that for any $x \in D_M(u_j, \xi_j/4)$ and $0 < \varepsilon < R < \xi_j/4$,

$$\begin{aligned} i^{-1}(D(i(x),(1-\delta)\varepsilon)) &\subseteq D_M(x,\varepsilon)\,, \\ i^{-1}(D(i(x),(1-\delta)R)) &\subseteq D_M(x,R)\,, \\ i^{-1}(D(i(x),(1-\delta)^{-1}R/e)) &\supseteq D_M(x,R/e)\,. \end{aligned}$$

Consequently, the left-hand side of (2.23) is bounded above by the probability that W_t does not hit $D(i(x),(1-\delta)\varepsilon)$ during n_ε excursions, each starting at $\partial D(i(x),(1-\delta)^{-1}R/e)$ and ending at $\partial D(i(x),(1-\delta)R)$. This results in (2.23) and hence in Lemma 2.3 holding, albeit with $1-\delta' = (1-\delta)(1+2\log(1-\delta))$ instead of $(1-\delta)$. Since M is a smooth, compact, two-dimensional manifold, there are at most $O(\varepsilon^{-2})$ points $x_j \in M$ such that $\inf_{\ell\neq j} d(x_\ell, x_j) \geq \varepsilon$. The upper bound in (1.3) thus follows by the same argument that concludes Section 2.

The complementary lower bound is next obtained by adapting the proof provided in Section 3. To this end, fixing $1/2 > \delta > 0$, let $\xi = \xi(\delta) > 0$ and $\rho = \rho(\delta) > 0$ be such that $D(0,\rho) \subset i(D_M(x_0, \xi/2))$ and (8.1) holds for the isothermal coordinate system i on $D_M(x_0,\xi)$, with $i(x_0) = 0$. It follows that

$$\bigcup_{x\in S} D(x,\varepsilon_1) \subset D(0,\rho) \subseteq i(D_M(x_0,\xi/2))\,,$$

provided $\varepsilon_1 < \rho/5$. Choosing $0 < \varepsilon_1 < \rho/5$ small enough so that $\varepsilon_1 < R_1(\delta)$ of Lemma 2.2, we say that $x \in S$ is n-successful if (3.2) applies. The probability $\overline{p}_k$ that a planar Brownian path B_t starting at any $z \in \partial D(x,\varepsilon_{n,k})$ hits $\partial D(x,\varepsilon_{n,k-1})$ prior to $\partial D(x,\varepsilon_{n,k+1})$, is independent of z and this is true even after an arbitrary random, path dependent, time change. With $x_0 \notin i^{-1}(D(x,\varepsilon_1))$, and $i^{-1}(\partial D(x,\varepsilon_{n,k})) \subset i^{-1}(D(0,\rho))$ for all $k = 1,\ldots,n$, we see that the identity (6.11) holds, resulting in the conclusion of Lemma 6.2. For $y \in S$, let $\mathcal{G}^y_l$ be the σ-algebra generated by the excursions $e^{(1)},\ldots,e^{(N^y_{n,l})}$, $e^{(N^y_{n,l}+1)}$ as defined in Section 6. Note that Lemma 6.4 applies to the law of a planar Brownian excursion $B_\cdot$ starting at $z \in \partial D(y, \varepsilon_{n,l-1})$, conditioned to first exit $D(y, \varepsilon_{n,l})$ at v, even after an arbitrary random, path dependent, time change (indeed, both sides of (6.13) are clearly independent of such a time change). Moreover, the upper bound in (6.13) is independent of the initial point $z \in \partial D(y, \varepsilon_{n,l-1})$. In case $N^y_{n,l} = m_l > 0$, since $x_0 \notin i^{-1}(D(y, \varepsilon_1))$ we have the representation (6.14), where conditioned upon $\mathcal{G}^y_l$, the random vectors $\{Z^{(j)}_k, k \in J_l\}$ are independent for $j = 1, 2, \dots, m_l$. Recall the above mentioned identity between the 'isomorphic image' of the path of X_t until first exiting $i^{-1}(D(y, \varepsilon_{n,l}))$ and the law of a time-changed planar Brownian path until its first exit of $D(y, \varepsilon_{n,l})$. This identity implies that each random vector $\{Z^{(j)}_k, k \in J_l\}$ has the conditional law of $\{Z_k, k \in J_l\}$ of Lemma 6.4 for some random $z_j \in \partial D(y, \varepsilon_{n,l-1})$ and $v_j \in \partial D(y, \varepsilon_{n,l})$, both measurable on $\mathcal{G}^y_l$. With (6.13) in force, we thus establish that the conclusion (6.12) of Lemma 6.3 applies here and can follow the proof of Lemma 7.1 to arrive at its conclusion. Thus establishing all estimates of Sections 6 and 7, we have that Lemma 3.1 holds and consequently the bound of (3.9) applies. It follows from (8.1) that

$$\begin{aligned} i^{-1}(\partial D(x, \varepsilon_{n,n-1})) &\subset D_M(i^{-1}(x), (1+\delta)\varepsilon_{n,n-1}), \\ i^{-1}(\partial D(x, \varepsilon_{n,n})) &\bigcap D_M(i^{-1}(x), (1-\delta)\varepsilon_{n,n}) = \emptyset, \\ D_M(i^{-1}(x), (1-\delta)\varepsilon_{n,1}) &\subseteq i^{-1}(D(x, \varepsilon_{n,1})). \end{aligned}$$

Consequently, if x is n-successful, it follows that

$$\mathcal{T}(i^{-1}(x), (1-\delta)\varepsilon_{n,1}) \geq \sum_{j=0}^{N} \tau^{(j)},$$

where $N = n_n = 3an^2 \log n$ and $\tau^{(j)}$ correspond now to excursions between the sets $\partial D_M(i^{-1}(x), (1-\delta)\varepsilon_{n,n})$ and $\partial D_M(i^{-1}(x), (1+\delta)\varepsilon_{n,n-1})$. The statement and proof of Lemma 3.2 then apply, except that we now use $\mathcal{T}(i^{-1}(x), (1-\delta)\varepsilon_{n,1})$ in (3.10). The lower bound in (1.3) follows by the same argument as in Section 3, now with $\mathcal{C}_{(1-\delta)\varepsilon_{n(j),1}}$ in (3.11). $\square$

9. Complements and unsolved problems

1. We have the following direct corollary of Theorem 1.2.

COROLLARY 9.1. *For $0 < \gamma < 1$ let $\mathcal{T}_n(\gamma)$ denote the time it takes until the largest disk unvisited by the simple random walk in $\mathbb{Z}^2_n$ has radius n^γ. Then,*

$$\lim_{n\to\infty} \frac{\mathcal{T}_n(\gamma)}{(n \log n)^2} = \frac{4(1-\gamma)^2}{\pi} \quad \textit{in probability.}$$

Equivalently, for $0 < \alpha < 1$ the logarithm to base n of the radius of the largest unvisited disk at time $\alpha\mathcal{T}_n$ converges in probability to $1 - \sqrt{\alpha}$.

Proof of Corollary 9.1 . The lower bound on $\mathcal{T}_n(\gamma)$ is derived in Section 4. By Theorem 1.2 also $\mathbf{P}(\mathcal{C}_{\varepsilon_n} < \frac{2(1-\gamma+\delta)^2}{\pi}(\log n)^2) \geq 1-\delta$ for $\varepsilon_n = \frac{1}{3}n^{\gamma-1}$ and all n large enough. Similarly to Section 4, this yields the upper bound on $\mathcal{T}_n(\gamma)$ by strong approximation (and tail estimates for the supremum of $|W_t - W_{k/2n^2}|$ over $t \in [k/2n^2, (k+1)/2n^2]$ and $k \leq 4n^2(\log n)^2$). □

2. Given a planar lattice $\mathcal{L}$, let $\mathcal{L}_\rho = \mathcal{L} \cap D(0,\rho)$, a finite connected graph of N_ρ vertices. Denote by $\mathcal{T}_\rho$ the covering time for a simple random walk on $\mathcal{L}_\rho$. The approach of Section 4 can be adapted so as to show that

$$\lim_{\rho\to\infty} \frac{\mathcal{T}_\rho}{N_\rho(\log N_\rho)^2} = C_{\mathcal{L}} := \frac{A}{2\pi(\det\Gamma)^{1/2}} \qquad \text{in probability,}$$

where

$$A = \lim_{\rho\to\infty}\left(\frac{\pi\rho^2}{N_\rho}\right)$$

is the area of a fundamental cell of $\mathcal{L}$ and

$$\Gamma = \lim_{n\to\infty}\frac{1}{n}\mathbb{E}(S_n S_n') \ ,$$

is the two dimensional stationary covariance matrix associated with the simple random walk on $\mathcal{L}$ (note that $C_{\mathcal{L}}$ is invariant under affine transformations of $\mathbb{R}^2$ and as such is an intrinsic property of $\mathcal{L}$). Of particular interest are the triangular (degree $d=3$) and the honey-comb (degree $d=6$) lattices for which it is easy to check that $\Gamma = \frac{1}{2}I$ and $A = \frac{d}{4}\tan(\frac{\pi}{d})$.

3. Jonasson and Schramm show in [18] the existence of universal constants $C_d > 0$ such that for any planar graphs G_N of N vertices and maximal degree $d_{\max}(G_N) \leq d$, one has

$$\liminf_{N\to\infty}\frac{\mathcal{T}(G_N)}{N(\log N)^2} \geq C_d \ ,$$

where $\mathcal{T}(G_N)$ is the covering time for the simple random walk on G_N. We believe that $C_d = \frac{d}{4\pi}\tan(\frac{\pi}{d})$ for $d = 3, 4$ and $d = 6$, corresponding to G_N taken from the triangular, square and honey-comb lattices, of degree $d = 3, 4$ and 6, respectively.

4. Recall that $\mathcal{T}_n$ denotes the (random) cover time for a simple random walk in $\mathbb{Z}_n^2$. A natural question, suggested to us by David Aldous, is to find a limit law for an appropriately normalized version of $\mathcal{T}_n$. The analogies with branching random walk lead us to suspect that perhaps the random variable $\mathcal{T}_n^{1/2}/n$, minus its median, will have a nondegenerate limit law.

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