ADVANCES IN APPLIED MATHEMATICS 1, 37–49 (1980)

Mass Renormalization for the $\lambda\phi^4$ Euclidean Lattice Field

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The inverse correlation length or physical mass of the $\lambda\phi_n^4 + \sigma\phi_n^2$ Euclidean lattice field is shown to be a continuous-increasing function of σ in the single-phase region. By a suitable choice of σ, the inverse correlation length can be set equal to any strictly positive value.

I. INTRODUCTION

The $\lambda\phi_n^4 + \sigma\phi_n^2$ Euclidean lattice field is a Markov random field on $\mathbb{Z}^n$. A random field ϕ on $\mathbb{Z}^n$ is determined by a probability measure on the set of functions $\phi : \mathbb{Z}^n \to \mathbb{R}$. The probability measure associated with a Markov random field has the property that for any $x \in \mathbb{Z}^n$ Borel set $B \in \mathbb{R}$, and finite $\Lambda \subseteq \mathbb{Z}^n$ containing Λ_x, the $2n$ nearest neighbors of x,

$$P(\phi(x) \in B \,|\, \phi(y), y \in \Lambda) = P(\phi(x) \in B \,|\, \phi(y), y \in \Lambda_x).$$

In the field theory literature this is referred to as the local Markov property [4]. In the Markov random field we consider, the associated measure is also stationary, or invariant under lattice translations and rotations.

When $n = 1$, a time-reversible stationary Markov process on $\mathbb{R}$ is typically characterized by a self-adjoint second-order ordinary differential operator, the infinitesimal generator of the process [10]. These operators have purely discrete spectra with unique lowest eigenstates. The lowest eigenstate determines the equilibrium measure of the process. Its uniqueness assures us that there is a unique stationary Markov process associated with a given infinitesimal generator.

The Markov random fields we consider are also intimately related to their infinitesimal generator, called the physical Hamiltonian, which is

*Supported in part by the National Science Foundation, Grant MPS 74-13252.

0196-8858/80/01037-13$05.00/0

formally an elliptic partial differential operator in an infinite number of variables. When $n \geq 2$, these operators may have a degenerate lowest eigenvalue, and therefore, there may be more than one stationary Markov random field associated with a given Hamiltonian. This is the phenomenon of multiple phases.

Nondegeneracy of the lowest eigenvalue is equivalent to the ergodicity of our field under lattice translations [4]. In this case

$$\lim_{\Lambda \to \mathbb{Z}^n} \frac{1}{|\Lambda|} \sum_{x \in \Lambda} f(\phi(x)) = \text{constant a.e.},$$

while in the degenerate case, when the field is not ergodic, the above limit will in general be a nonconstant random variable, the value of which depends on the phase we are observing. If the field is nonergodic, it can always be decomposed into ergodic pure phases; see, for example, [4].

In the Markov random fields considered here, there is a simple criteria for ergodicity: the vanishing of the long-range order.

Explicitly, let $E_\sigma(\cdot)$ denote expectations for the Markov random field associated with $\lambda\phi_n^4 + \sigma\phi_n^2$, emphasizing the σ dependence. $E_\sigma(\cdot)$ is obtained as a limit of expectations $E_{\sigma, L}(\cdot)$ for the theory with half-Dirichlet boundary conditions in the region L. For details consult [1].

The long-range order $\mathcal{L}(\sigma) \geq 0$ is defined by

$$\mathcal{L}(\sigma)^2 = \lim_{t \to \infty} E_\sigma(\phi(0, \ldots, 0)\phi(t, 0, \ldots, 0)).$$

If $n \geq 2$, the set of σ with $\mathcal{L}(\sigma) = 0$, is a proper right half-line, since $\mathcal{L}(\sigma)$ is decreasing [2], and $\mathcal{L}(\sigma) \neq 0$ for σ sufficiently negative [3]. The right half-line, where $\mathcal{L}(\sigma) = 0$, which is the region of ergodicity [1], is therefore called the single-phase region.

In the single-phase region, the gap m between the first two eigenvalues is called the inverse correlation length, since events which are separated by distances greater than m^{-1} are "essentially" uncorrelated. Explicitly, if $\mathcal{O}_j^\pm$ denotes the algebra generated by $\phi(x)$, $x_1 \geq j$ or $x_1 \leq j$, respectively,

$$|E_\sigma(fg) - E_\sigma(f)E_\sigma(g)| \leq \exp(-m|j-k|)\|f\|_2\|g\|_2$$

for any $f \in \mathcal{O}_j^-$, $g \in \mathcal{O}_k^+$; $j \leq k$. In analogy with continuum field theory, the inverse correlation length is also called the physical mass. Let us define

$$m(\sigma) = \lim_{t \to \infty} -\log(E_\sigma(\phi(0, \ldots, 0)\phi(t, 0, \ldots, 0)))/t.$$

In the single-phase region $m(\sigma)$ is the physical mass [1]. $m(\sigma)$ is always monotone increasing. The critical point σ_c is defined as $\sigma_c = \sup\{\sigma | m(\sigma) = 0\}$. Since $\mathcal{L}(\sigma) \neq 0$ implies $m(\sigma) = 0$, we see that the region $\{\sigma | \sigma > \sigma_c\}$ is in the single-phase region.