Self-collisions of superprocesses: renormalization and limit theorems

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Abstract

In this paper, we study a renormalized self-collision local time for superprocesses over stable processes and classical diffusions. When the renormalization breaks down, we obtain limit theorems. © 1999 Elsevier Science B.V. All rights reserved.

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1. Introduction

With any nice Markov process \( z_t \) in \( \mathbb{R}^d \) we associate a new Markov \( Z_t \) taking values in the space of finite measures on \( \mathbb{R}^d \). The process \( Z_t \) is called the superprocess over \( z_t \), and we refer to Dynkin (1988a) for an introduction to superprocesses and for further references.

We will use the notation

\[
\langle \varphi, Z_t \rangle = \int \varphi(x) Z_t(dx),
\]

\[
\langle f(x, y), Z_t(dx)Z_t(dy) \rangle = \int \int f(x, y) Z_t(dx)Z_t(dy).
\]

Throughout this paper we assume that the initial measure \( Z_0 = \mu \) has a bounded and integrable density with respect to Lebesgue measure. Also we use \( |v| \) for the mass of a measure \( v \). Our starting point is the formal expression

\[
\int_0^T \langle \delta(x - y), Z_t(dx)Z_t(dy) \rangle \, ds
\]

which intuitively should measure the ‘self-collisions’ of \( Z_t \). In Eq. (1.1), \( \delta \) is the Dirac delta ‘function’. In an attempt to make Eq. (1.1) rigorous, we replace \( \delta \) by

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an approximate delta function. Let \( f(x) \geq 0 \) be a continuous symmetric function with support in the unit ball, and such that \( \int f(x) \, dx = 1 \). Set

\[
f_{\varepsilon}(x) = \frac{1}{\varepsilon^d} f\left(\frac{x}{\varepsilon}\right)
\]

and replace Eq. (1.1) by

\[
\int_0^T \langle f_{\varepsilon}(x - y), Z_s(dx) \, Z_s(dy) \rangle \, ds.
\] (1.2)

We will describe the behavior of Eq. (1.2) as \( \varepsilon \to 0 \).

To be specific, let us first take \( Z_t \) to be the superprocess over Brownian motion in \( \mathbb{R}^d \). If \( d = 1 \) there are no problems with the \( \varepsilon \to 0 \) limit. However, if \( d \geq 2 \), the case we study here, Eq. (1.2) will typically blow up as \( \varepsilon \to 0 \). In Theorem 1 we will show that if \( d = 2 \) or 3, then Eq. (1.2) can be renormalized, i.e. by subtracting a singular term which does not involve collisions, we can obtain a nontrivial limit.

This is the analogue of a result for self-intersections of Brownian motion in \( \mathbb{R}^2 \) which goes back to Varadhan (1969); see also Le Gall (1985), Rosen (1986), Yor (1985a) and Dynkin (1988b). See also our work on the self-intersections of superprocesses (Rosen, 1992), which initiated many of the techniques used in this paper. The renormalized intersection local time for Brownian motion in \( \mathbb{R}^2 \) turns out to be the right tool for analyzing fluctuations of the Wiener Sausage (see Le Gall, 1986b; Chavel et al., 1991; Wienryb, 1987) and the range of random walks (Le Gall, 1986a; Le Gall-Rosen, 1991). It is our hope that the renormalized collision local time of Theorem 1 will find similar applications to the study of measure-valued processes. In this regard see the recent paper of Evans and Perkins (1997).

When \( d = 4 \), we can no longer obtain a renormalized collision local time. However, Theorem 1 shows that a suitably scaled version converges in distribution. This is the analogue of Yor’s theorem for Brownian motion in \( \mathbb{R}^3 \), (Yor, 1985b; Rosen, 1988).

We use \( B_t \) to denote a real Brownian motion independent of our superprocess.

**Theorem 1.** Let \( Z_t \) denote the superprocess over Brownian motion in \( \mathbb{R}^d \), and set

\[
\gamma_\varepsilon(T) = \int_0^T \langle f_{\varepsilon}(x - y), Z_s(dx)Z_s(dy) \rangle \, ds - 2 \int_0^T \varphi_\varepsilon(T - s)|Z_s| \, ds,
\] (1.3)

where

\[
\varphi_\varepsilon(t) = \int_0^t \left( \int p_\varepsilon(x)f_{\varepsilon}(x - y)p_\varepsilon(y) \, dx \, dy \right) \, dr
\] (1.4)

and

\[
p_\varepsilon(y) = \frac{e^{-y^2/2s}}{(2\pi s)^{d/2}}
\]

is the transition density for Brownian motion in \( \mathbb{R}^d \).

If \( d = 2 \) or 3, then \( \gamma_\varepsilon(T) \) converges in \( L^2 \) as \( \varepsilon \to 0 \).
If \( d = 4 \), then \( \gamma_\varepsilon(T)/\lg(1/\varepsilon) \) converges weakly to \( B_{M_\varepsilon} \), where
\[
M_\varepsilon = \frac{1}{\pi^4} \int_0^T |Z_s| \, ds \tag{1.5}
\]

**Remarks on Theorem 1.**

* More generally, if \( h \in C_0^\infty(\mathbb{R}^d) \) and we set
\[
\gamma_\varepsilon(T, h) = \int_0^T \langle h(x) f_\varepsilon(x - y), Z_s(dx)Z_s(dy) \rangle - 2 \int_0^T \langle \varphi_{e,-}, Z_s \rangle \, ds,
\]
where now
\[
\varphi_{e,1}(z) = \int_0^T \left( \int \int h(z + x) p_r(x) f_\varepsilon(x - y) p_r(y) \, dx \, dy \right) \, dr,
\]
then \( \gamma_\varepsilon(T, h) \) converges in \( L^2 \) for \( d = 2, 3 \) while if \( d = 4 \)

\[
\frac{\gamma_\varepsilon(T, h)}{\lg(1/\varepsilon)}
\]
converges weakly to \( B_{M_\varepsilon(h)} \) where
\[
M_\varepsilon(h) = \frac{1}{\pi^4} \int_0^T \langle h^2, Z_s \rangle \, ds.
\]

* Adler and Lewin (1991) have developed a Tanaka-like formula for the renormalized intersection local time of super-Brownian motion. Formally applying Lemma 1.3 of Adler and Lewin (1991), (Ito’s formula), as in Lemma 1.4 of Adler and Lewin (1991) we obtain
\[
\langle G_\varepsilon(x - y), Z_T(dx)Z_T(dy) \rangle
\]
\[
= \langle G_\varepsilon(x - y), Z_0(dx)Z_0(dy) \rangle + \int_0^T \langle \Delta G_\varepsilon(x - y), Z_s(dx)Z_s(dy) \rangle \, ds
\]
\[
+ 2 \int_0^T \langle G_\varepsilon(0), Z_s(dx) \rangle \, ds + 2 \int_0^T \langle G_\varepsilon(x - y), Z_s(dx)M(ds, dy) \rangle,
\]
where \( G(x) \) (see Eqs. (2.2) and (2.3)), is the 1-potential for Brownian motion in \( \mathbb{R}^d \), \( G_\varepsilon(x) = f_\varepsilon * G(x) \) and \( M(ds, dy) \) is the martingale measure associated with super-Brownian motion. Setting
\[
\hat{\gamma}(T) = \lim_{\varepsilon \to 0} \int_0^T \langle f_\varepsilon(x - y), Z_s(dx)Z_s(dy) \rangle \, ds - G_\varepsilon(0) \int_0^T |Z_s| \, ds,
\]
which is very similar to \( \gamma(T) \) (see e.g. Eq. (2.8)), and using the fact that \((-\Delta/2 + 1)G_\varepsilon = f_\varepsilon \), Eq. (1.6) suggests that if \( d = 2 \) or 3 we will get the Tanaka-like formula
\[
2\hat{\gamma}(T) = \langle G(x - y), Z_0(dx)Z_0(dy) \rangle - \langle G(x - y), Z_T(dx)Z_T(dy) \rangle
\]
\[
+ 2 \int_0^T \langle G(x - y), Z_s(dx)Z_s(dy) \rangle \, ds
\]
\[
+ 2 \int_0^T \langle G(x - y), Z_s(dx)M(ds, dy) \rangle.
\]
It would be interesting to justify such a formula. (We caution the reader that the super-Brownian motion in Adler and Lewin (1991) and Adler (1993) is somewhat different from the super-Brownian motion considered here which follows Dynkin (1988a).)

- Adler (1993) gives a particle picture interpretation for the renormalized intersection local time of super-Brownian motion. Using his notation we have for \( \alpha \neq \beta, \ |x| = |\beta| \) and \( t \sim \alpha \),

\[
G(\alpha, x^\alpha \beta - X_t^\beta) = G(\alpha, x^\alpha \beta - X_s^\beta) + \int_0^t \nabla G(\alpha, x^\alpha \beta - X_s^\beta) \, dX_s^\alpha \\
- \int_0^t \nabla G(\alpha, x^\alpha \beta - X_s^\beta) \, dX_s^\beta + \int_0^t \Delta G(\alpha, x^\alpha \beta - X_s^\beta) \, ds. \tag{1.9}
\]

Set

\[
\gamma_{\alpha}(T) = \frac{1}{\mu} \sum_{\alpha, \beta} \int_0^T f_\alpha(x^\alpha \beta - X_s^\beta) \, ds - \frac{1}{\mu} \sum_{\alpha} \int_0^T 1_{[x]}(X_s^\alpha) \, ds. \tag{1.10}
\]

Arguing as in Adler (1993), Eq. (1.9) leads to

\[
2\gamma_{\alpha}(T) = \langle G(\alpha, x - y), Z^\alpha(\alpha, \sigma^\alpha(\alpha, x)) \rangle - \langle G(\alpha, x - y), Z^\beta(\beta, \sigma^\beta(\beta, y)) \rangle \\
+ 2 \int_0^T \langle G(\alpha, x - y), Z^\alpha(\alpha, \sigma^\alpha(\alpha, x)) \rangle \, ds \\
+ 2 \int_0^T \langle G(\alpha, x - y), Z^\beta(\beta, \sigma^\beta(\beta, y)) \rangle \, ds - \int_0^T \Delta G(\alpha, x^\alpha \beta - X_s^\beta) \, ds, \tag{1.11}
\]

where \( M^\mu \) is the martingale measure analogous to that which is denoted \( Z^\mu \) in Adler and Lewin (1991) and we used \( (-\Delta + 2)G = 2f \) to handle the case of \( \alpha = \beta \).

Comparing Eqs. (1.8) and (1.11) suggests that \( \gamma_{\alpha}(T) \to \gamma(T) \), where, as in Adler and Lewin (1991), we take \( \mu \to \infty \) and \( \varepsilon = \mu^{-c} \) for appropriate \( c > 0 \). If indeed this could be proven, it would indicate that the renormalization term is needed only to control the spurious collisions which arise from including \( \beta = \alpha \) in Eq. (1.10).

Theorem 1 will be derived with the aid of the following very explicit theorem.

**Theorem 2.** Let \( x_t \) be Brownian motion in \( \mathbb{R}^d \) killed at an independent exponential time, and let \( X_t \) be the superprocess over \( x_t \).

(a) If \( d = 2 \), then

\[
\int_0^\infty \langle f_\alpha(x - y), X_t(\sigma^\alpha(\sigma^\beta(\beta, x))) \rangle \, ds = \frac{1}{\pi} \log \left( \frac{1}{\varepsilon} \right) \int_0^\infty |x| \, ds \tag{1.12}
\]

converges in \( L^2 \) as \( \varepsilon \to 0 \).

(b) If \( d = 3 \), then

\[
\int_0^\infty \langle f_\alpha(x - y), X_t(\sigma^\alpha(\sigma^\beta(\beta, x))) \rangle \, ds = \frac{1}{4\pi} \frac{c(f)}{\varepsilon} \int_0^\infty |x| \, ds \tag{1.13}
\]

where \( c(f) = \int f(x)(1/|x|) \, dx \), converges in \( L^2 \) as \( \varepsilon \to 0 \).
(c) If \( d = 4 \), and

\[
\gamma_z = \int_0^\infty \langle f_z(x - y), X_z(dx)X_z(dy) \rangle \, ds - a(z) \int_0^\infty |X_z| \, ds,
\]

where

\[
a(z) = \frac{1}{2\pi^2} \left( \frac{1}{e^2} \int f(y) \frac{1}{y^2} \, dy - \log \left( \frac{1}{z^2} \right) \right),
\]

then \( \gamma_z/\lg(1/z) \) converges in distribution and we have

\[
E \mu(\epsilon^{-\langle \gamma_z/\lg(1/\epsilon) \rangle}) \rightarrow e^{i\mu(1/2(1 - \sqrt{1 - 2\lambda^2/\pi^2}))}
\]

for \( \lambda \) small, as \( \epsilon \to 0 \).

**Remark.** \( X_t \) is not the same as \( Z_t \) killed at an independent exponential time!

Theorem 1 can be generalized to nice diffusions in \( \mathbb{R}^d \). Let \( z_t \) be a diffusion with generator

\[
\frac{1}{2} \sum_{i,j=1}^d a_{ij}(x) \frac{\partial}{\partial x_i} \frac{\partial}{\partial x_j} + \sum_{i=1}^d b_i(x) \frac{\partial}{\partial x_i}.
\]

If \( a_{ij}, b_i \) are smooth and uniformly bounded together with their derivatives, and

\[
\sum_{i,j=1}^d a_{ij}(x) \lambda_i \lambda_j \geq \delta \sum_{i=1}^d \lambda_i^2
\]

for some \( \delta > 0 \), uniformly in \( x \) and \( \lambda_i \), we will say that \( z_t \) is a smooth uniformly elliptic diffusion.

**Theorem 3.** Let \( Z_t \) denote the superprocess over \( z_t \), a smooth uniformly elliptic diffusion in \( \mathbb{R}^d \) and set

\[
\gamma_z(T) = \int_0^T \langle f_z(x - y), Z_z(dx)Z_z(dy) \rangle \, ds - 2 \int_0^T \langle \varphi_z, t-x, Z_z \rangle \, ds,
\]

where

\[
\varphi_z(z) = \int_0^T \left( \int \int p_t(z,x)p_t(z,y)f_z(x - y) \, dx \, dy \right) \, dt
\]

and \( p_t(x,y) \) is the transition density for \( z_t \).

If \( d = 2 \) or \( 3 \), then \( \gamma_z(T) \) converges in \( L^2 \) as \( \epsilon \to 0 \).

If \( d = 4 \) then \( \gamma_z(T)/\lg(1/\epsilon) \) converges weakly to \( B_{M_T} \) where

\[
M_T = \frac{1}{\pi^4} \int_0^T \langle \psi, Z_z \rangle \, ds
\]

and

\[
\psi(x) = \frac{1}{\det a_{ij}(x)}.
\]
We now generalize Theorem 1 to symmetric stable processes of order $\beta$ in $\mathbb{R}^d$. As before, only the case $\beta \leq d$ is of interest, since if $\beta > d$, the $\varepsilon \to 0$ limit exists.

**Theorem 4.** Let $Y_t$ denote the superprocess over the symmetric stable process $y_t$ of order $\beta$ in $\mathbb{R}^d$, and

$$\gamma_\varepsilon(T) = \int_0^T \langle f_\varepsilon(x - y), Y_s(dy) \rangle ds - 2 \int_0^T \varphi_\varepsilon(T - s) |Y_s| ds,$$

(1.23)

where

$$\varphi_\varepsilon(t) = \int_0^t \left( \int \int p_\varepsilon(x) f_\varepsilon(x - y) p_\varepsilon(y) dx dy \right) dr$$

(1.24)

and $p_\varepsilon(y)$ denotes the transition density for $y_t$.

If $d/2 < \beta \leq d$, then $\gamma_\varepsilon(T)$ converges in $L^2$ as $\varepsilon \to 0$.

If $\beta = d/2$, then $\gamma_\varepsilon(T)/\lg(1/\varepsilon)$ converges weakly to $B_{M_T}$ where

$$M_T = a(d) \int_0^T |Y_s| ds$$

(1.25)

and

$$a(d) = \frac{2^{4-2d}}{\pi^d} \frac{1}{\Gamma^2(d/2)}$$

Sections 2–6 are devoted to Theorem 2, i.e. the superprocess over killed Brownian motion. In Section 7, we derive Theorem 1 with the aid of Theorem 2. The necessary modifications for the proofs of Theorems 3 and 4 are explained in Sections 8 and 9.

The present paper is a sequel to Rosen (1992) which studied renormalization and limit theorems for self-intersections of superprocesses. (Collisions occur at the same time, while self-intersections of the path can occur at different times.) We have tried to adhere to the structure of that paper to allow the reader easy reference. Some arguments needed for the present paper are almost identical to those in Rosen (1992), and in such cases we have simply referred the reader to that paper.

2. Theorem 2: preliminaries

Our proofs involve the calculation of moments, and in this section we derive a formula for moments of the approximate renormalized collision local time. Our starting point is Dynkin’s formula (1988a)

$$E_\mu \left( \prod_{i=1}^n (f_i, X_t) \right)$$

$$= \sum_{D_n} \int_{V_n} \mu(dy) \prod_{a \in A} p_{s(a)-s(a)}(y_{f(a)} - y_{g(a)}) \prod_{i \in V_i} ds_i dy_i \prod_{i=1}^n f_i(z_i) dz_i.$$  (2.1)
In Eq. (2.1),
\[ p_s(x) = e^{-s} e^{-x^2/2s} \frac{1}{(2\pi s)^{d/2}} \]  
(2.2)
is the transition density for exponentially killed Brownian motion in \( \mathbb{R}^d \), where by convention \( p_s(x) = 0 \) if \( s < 0 \). \( D_n \) is the set of directed binary graphs with \( n \) exits marked \( 1, 2, \ldots, n \). Given such a graph, \( A \) is the set of arrows, and if the arrow \( a \in A \) goes from the vertex \( v \) to \( w \), we write \( v = i(a), w = f(a) \). To each vertex \( v \) we associate two variables \((s_v, y_v)\) which we refer to as the time and space coordinates of \( v \). \( V^- \) denotes the set of entrances for our graph, and if \( v \in V^- \), we set \( s_v = 0 \). If \( v \) is the exit labelled by \( j \), \( i \neq j \leq n \), we set \((s_v, y_v) = (t_j, z_j)\).

Finally, \( V_0 \) denotes the set of internal vertices, i.e. those vertices which are neither entrances nor exits.

Let
\[ G(x) = \int_0^\infty p_s(x) \, ds \]  
(2.3)
denote the Green’s function for exponentially killed Brownian motion in \( \mathbb{R}^d \). From Eq. (2.1) we see that
\[ E_\mu \left( \int_0^\infty \langle f_i(X_t), X_t \rangle \, dt \right) \]
\[ = \sum_{D_n} \int \prod_{v \in V^-} \mu(dy_v) \prod_{a \in A} G(y_{f(a)} - y_{i(a)}) \prod_{v \in V^n} dy_v \prod_{i=1}^n f_i(z_i) \, dz_i. \]  
(2.4)
From Eq. (2.1) it follows that
\[ E_\mu \left( \left[ \int_0^\infty \langle f_c(x - y), X_t(dz)X_t(dy) \rangle \, ds \right]^n \right) \]
\[ = \sum_{D_n} \int \prod_{v \in V^-} \mu(dy_v) \prod_{a \in A} p_{t_{f(a)} - t_{x(a)}}(y_{f(a)} - y_{i(a)}) \prod_{v \in V^n} dx_v dy_v \]
\[ \times \prod_{i=1}^n f_i(z_{2i} - z_{2i-1}) \, dz_{2i} \, dz_{2i}, \]  
(2.5)
where now the times \( t_{2i-1}, t_{2i} \) associated with the exits labeled \( 2i - 1, 2i \) are both replaced by \( r_s \), and we integrate \( r_s \) over \([0, \infty)\).

We will say that the pair of exits \( v, w \) are coupled if for some \( k \) we have
\[ z_{2k} = y_v, \quad z_{2k-1} = y_w. \]
or
\[ z_{2k} = y_w, \quad z_{2k-1} = y_v. \] (2.6)

We will say that a pair of exits \( v, w \) are a twin if they have the same immediate predecessor, i.e., if we can find \( a, b \in A \) and a vertex \( u \) such that
\[ i(a) = i(b) = u \] (2.7)

and
\[ f(a) = v, \quad f(b) = w. \]

If a twin \( v, w \) are coupled, and e.g. \( z_{2k} = y_v, \quad z_{2k-1} = y_w \) and Eq. (2.7) holds, then we get a factor in Eq. (2.5) of the form
\[
\int_0^\infty \int_0^\infty p_{t-s}(y_v - y_w) p_{t-s}(y_v - y_w) f_v(y_v - y_w) dy_v dy_w dt
\]
\[ = \int_0^\infty \int_0^\infty p_{t-s}(y_v) p_{t-s}(y_w) f_v(y_v - y_w) dy_v dy_w dt
\]
\[ = \int_0^\infty \int p_{2t}(y) f_v(y) dy dt
\]
\[ = \frac{1}{2} \int f_v(y) G(y) dy. \] (2.8)

Set
\[ c(\varepsilon) = \int f_v(y) G(y) dy. \] (2.9)

Then it is easy to check that
\[
E_p \left( \left[ \int_0^\infty \langle f_v(x - y), X_s(dx) X_s(dy) \rangle ds - c(\varepsilon) \int_0^\infty (1, X_s) ds \right]^n \right)
\]
\[ = \sum_{C_{2n}} \int \prod_{v \in V^+} \mu(dy_v) \prod_{a \in A} p_{t_{i(a)} - t_{i(a)}}(y_{f(a)} - y_{i(a)}) \prod_{v \in V^-} dx_v dy_v
\]
\[ \times \prod_{i=1}^n f_v(z_{2i} - z_{2i-1}) dz_{2i} dz_{2i} dr_i, \] (2.10)

where \( C_{2n} \) is the set of binary graphs with \( 2n \) labeled exits; \( 1, 2, \ldots, 2n \), such that no twin exits are coupled, i.e. no twin exits are labeled \( 2i - 1, 2i \) for any \( i \).

Thus, the effect of the subtraction term in Eq. (2.10) is to eliminate all coupled twins. The factor 2 comes from the two possibilities in Eq. (2.6).

We now calculate the asymptotics of \( c(\varepsilon) \). We first note that
\[
G * G(y) = \int \int_0^\infty p_s(y - x) p_t(x) ds dt
\]
\[ = \int_0^\infty \int_0^\infty p_{s+t}(y) ds dt
\]
\begin{align*}
&= \int_0^\infty \int_0^t p_t(y) \, ds \, dt \\
&= \int_0^\infty t p_t(y) \, dt \\
&= \frac{1}{2\pi} \int_0^\infty e^{-t} \frac{e^{-|y|^2/2r}}{(2\pi t)^{d/2}} \\
&= \frac{1}{2\pi} g(y)
\end{align*}

(2.11)

where \( g(y), y \in \mathbb{R}^d \), with obvious notation, corresponds to the Green’s function for killed Brownian motion in \( d - 2 \) dimensions.

If \( d = 2 \), it is known that for \( |y| \leq \frac{1}{2} \),

\[
G(y) = \frac{1}{\pi} \left[ \log \left( \frac{1}{|y|} \right) + \log(\sqrt{2}) - \kappa \right] + O(|y|),
\]

(2.12)

where \( \kappa \) is Euler’s constant. Hence

\[
c(\varepsilon) = \int f_\varepsilon(y) G(y) \, dy
\]

\[
= \frac{1}{\pi} \int f_\varepsilon(y) \left( \log \left( \frac{1}{|y|} \right) \, dy + \log(\sqrt{2}) - \kappa \right) + O(|y|) \, dy
\]

\[
= \frac{1}{\pi} \left( \log \left( \frac{1}{\varepsilon} \right) + \int f(y) \log \left( \frac{1}{|y|} \right) \, dy + \log(\sqrt{2}) - \kappa \right) + O(\varepsilon).
\]

(2.13)

If \( d = 3 \), it is known that

\[
G(y) = \frac{1}{2\pi} \frac{e^{-|y|}}{|y|}.
\]

(2.14)

Hence

\[
c(\varepsilon) = \frac{1}{2\pi} \int f_\varepsilon(y) \frac{e^{-|y|}}{|y|} \, dy
\]

\[
= \frac{1}{2\pi} \int \frac{e^{-|y|}}{|y|} f(y) \, dy
\]

\[
= \frac{1}{2\pi} \int f(y) \frac{dy}{|y|} - \frac{1}{2\pi} + O(\varepsilon).
\]

(2.15)

Finally, for \( d = 4 \), let us analyze \( G(x) \) using \( G_0(x) = (1/2\pi^2)1/|x|^2 \) the zero-potential for Brownian motion in \( \mathbb{R}^4 \). Iterating the resolvent equation we find

\[
G_0(x) - G(x) = G \ast G_0(x)
\]

\[
= G \ast G(x) + G \ast G \ast G_0(x).
\]

(2.16)
By Eqs. (2.11) and (2.12), we know that

\[ G \ast G(x) = \frac{1}{2\pi^2} \lg \left( \frac{1}{|x|} \right) + O(1), \quad |x| < 1 \]  

(2.17)

and it is easy to see that

\[ G \ast G \ast G_0(x) = O(1) \]

so that for \(|x| < \frac{1}{2}\)

\[ G(x) = G_0(x) - \frac{1}{2\pi^2} \lg \left( \frac{1}{|x|} \right) + O(1) \]

\[ = \frac{1}{2\pi^2} \frac{1}{|x|^2} - \frac{1}{2\pi^2} \lg \left( \frac{1}{|x|} \right) + O(1). \]  

(2.18)

Hence

\[ c(\varepsilon) = \int f_0(y)G(y)\,dy \]

\[ = \frac{1}{2\pi^2} \frac{1}{\varepsilon^2} \int \frac{f(y)}{y^2}\,dy - \frac{1}{2\pi^2} \lg \left( \frac{1}{\varepsilon} \right) + O(1), \quad d = 4. \]  

(2.19)

We also note for future reference that, as in Eq. (2.1),

\[ G \ast G \ast G(y) = \int_0^\infty \int_0^\infty \int_0^\infty p_{r+s+t}(y)\,dr\,ds\,dt \]

\[ = \int_0^\infty \int_0^\infty \int_0^\infty p_r(y)\,dr\,ds\,dt \]

\[ = \int_0^\infty \frac{t^2}{2} \, dt. \]  

(2.20)

3. Theorem 2: the second moment

In this section, we compute the asymptotics of

\[ I(\varepsilon) = E_\mu \left[ \left( \int_0^\infty \langle f_\mu(x - y), \, X_\mu(dx)X_\mu(dy) \rangle \, ds - c(\varepsilon) \int_0^\infty \langle 1, X_\mu \rangle \, ds \right)^2 \right]. \]  

(3.1)

By Eq. (2.10) we obtain a contribution from each binary graph with four exits, such that no twin exits are coupled.

We first sketch the possible graphs and write down their contribution. Later we will work out the combinatoric factors. We sometimes use the abbreviation \(dx \ldots\) to indicate
integration over all variables.

\[
\int \mu(du) p_r(z - u) p_{s-r}(x - z) p_{t'-r}(y - z) p_{t'-s}(x - z_1) p_{t'-s}(y - z_2) p_{t'-s'}(y - z_3) f_s(x) f_s(y) d\mu(u) p_{s-r}(x) p_{t'-r}(y) p_{t-r'-s} * f_s(x) p_{t'-s'-t'-s} d\mu(u) * f_s(x - y) dx \ldots
\]

\[
= |\mu| \int p_{s-r}(x + y) p_{s-r}(y) p_{t-r'-s} * f_s(x) p_{t'-s'-t'-s} * f_s(x) dx \ldots
\]

\[
= |\mu| \int p_{s-r+s'-r}(x) p_{t'-r'-s'-s} * f_s(x) p_{t'-s'-s'-t'-s} * f_s(x) dx \ldots
\]

\[
= 2 |\mu| \int p_{2s+(s'-s)}(x) p_{2(t-s)+(s'-s)} * f_s(x) p_{2(t'-s)+(s'-s)} * f_s(x) dx \ldots
\]

\[
= \frac{1}{4} |\mu| \int G * p_s(x)(G * p_t * f_s(x))^2 dx dv,
\]

(3.2)
\[
\int \mu(du)p_s(z - u)p_{t - r}(z_1 - z)p_{t' - r'}(y - z)p_{t - s}(x - z_2)p_{t' - s}(x - z_3) \\
p_{t' - s'}(y - z_4)f_z(z_1 - z_2)f_z(z_3 - z_4)\ dx\ dy\ dz\ dz'\ dr\ ds\ ds'\ dt\ dt' \\
= |\mu| \int p_s(y)p_{t - r}(x - y)p_{2t'(t' - s - s')}f_y(x - y)p_{t - s' + t}f_y(x)\ dx\ dy \\
= |\mu| \int p_s(y)p_{t - r'}(x)p_{2t'(t' - s - s')}f_y(x)p_{t - s + t'}f_y(x)\ dx\ dy \\
= |\mu| \int p_s(x)p_{2t'(t' - s - s')}f_y(x)p_{t - s + t'}f_y(x)\ dx \\
= |\mu| \int p_s(x)p_{2t'(t' - s - s')}f_y(x)p_{2t'(t' + t' - s') + t'}f_y(x)\ dx \\
= \frac{1}{8} |\mu| \int p_s(x)G * p_y * f_y(x)G * p_y * f_y(x)\ dx\ dy, \quad (3.3)
\]

\[
\int \hat{u}(du)\mu(du)\mu(du)p_s(u - x)p_{t - r}(v - y)p_{t - s}(x - z_1)p_{t - s}(x - z_3)p_{t - s}(y - z_2) \\
\times p_{t' - s'}(y - z_4)f_z(z_1 - z_2)f_z(z_3 - z_4)\ dx\ dy\ dz\ dz'\ dr\ ds\ ds'\ dt\ dt' \\
= \int \hat{u}(du)\mu(du)p_s(u - x)p_{t - r}(v - y)p_{t - s}(x - z_1)p_{t - s}(x - z_3)p_{t - s}(y - z_2) \\
\times p_{t' - s'}(y - z_4)f_z(z_1 - z_2)f_z(z_3 - z_4)\ dx\ dy\ dz\ dz'\ dr\ ds\ ds'\ dt\ dt' \\
= \frac{1}{4} \int \hat{u}(du)\mu(du)G * p_r(x - (u - v))(G * p_r * f_s(x))^2\ dx\ dr, \quad (3.4)
\]
\[ \int \mu(du)\mu(dv)p_t(u - z_1)p_t(v - x)p_{t' - s}(x - z_3)p_{t' - s'}(y - z_2) \]
\[ \times p_{t' - s'}(y - z_4)f_s(z_1 - z_2)f_s(z_3 - z_4)\,dx\,dy\,dz\,ds\,ds'\,dr\,dr' \]
\[ = \int \mu(du)\mu(dv)p_t(v - x)p_{t' - s + t} * f_s(u - y)p_{t' - s}(x - y)p_{t' - s + t'} * f_s(x - y)\,dx \ldots \]
\[ = \frac{1}{8} \int \mu(du)\mu(dv)G * p_r * f_s(x - (u - v))p_s(x)G * p_r * f_s(x)\,dx\,dr, \]

(3.5)

Graph 5

\[ \int \mu(du)\mu(dv)\mu(dw)p_t(x - w)p_{t' - s + t} * f_s(x - u)p_{t' - s + t'} * f_s(x - v)\,dx\,ds\,dr\,dr' \]
\[ = \frac{1}{4} \int \mu(du)\mu(dv)\mu(dw)p_s(x - w)G * p_r * f_s(x - u)G * p_r * f_s(x - v)\,dx\,ds, \]

(3.6)

Graph 6

\[ \frac{1}{4} \int \mu(du)\mu(dv)\mu(dw)\mu(dz)G * f_s(u - v)G * f_s(w - z). \]

(3.7)

Let \( u(x) \) denote a generic measurable function which falls off exponentially and monotonically in \(|x|\), and such that \(|x| \to \infty\), and

\[ |u(x)| \leq c\frac{1}{|x|^2}, \]

and let \( u_{x,\varepsilon}(x) \) denote a generic measurable function which falls off exponentially and monotonically in \(|x|\), and such that

\[ u_{x,\varepsilon}(x) \leq c|x|^{-\varepsilon}, \quad |x| \geq \varepsilon, \]
\[ u_{x,\varepsilon}(x) \leq cz^{-\varepsilon}, \quad |x| \leq \varepsilon. \]

With \( u_{0,\varepsilon} \) we associate \( \log(1/|x|) \) instead of \(|x|^{-\varepsilon} \).
We will use the following simple lemma from Rosen (1992):

**Lemma 1.** If \( x < d \) then \( u_x \cdot f_x(x) \) has exponential falloff as \( |x| \to \infty \), and

\[
|u_x \cdot f_x(x)| \leq \begin{cases} 
\bar{c} \cdot \frac{1}{|x|^p}, & |x| \geq \varepsilon, \\
\bar{c} \cdot \frac{1}{\varepsilon^p}, & |x| \leq \varepsilon,
\end{cases}
\]

i.e. \( u_x \cdot f_x(x) = u_{x,\varepsilon}(x) \).

The functions \( G, G \ast G \) are of the above form as we saw in Section 2. They all have exponential falloff as \( |x| \to \infty \), while for small \( x \) we have the bounds:

\[
G(x) \quad \frac{d = 2}{c \lg(\frac{1}{|x|})} \quad \frac{d = 3}{cx^{-1}} \quad \frac{d = 4}{cx^{-2}}
\]

\[
G \ast G(x) \quad c \quad c \quad c \lg \left( \frac{1}{|x|} \right)
\]

Using Eq. (3.9) and Lemma 4, it is easy to check that all the integrals in formulas (3.2)–(3.7) are uniformly bounded as \( \varepsilon \to 0 \) when \( d = 2 \) or 3.

We thus concentrate on \( d = 4 \). The integrals for graphs 3, 5 and 6 are uniformly bounded as \( \varepsilon \to 0 \), while the above shows that the integral for graph 4 is \( O(\lg(1/\varepsilon)) \). To obtain a similar bound on the integral for Graph 1 we first note that

\[
\int \prod_{i=1}^3 p_x(y_i) \, dv \leq \prod_{i=1}^3 \left( \int p_{x,\varepsilon}^2(y_i) \, dv \right)^{1/3} = \prod_{i=1}^3 u_{10/3}(y_i)
\]

and that \( u_{10/3} \ast u_2(y) = u_{4/3}(y) \) hence

\[
\frac{1}{4} |\mu| \int G \ast p_x(x)(G \ast p_x \ast f_x(x))^2 \, dx \, dv
\]

\[
\leq c \int u_{4/3}(x) u_{4/3,\varepsilon}(x) \, dx
\]

\[
\leq c \lg \left( \frac{1}{\varepsilon} \right).
\]

We now carefully compute the integral (3.3) corresponding to Graph 2. We will show that it is \( \sim c(\lg(1/\varepsilon))^2 \).

Using Eqs. (2.18) and (2.17) we first obtain an upper bound:

\[
J(\varepsilon) = \frac{1}{8} \int p_x(x) G \ast p_x \ast f_x(x) G \ast p_x \ast f_x(x) \, dx \, dv
\]

\[
\leq \frac{1}{8} \int G(x) G \ast f_x(x) G \ast f_x(x) \, dx
\]

\[
= \frac{1}{8} \int_{|x| \leq 1/2} G(x) G \ast f_x(x) G \ast f_x(x) \, dx + 0(1)
\]

\[
= \frac{1}{8} \int_{|x| \leq 1/2} \frac{1}{2\pi^2} x^{-2} \frac{1}{2\pi^2} (x^{-2} \ast f_x)(x)
\]

\[
\sim c(\lg(1/\varepsilon))^2.
\]
\[
\frac{1}{2\pi^2} \left( \log \left( \frac{1}{|x|} \right) * f_\varepsilon \right)(x) + 0 \left( \log \left( \frac{1}{\varepsilon} \right) \right) \\
= \frac{1}{64\pi^6} \int_{|2\varepsilon| \leq |x| \leq 1/2} x^{-2}(x^{-2} * f_\varepsilon)(x) \\
\left( \log \left( \frac{1}{|x|} \right) * f_\varepsilon \right)(x) + 0 \left( \log \left( \frac{1}{\varepsilon} \right) \right) \\
= \frac{1}{64\pi^6} \int_{|x| \leq 1/2} \int \frac{1}{x^2} \frac{1}{|x-z|^2} \log \left( \frac{1}{|x|} \right) f(y)f(z) \, dx \, dy \, dz \\
+ 0 \left( \log \left( \frac{1}{\varepsilon} \right) \right) \\
= \frac{1}{64\pi^6} \int_{|x| \leq 1/2} \frac{1}{x^2} \left( \log \left( \frac{1}{|x|} \right) + \log \left( \frac{1}{|x|} \right) \right) \, dx + O \left( \log \left( \frac{1}{\varepsilon} \right) \right) \\
= \frac{1}{64\pi^6} 2\pi^2 \int_2^{1/2c} \frac{1}{r} \left( \log \left( \frac{1}{\varepsilon} \right) - \log(r) \right) \, dr + O \left( \log \left( \frac{1}{\varepsilon} \right) \right) \\
= \frac{1}{64\pi^4} \log^2 \left( \frac{1}{\varepsilon} \right) + O \left( \log \left( \frac{1}{\varepsilon} \right) \right) . \tag{3.11}
\]

We now obtain a lower bound whose leading term is the same as that obtained in the upper bound.

\[
J(\varepsilon) = \frac{1}{8} \int \int_0^\infty p_\varepsilon(x)G * p_\varepsilon * f_\varepsilon(x)G * G * p_\varepsilon * f_\varepsilon(x) \, dx \, dx \\
\geq \frac{1}{8} \int \int_0^{c^2} p_\varepsilon(x)G * p_\varepsilon * f_\varepsilon(x)G * G * p_\varepsilon * f_\varepsilon(x) \, dx \, dx \\
\geq \frac{1}{8} \int G * p_{c^2} * f_\varepsilon(x)G * G * p_{c^2} * f_\varepsilon(x) \left( \int_0^{c^2} p_\varepsilon(x) \, dx \right) \, dx \\
= \frac{1}{8} \int_{|x| \leq 1/2} G * p_{c^2} * f_\varepsilon(x)G * G * p_{c^2} * f_\varepsilon(x) \left( \int_0^{c^2} p_\varepsilon(x) \, dx \right) \, dx - O(1) \\
= \frac{1}{8} \int_{2c \leq |x| \leq 1/2} \frac{1}{2\pi^2|x|^2} * p_{c^2} * f_\varepsilon(x) \frac{1}{2\pi^2} \log \left( \frac{1}{|x|} \right) \\
* p_{c^2} * f_\varepsilon(x) \left( \int_0^{c^2} p_\varepsilon(x) \, dx \right) \, dx - O \left( \log \left( \frac{1}{\varepsilon} \right) \right) \\
= \frac{1}{32\pi^4} \int_{2c \leq |x| \leq 1/2} |x|^{-2} * p_1 * f(x) \log \left( \frac{1}{|x|} \right) \\
* p_1 * f(x) \left( c^2 \int_0^{c^2} p_\varepsilon(cx) \, dx \right) \, dx - O \left( \log \left( \frac{1}{\varepsilon} \right) \right) 
\]
\[
\int_2^{2e} \frac{\ln \left( \frac{1}{e} \right)}{x^2} \, dx - O \left( \ln \left( \frac{1}{e} \right) \right) = \frac{1}{64\pi^4} \ln^2 \left( \frac{1}{e} \right) - O \left( \ln \left( \frac{1}{e} \right) \right)
\]

as in the calculation of Eq. (3.11). Here we have used

\[
\int_0^{e^2} p_{\varepsilon}(\varepsilon x) \, \varepsilon^d \, dx = \int_0^1 e^{-\varepsilon^2/2t} \, dt = \int_0^1 \frac{e^{-\varepsilon^2/2t}}{t^2} \, dt - \int_0^1 \frac{(1 - e^{-\varepsilon^2})}{t^2} \, dt = G_0(x) - \int_1^\infty \frac{e^{-\varepsilon^2/2t}}{t^2} \, dt - O \left( e^2 \int_0^1 \frac{e^{-\varepsilon^2/2t}}{t^2} \, dt \right)
\]

for \( 2 \leq |x| \leq 1/2e \).

It can be shown as in Rosen (1992) that the number of graphs in \( C_4 \) which give rise to contributions (3.3) is precisely 43.

Thus

\[
I(\varepsilon) = 4^3 |\mu| I(\varepsilon) + O \left( \ln \left( \frac{1}{e} \right) \right) = \frac{1}{\pi^4} |\mu| \ln^2 \left( \frac{1}{e} \right) + O \left( \ln \left( \frac{1}{e} \right) \right),
\]

(3.13)

4. Proof of Theorem 2(a) and (b)

Let

\[
\gamma_{\varepsilon} = \int_0^\infty \langle f_{\varepsilon}(x-y), X_s(dx)X_s(dy) \rangle \, ds - c(\varepsilon) \int_0^\infty \langle 1, X_s \rangle \, ds,
\]

where

\[
c(\varepsilon) = \begin{cases} 
\frac{1}{\pi} \ln \left( \frac{1}{\varepsilon} \right), & d = 2, \\
\frac{1}{2\pi} \frac{1}{\varepsilon} \int \frac{f(y)}{|y|} \, dy, & d = 3.
\end{cases}
\]

Almost precisely as in Rosen (1992) we can show that

\[
E_\mu [(\gamma_{\varepsilon} - \gamma_{\varepsilon})^2] \to 0 \quad \text{as} \; \varepsilon, \bar{\varepsilon} \to 0
\]

(4.2)

and this completes the proof of Theorem 2(a) and (b).

5. Proof of Theorem 2(c): Combinatorial aspect

Our proof is by the method of moments.
Recall that
\[ \gamma_z = \int_0^\infty \langle f_z(x - y), X_x(dx)X_y(dy) \rangle \, ds - c(\varepsilon) \int_0^\infty \langle 1, X_x \rangle \, ds, \]  
where
\[ c(\varepsilon) = \int f_z(x)G(x) \, dx = \frac{1}{2\pi^2 \varepsilon^2} \int \frac{f(y)}{y^2} \, dy - \frac{1}{2\pi^2} \log \left( \frac{1}{\varepsilon} \right) + o(1). \]

By Eq. (2.10) we know that
\[ E_\mu(\gamma_x^{2m}) \]  
is a sum of contributions from the graphs of \( C_{4m} \), i.e. the set of binary graphs with \( 4m \) labeled exits, 1, 2, ..., \( 4m \) with no twin exits coupled — i.e. no twin exits are ever labeled \( 2i - 1, 2i \) for any \( i \).

The basic idea which we explain in this and the next section is that the dominant contribution to Eq. (5.3) comes from graphs which effectively break Eq. (5.3) up into a product of \( m \) second moments.

Let \( A_{4m} \subset C_{4m} \) denote those binary graphs in \( C_{4m} \) for which there is a complete pairing \((i_1, j_1), \ldots, (i_m, j_m)\) of the \( 2m \) integers 1, 2, ..., \( 2m \) and such that for each such pair \((i, j)\) the exits labeled \( 2i - 1, 2i \), \( 2j - 1, 2j \) are arranged as in Graph 2 of Section 3:

![Graph 2](image)

or one of its \( 4^3 \) variants as described at the end of Section 3.

We will see later that the dominant contribution to Eq. (5.3) comes from the graphs in \( A_{4m} \), and is of order \( \lg^{2m}(1/\varepsilon) \), while any other graph in \( C_{4m} \) will give a contribution which is \( O(\lg^{2m-1}(1/\varepsilon)) \).

Let us compute the contribution from the graphs in \( A_{4m} \). Consider the subgraph (5.4). The partial integral with respect to \( dx \, dy \, dz_{2i_{j-1}} \, dz_{2i_j} \) is described in Eq. (3.3). It is crucial that this partial integral is independent of \( x \) and \( r \) (a consequence of the translation invariance of Brownian motion), and is simply the constant (see Eq. (3.11))
\[ J(\varepsilon) = \frac{1}{4\pi} \log \left( \frac{1}{\varepsilon} \right) + O \left( \log \left( \frac{1}{\varepsilon} \right) \right). \]
As we saw at the end of Section 3, there are 43 variants of Eq. (5.4). Thus the partial integration corresponding to all \( m \) pairs \((i';j')\) and all the 43 variants for each pair gives rise to the factor
\[
\left( \frac{1}{\pi^2} \lg^2 \left( \frac{1}{\varepsilon} \right) \right)^m + O \left( \lg^{2m-1} \left( \frac{1}{\varepsilon} \right) \right). \tag{5.6}
\]

After this partial integration, we are simply left with a binary graph with \( m \) exits. Since any graph in \( D_m \) can arise in this fashion, and since there are \((2m)!/m!2^m\) ways to pair the integers \( 1, 2, \ldots, 2m \) we see that (see Eq. (2.4)) the contribution to Eq. (5.3) from \( A_{4m} \) is
\[
\frac{(2m)!}{m!2^m} \left( \frac{1}{\pi^2} \lg^2 \left( \frac{1}{\varepsilon} \right) \right)^m E_{\mu} \left( \int_0^{\infty} (1, X_s) \ ds \right)^m + O \left( \lg^{2m-1} \left( \frac{1}{\varepsilon} \right) \right). \tag{5.7}
\]

We will show in the next section that the contribution of all graphs in \( C_{4m} - A_{4m} \) is \( O(\lg^{2m-1}(1/\varepsilon)) \). This will give
\[
E_{\mu} \left( \frac{\gamma_{\varepsilon}}{\lg(1/\varepsilon)} \right)^{2m} \longrightarrow \frac{(2m)!}{m!} \left( \frac{1}{2\pi^2} \right)^m E_{\mu} \left( \int_0^{\infty} (1, X_s) \ ds \right)^m \quad \text{as } \varepsilon \to 0. \tag{5.8}
\]

Furthermore, the next section will show that
\[
E_{\mu} \left( \frac{\gamma_{\varepsilon}}{\lg(1/\varepsilon)} \right)^{2m-1} \longrightarrow \quad \text{as } \varepsilon \to 0. \tag{5.9}
\]

Let \( M_{2m} \) denote the right-hand side of Eq. (5.8). A simple combinatoric argument spelled out in Rosen (1992) shows that for \( |\lambda| \) small,
\[
\sum_{m=0}^{\infty} \frac{2^m M_{2m}}{(2m)!} = e^{\mu [(1/2)(1 - \sqrt{1 - 2\lambda^2/\pi^2})].} \tag{5.10}
\]

This shows at once that any limit distribution of \( \gamma_{\varepsilon}/\lg(1/\varepsilon) \) is determined by its moments, hence unique, and also shows that its Laplace transform is given by Eq. (5.10), which establishes Theorem 2(c).

6. Proof of Theorem 2(c): analytic aspect

We recall from Eq. (2.10) that
\[
E_{\mu}(\gamma_{\varepsilon}^n) = \sum_{C_{4m}} \int_{\mathbb{R}} \prod_{v \in V_\mu} \mu(dy_v) \prod_{a \in A} p_{x_{f(a)} - x_{y(a)}}(y_{f(a)} - y_{i(a)}) \prod_{v \in R_0} dy_v dy_v
\]
\[
\times \prod_{j=1}^{n} f_j(z_{2j} - z_{2j-1}) \ dz_{2j} \ dz_{2j} \ dr_j. \tag{6.1}
\]

In this section, we show that unless \( n = 2m \) and the graph \( C \) is in \( A_{4m} \), then the contribution of \( C \) to Eq. (6.1) is
\[
0 \left( \lg^{n-1} \left( \frac{1}{\varepsilon} \right) \right). \tag{6.2}
\]

As discussed in Section 5, this will complete the proof of Theorem 2(c).
We can think of the integral in Eq. (6.1) as obtained by assigning a factor \( p_{y'f(a) - y(a)} \) to each arrow \( a \in A \). We must integrate out all internal variables \( dy_v, v \in V_o \), all entrances with respect to \( d\mu \), all exits with \( \prod_{i=1}^{n} f_i(z_{2i} - z_{2i-1}) \) as well as all time variables.

Our approach to Eq. (6.2) is to successively integrate out the variables, at each stage replacing the graph \( C \) by a different graph \( C' \) (not necessarily a directed or binary graph).

The arrows of \( C' \) are associated with factors described below, such that the contribution of \( C \) is bounded by that of \( C' \). In this process we will be able to associate a factor \( 0(\lg(1/\epsilon)) \) to each \( f \) in Eq. (6.1) in such a way that these factors will bound all divergences as \( \epsilon \to 0 \), and we will show that unless \( n = 2m \) and \( C \subseteq A_{4m} \), at least one of the factors associated to some \( f \) will be \( 0(1) \).

Here are the details:

We begin by integrating the exit variables \( z_1, \ldots, z_{2n} \). We obtain \( n \) factors of the form

\[
\int p_{a - z_{2i-1}}(a - z_{2i-1}) f_i(z_{2i} - z_{2i-1}) p_{b - z_{2i}}(b - z_{2i}) \, dz_{2i-1} \, dz_{2i} \leq cG \ast f_c(b - a). \tag{6.3}
\]

We know from the fact that \( C \subseteq C_{2m} \), that \( a \neq b \). Form a new graph \( C' \) obtained by putting an edge between \( i(u) \) and \( i(v) \) whenever \( f(u) = z_{2i-1}, f(v) = z_{2i} \), i.e. we connect the vertices associated with \( a, b \) in Eq. (6.3). With this new edge, called a ‘leading edge’, we associate the factor \( G \ast f_c \).

Assume that \( C' \) has a subgraph of the form

\[
\begin{align*}
\text{c} & \quad \text{x} \\
& \quad \text{a} \\
& \quad \text{b}
\end{align*}
\tag{6.4}
\]

where \( (x, a), (x, b) \) are both leading edges. We distinguish three possibilities:

1. \( a = c, \) or \( b = c \) (We cannot have both.)
2. \( a \equiv b \)
3. \( a, b, c \) are distinct.

We analyze each in turn:

(i) Assume that \( b \equiv c \). This can only have occurred if \( C \) contained the subgraph

\[
\begin{align*}
d & \quad c \\
& \quad \text{x} \\
& \quad \text{z}_{2i} \\
& \quad \text{z}_{2i-1}
\end{align*}
\tag{6.5}
\]
Since we think of $z_{2i}, z_{2i-1}$ as connected by $f_c$, we refer to the situation in Eq. (6.5) as a simple loop.

The partial integral over $x$ in this case is bounded by

$$
\int G(c - x)G * f_c(c - x)G * f_c(a - x) \, dx = \int G(x)G * f_c(x)G * f_c(a - c - x) \, dx.
$$

(6.6)

We know from Lemma 1 that

$$G * f_c \leq u_{2,c}.$$

If $|x| > \frac{1}{2}|a - c|$, Eq. (6.6) is bounded by

$$u_{2,c}(a - c) \int G(x)G * f_c(a - c - x) \, dx = u_{2,c}(a - c)u_0,c(a - c)
$$

(6.7)

as we see from Eq. (3.9). While if $|x| \leq \frac{1}{2}|a - c|$, so that $|a - c - x| > (|a - c|)/2$, Eq. (6.6) is bounded by

$$u_{2,c}(a - c) \int G(x)G * f_c(x) \, dx = \lg \left( \frac{1}{e} \right) u_{2,c}(a - c).
$$

(6.8)

In any event, Eq. (6.6) is bounded by $\lg(1/e)u_{2,c}(a - c)$. (It is important to recall that we cannot have $a \equiv c$.) We then form a new graph $C''$, with an edge between the vertices associated with $a$ and $c$. We consider the factor $\lg(1/e)$ as associated with $f_c(z_{2i} - z_{2i-1})$, and associate $u_{2,c}$ to our new edge, now called a leading edge.

Because Eq. (6.5) refers to a binary graph, in $C''$, aside from our new edge connecting $a$ and $c$, there is only one other arrow connecting $c$, with a factor $G(c - d)$. We now integrate

$$
\int G(c - d)u_{2,c}(a - c) \, dc = u_0,c(d - a).
$$

(6.9)

(This integral was already computed in Eq. (6.7).)

If $a \equiv d$, we are in the situation of Section 3, Graph 2, i.e. our subgraph (6.5) was precisely of the form making up $A_{4m}$, contributing $\log^2(1/e)$ which we associate with the two $f_c$ factors for that subgraph, which now have no further influence.

If $a \not\equiv d$, we form a new graph $C''$ linking $a$ and $d$, and with the factor $u_{0,c}(d - a)$. We have a subgraph

$$
\begin{array}{c}
\bullet \\
\bullet \\
\bullet \\
\bullet \\
\bullet \\
\bullet \\
\bullet \\
\bullet \\
\bullet \\
\bullet
\end{array}
\begin{array}{c}
e \\
d \\
a \\
h
\end{array}
$$

(6.10)

which looks like Eq. (6.4), except that instead of the factor $u_{2,c}$ associated to $(d,a)$, we have $u_{0,c}$. 
We will see after analyzing cases (ii) and (iii) that the worst possible case comes from the two loop subgraph

\[ (6.11) \]

i.e. \( h \equiv e \) (so that \( a \neq e \)) and the partial integral over \( d \) is bounded by

\[ \int G(e - d)u_{2,\lambda}(e - d)u_{0,\lambda}(d - a)d(d) = \int G(x)u_{2,\lambda}(x)u_{0,\lambda}(a - e - x) \, dx \quad (6.12) \]

with \( a \neq e \).

As in the analysis of Eq. (6.6) we find Eq. (6.12) bounded by

\[ \lg \left( \frac{1}{e^2} \right) u_{0,\lambda}(a - e) + u_{2,\lambda}(a - e) \quad (6.13) \]

so that the \( de \) integral is

\[ O\left( \lg \left( \frac{1}{e^2} \right) \right). \quad (6.14) \]

Thus, three factors of \( f_e \) give rise only to a \( \lg^2(1/e) \) contribution (as opposed to \( \lg^3(1/e) \)).

In particular, a subgraph of the form

with \( i \geq 2 \) loops, gives a contribution which is \( O(\lg^{i-1}(1/e)) \), unless \( i = 2 \).

(ii) This case arises from the subgraph
The partial integral with respect to \(z_1, \ldots, z_4, x, y\) is bounded by

\[
\int \int p_t(x-c)p_t-r-r'-s \ast \tilde{f}_t(x-y)p_{t'-r'-r''-s} \ast \tilde{f}_t(y-d) \, dx \, dy \, dr \, ds \, dt \, dt'
\]

\[
= \int p_{r+s}(c-d-x)p_{t-r-r'-s} \ast \tilde{f}_t(x)p_{t'-r'-r''-s} \ast \tilde{f}_t(x) \, dx \, dr \, ds \, dt \, dt'
\]

\[
= \frac{2}{8} \int G \ast p_t(c-d-x)(G \ast p_t \ast f_t(x))^2 \, dx \, dv'
\]

\[
\leq u_{0, \varepsilon}(c-d)
\]

(6.15)

as in Eq. (6.7) after using Holder’s inequality in the \(dv\) integral as in Eq. (3.10).

If \(c \equiv d\) (which is the situation of Section 3, Graph 1), we have a \(\lg(1/\varepsilon)\) for two factors of \(f_\varepsilon\), while if \(c \not= d\) we can also bound

\[
u_0, \varepsilon(c-d) \leq \lg \left( \frac{1}{\varepsilon} \right) u(c-d),
\]

where \(u(c-d)\) is bounded, and falls off exponentially as \(|c-d| \to \infty\). We have a factor \(\lg(1/\varepsilon)\) for the two \(f_\varepsilon\)’s, and a new graph with an edge connecting the vertices associated with \(c\) and \(d\), and associated factor \(u(c-d)\).

(iii) If \(a, b, c\) are distinct, the partial \(x\) integral is

\[
\int G(c-x)G \ast f_t(x-a)G \ast f_t(x-b) \, dx.
\]

(6.16)

If the variable \(a\) or \(b\) no longer appears in any other factors associated with edges of our graph – we perform the \(da\) or \(db\) integral. If, e.g., we first do the \(da\) integration, then Eq. (6.16) is bounded by

\[
u_0, \varepsilon(c-b) \leq \lg \left( \frac{1}{\varepsilon} \right) u(c-b)
\]

(6.17)

and as in the discussion of (ii), we associate \(\lg(1/\varepsilon)\) with two \(f_\varepsilon\) factors.

If both \(a\) and \(b\) appear in other factors, we use

\[
u \leq \frac{1}{2}(u^2 + v^2)
\]

to bound Eq. (6.16) by

\[
G \ast u_{4, \varepsilon}(a-c) + G \ast u_{4, \varepsilon}(b-c) \leq \lg \left( \frac{1}{\varepsilon} \right) (u_{2, \varepsilon}(a-c) + u_{2, \varepsilon}(b-c)).
\]

(6.18)

We now form two new graphs, one with a new edge connecting \(a\) and \(c\), with a factor \(u_{2, \varepsilon}(a-c)\) – and analogously for the other graph.

It suffices to consider the first graph. Notice that the factor \(u_{2, \varepsilon}(a-c)\) is the type of factor we obtained from the initial integration over exits – hence we can continue our analysis as if it arose in the latter manner – with the difference that we have actually used up two \(f_\varepsilon\) factors at the cost of one \(\lg(1/\varepsilon)\) factor. This could only lead to problems if our new \(u_{2, \varepsilon}(a-c)\) were part of a two-loop graph, and it is easily seen that that is impossible because of the \(dx\) integration in Eq. (6.16).
We can now return to the end of our discussion of case (i), and see that indeed the worst possible case for Eq. (6.10) is as described there – i.e. Eq. (6.11).

By iterating (i)–(iii), applied to leading edges, we see that Eq. (6.2) holds.

7. Proof of Theorem 1

In analogy with Eq. (2.10) we find

\[ E_{\gamma}(\gamma_{\tau}(T)) = \sum_{C_{20}} \int \prod_{r \in V_{2}} \mu(dy_{r}) \prod_{a \in A} p_{s_{x}(a)-s_{y}(a)}(y_{f(a)} - y_{i(a)}) \int_{s_{x}} dy_{r} \]

\[ \times 1_{\{r_{i} \leq \tau \}} \prod_{i=1}^{n} f_{\delta}(z_{2j-1} - z_{2i}) \, dz_{2j-1} \, dz_{2i} \, dr_{i} \quad (7.1) \]

where now

\[ p_{s}(y) = e^{-y^{2}/2t} \]

is the transition density for Brownian motion in \( \mathbb{R}^{d} \).

Note that by inserting factors \( e^{-\left(s_{x}(a)-s_{y}(a)\right)} \geq e^{-T} \), we can bound the contribution to Eq. (7.1) of any graph \( C \), by its contribution to Eq. (2.10). This immediately shows that if \( d = 2, 3 \) and \( n = 2 \), then Eq. (7.1) is uniformly bounded in \( n \), while if \( d = 4 \), we can bound the contribution of each graph to Eq. (7.1) by \( c lg^{n-1}(1/e) \), and in fact, unless \( n = 2m \) and our graph belongs to \( A_{4m} \), then its contribution can be bounded by \( c lg^{n-1}(1/e) \).

The \( L^{2} \) convergence for \( d = 2, 3 \) follows easily by using such a domination together with Eq. (4.2). The case of \( d = 4 \) is more subtle.

We consider in detail the contribution of a subgraph of the type described by Graph 2 of Section 3. This contribution is

\[ J(r, \epsilon) = |\mu| \int A \int p_{s-s^{*}}(x) p_{2(s-s^{*})+(s-s^{*})} \ast f_{\epsilon}(x) p_{2(t-r)+2(s-s^{*})+(t-s^{*})} \]

\[ \ast f_{\epsilon}(x) \, ds' \, ds \, dt' \, dx \]

\[ = |\mu| \int A \int p_{s-s^{*}}(x) p_{2(s-s^{*})} \ast p_{t-r} \ast f_{\epsilon}(x) p_{2(t-r)} \ast p_{2(s-s^{*})} \ast p_{s-s^{*}} \]

\[ \ast f_{\epsilon}(x) \, ds' \, ds \, dt' \, dx \]

where \( A = \{(s', s, t, t') \mid r \leq s' \leq s \leq t, t' \leq T\} \).

Recall from Eqs. (3.11) and (3.12) that

\[ J(\epsilon) = \frac{1}{8} \int p_{\epsilon}(x) G \ast p_{\epsilon} \ast f_{\epsilon}(x) G \ast G \ast p_{\epsilon} \ast f_{\epsilon}(x) \, dx \, dv \]

\[ = \frac{1}{64 \pi^{4}} \log^{2} \left( \frac{1}{\epsilon} \right) + 0 \left( \log \left( \frac{1}{\epsilon} \right) \right). \quad (7.2) \]

We now show that for any fixed \( \delta > 0 \),

\[ J(r, \epsilon) = J(\epsilon) + 0 \left( \log \left( \frac{1}{\epsilon} \right) \right), \quad T - r \geq 3\delta \quad (7.3) \]
Let
\[ q_t = e^{-t} p_t(x), \quad G'(x) = \int_0^t e^{-t'} p_t(x) \, dt. \]

Using
\[ |1 - e^{-t}| \leq 2t \]
we easily check that
\[
J(r, \varepsilon) = |\mu| \int_0^r \int_A q_{s-r'}(x) q_{2s(1-s)} * p_{s-s'} * f_s(x) q_{2(s-r')} * p_{s-r'} * f_s(x) \, ds' \, ds \, dt \, dt' \, dx + 0 \left( \log \left( \frac{1}{\varepsilon} \right) \right).
\]
(7.4)

Note that under our assumption that \( T - r \geq 3\delta \) we have that
\[
A = \{(s', s, t, t') \mid r \leq s' \leq s \leq t, t' \leq T \}
\]
\[
\supset B = \{(s', s, t, t') \mid 0 \leq s' \leq s \leq s' \leq \delta, 0 \leq t - s \leq \delta, 0 \leq t' - s \leq \delta, \}
\]
Using the bound
\[ q_r(x) \leq c e^{-x^2/2r} \]
if
\[ r \geq \delta, \]
we see that in Eq. (7.4) we can assume that the integral is over the region B and using the bound
\[ |G(x) - G'(x)| \leq c \int_r^\infty \frac{e^{-x^2/2t}}{t^2} \, dt = u(x) \]
and the methods used to obtain Eq. (7.2) we see that
\[
J(r, \varepsilon) = \frac{1}{8} |\mu| \int_0^\delta q_r(x) G * p_r * f_r(x) G * G * p_r * f_r(x) \, dx + 0 \left( \log \left( \frac{1}{\varepsilon} \right) \right)
\]
\[ = \int_0^\delta q_r(x) G * p_r (y^f - y^a_t) \, dx + 0 \left( \log \left( \frac{1}{\varepsilon} \right) \right) \]
(7.5)

which proves Eq. (7.3).

The rest of the proof now follows as in Rosen (1992).

8. Theorem 3: superprocesses over diffusions

Let \( z_t \) to a smooth uniformly elliptic diffusion in \( \mathbb{R}^d \), with transition density \( p_s(x, y) \). It is easy to write down the analogue of Eq. (7.1) for \( Z_t \), the superprocess over \( z_t \): simply replace
\[ p_{s(s-a)}(Y^a_t - Y^a_t) \]
by

\[ p_{x(t)}(y_0) = p_{x(t)}(y_0), y_{y(0)}). \]

Since, for some \( M, x > 0 \)

\[ p_t(x, y) \leq M e^{-d(x-y)^2/2t} \] (8.1)

we can apply all the results of the previous sections to obtain bounds on the moments of \( \gamma_{\epsilon}(T) \).

In particular, if \( d = 2 \) or \( 3 \), \( E_\mu(\gamma_{\epsilon}^2(T)) \) is uniformly bounded in \( \epsilon \), and convergence in \( L^2 \) follows using Rosen (1987), (2.4), (3.16)).

When \( d = 4 \), the same reasoning shows that we can bound the contribution of any graph \( C \) to \( E_\mu(\gamma_{\epsilon}^2(T)) \) by \( \log^{n-1}(1/\epsilon) \) unless \( n = 2m \) and \( C \subseteq A_{4m} \).

As in the previous section, it suffices to show that

\[
\int \mu(du) p_t(z, u) p_{t-s}(z_1, z) p_{t-s}(y, z) p_{t-s}(x, y) p_{t-s}(x, z_2) p_{t-s}(x, z_3) p_{t-s}(y, z_4) f(z_1 - z_2) f(z_3 - z_4) d\mu_d \, dy \, d\bar{z} \, dr \, ds \, ds' \, dr' \\
= |\mu| \frac{1}{64\pi^4} \log^2 \left( \frac{1}{\epsilon} \right) + 0 \left( \log \frac{1}{\epsilon} \right). \tag{8.2}
\]

However, using the bounds just described, we know that up to errors of order \( \log(1/\epsilon) \), we can restrict integration to the region where \( z_1, x, y, z_1, \ldots, z_4 \) are close together. It is known that for \( x \) near \( y \)

\[ p_t(x, y) = \frac{e^{-|C^{-1}(x-y)|^2/2t}}{(2\pi)^d/2 \det(C)} + 0 \left( \frac{1}{\epsilon} \right) \tag{8.3}
\]

for some \( \beta > 0 \), where \( C = \sqrt{A(x)} \) and \( A(x) \) in the matrix \( a_{ij}(x) \).

We thus see that up to errors \( 0(\log(1/\epsilon)) \), the integral in Eq. (8.2) is equal to

\[
\int \mu(du) q_t(D(z-u)) q_{t-s}(D(z_1 - z)) q_{t-s}(D(y - z)) q_{t-s}(D(x - y)) q_{t-s}(D(x - z_2)) q_{t-s}(D(y - z_3)) f(z_1 - z_2) f(z_3 - z_4) \quad \text{(det)(D)}^7 \, d\mu \, dy \, d\bar{z} \, dr \, ds \, ds' \, dr' |\mu| \int \mu(du) q_t(z) q_{t-s}(z_1 - z) q_{t-s}(y - z) q_{t-s}(x - y) q_{t-s}(x - z_3) q_{t-s}(y - z_4) f_{x}(z_1 - z_2) f_{x}(z_3 - z_4) d\mu_d \, dy \, d\bar{z} \, dr \, ds \, ds' \, dr', \tag{8.4}
\]

where \( q \) denotes the Brownian transition density,

\[ D = \sqrt{A^{-1}(z)}, \quad A(z) = \{a_{ij}(z)\} \]

and

\[ f^D(x) = f(D^{-1}x) \]
Comparing Eq. (8.4) to the calculations in Eq. (3.11) we see that our last integral is
\[
\frac{1}{64\pi^2} \log^2 \left( \frac{1}{\varepsilon} \right) \left( \int f^D(y) \, dy \right) \left( \int f^D(z) \, dz \right) + O \left( \log \left( \frac{1}{\varepsilon} \right) \right)
\]
\[
= \frac{1}{64\pi^2} \log^2 \left( \frac{1}{\varepsilon} \right) \left( \det(D) \right)^2 + O \left( \log \left( \frac{1}{\varepsilon} \right) \right),
\]
which proves Eq. (8.2); hence Theorem 3.

9. Superstable processes: Theorem 4

Let \( y_t \) denote the symmetric stable process in \( \mathbb{R}^d \) of index \( \beta \) with transition density
\[
p_t^{(\beta)}(y) = \frac{1}{(2\pi)^d} \int e^{yp} e^{-\varepsilon p^\beta} \, dp
\]
so that
\[
p_t^{(\beta)}(0) = \frac{1}{(2\pi)^d} \int e^{-\varepsilon p^\beta} \, dp = \frac{1}{\beta 2^{d-1} \pi^{d/2}} \frac{\Gamma(d/\beta)}{\Gamma(d/2)}.
\]
With
\[
G_t^{(\beta)}(y) = \int_0^\infty e^{-\varepsilon s} p_t(y) \, ds,
\]
we have
\[
G_0^{(\beta)}(y) = \frac{\Gamma((d - \beta)/2)}{\Gamma(\beta/2)} \frac{1}{2^{d-1} \pi^{d/2}} \frac{1}{|y|^{d-\beta}}.
\]
Our normalization has the property that
\[
G_0^{(x)} \ast G_0^{(\beta)} = G_0^{(x + \beta)}.
\]
Notice that with our normalization
\[
p_t^{(2)}(y) = p_{2t}(y),
\]
where \( p_t \) is the Brownian transition density, hence the Brownian Green’s function is twice \( G_0^{(2)} \).

When \( \beta \) is fixed, we often suppress it and set \( G_1 \equiv G_1^{(\beta)} \). Theorem 4 will follow from Theorem 5 in the same manner that Theorem 1 followed from Theorem 2.

Theorem 5. Let \( X_t \) be the superprocess over \( x_t \), the symmetric stable process in \( \mathbb{R}^d \) of index \( \beta \) killed at an independent exponential time.

1. If \( d/2 < \beta \leq d \) then as \( \varepsilon \to 0 \)
\[
\int_0^\infty \langle f_s(x - y), X_s(dx)X_s(dy) \rangle \, ds - \zeta(\varepsilon) \int_0^\infty |X_s| \, ds
\]
converges in $L^2$, where

\[
\zeta(\varepsilon) = \beta p_1(0) \log \left( \frac{1}{\varepsilon} \right) \quad \text{if} \quad \beta = d,
\]

\[
\zeta(\varepsilon) = \frac{1}{\varepsilon^d - \beta} \int G_0^{(\beta)}(x) f(x) \, dx \quad \text{if} \quad \beta < d.
\]

(9.3)

2. If $\beta = d/2$, then if

\[
\gamma_\varepsilon = \int_0^\infty \left( f_\varepsilon(x - y), X_\varepsilon(dx) X_\varepsilon(dy) \right) \, ds - \alpha(\varepsilon) \int_0^\infty |X_\varepsilon| \, ds,
\]

where

\[
\alpha(\varepsilon) = \frac{1}{\varepsilon^d} \int G_0^{(\beta)}(x) f(x) \, dx - \beta p_1(0) \log \left( \frac{1}{\varepsilon} \right).
\]

Then as $\varepsilon \to 0$, $\gamma_\varepsilon/\log(1/\varepsilon)$ converges in distribution and

\[
E_\mu(e^{-\gamma_\varepsilon/\log(1/\varepsilon)}) \to e^{\mu(1/2)(1 - \sqrt{1 - \Lambda(d)^{d/2}})}
\]

for $\Lambda$ small, where

\[
\Lambda(d) = \frac{2^{d-2d}}{\pi^d} \frac{1}{T^2(d/2)}
\]

(9.5)

Theorem 5 will follow as in the proof of theorem 2 once we have computed the asymptotics of $G$ and $G \ast G$, see the appendix of Rosen (1990).

We first use the resolvent equation to find

\[
G = G_0 - G_0 \ast G.
\]

(9.6)

If $d/2 < \beta < d$, then it is easy to see that the last two terms are continuous, and then Eq. (9.2) gives that half of Eq. (9.3) referring to $\beta < d$.

If $\beta = d/2$, we use $G = G_0 - G \ast G + G \ast G \ast G_0$ and proceed as in Eq. (2.11):

\[
G \ast G(x) = \int_0^\infty e^{-t} p_\varepsilon(x) \, dt
\]

\[
= \int_0^1 e^{-t} p_\varepsilon(x) \, dt + \int_1^\infty e^{-t} p_\varepsilon(x) \, dt
\]

(9.7)

and

\[
\int_0^1 e^{-t} p_\varepsilon(x) \, dt = \int_0^1 t p_\varepsilon(x) \, dt + \int_0^1 (e^{-t} - 1) t p_\varepsilon(x) \, dt.
\]

(9.8)

Then, using scaling

\[
\int_0^1 t p_\varepsilon(x) \, dt = \frac{1}{\Lambda^{d}} \int_0^1 t p_{\varepsilon \Lambda}(1) \, dt
\]

\[
= \int_0^{1/\Lambda^{d}} t p_1(1) \, dt
\]


\[
\int_1^{1/x^\beta} tp_1(0) \, dt = \int_0^1 tp_1(0) \, dt \\
= \int_1^{1/x^\beta} tp_1(0) \, dt + \int_1^{1/x^\beta} t(p_1(1) - p_1(0)) \, dt + \int_0^1 tp_1(1) \, dt
\]

and finally
\[
\int_1^{1/x^\beta} tp_1(0) \, dt = p_1(0) \int_1^{1/x^\beta} \frac{1}{t} \, dt = \beta p_1(0) \log\left(\frac{1}{|x|}\right)
\]

Thus, from Eqs. (9.7)-(9.10) we find that for \( \beta = d/2 \),
\[
G * G(x) = \beta p_1(0) \log\left(\frac{1}{|x|}\right)
\]
+ terms continuous in \( x \) which leads to \( \alpha(\varepsilon) \).

Finally, we need the analogue of Eq. (3.11) when \( \beta = d/2 \):
\[
K(\varepsilon) \approx \frac{1}{2^3} \int G(x)G * f_\varepsilon(x)G * G * f_\varepsilon(x)
\]
\[
= \frac{1}{2^3} \int_{2\varepsilon < |x| < 1/2} G_0 G_0 * f_\varepsilon G * G * f_\varepsilon \, dx + O\left(\frac{1}{\varepsilon}\right)
\]
\[
= \frac{1}{2^3} C^2(\beta) \beta p_1(0) \int_{2\varepsilon < |x| < 1/2} \frac{1}{x^{d-\beta}} \frac{1}{|x-y|^{d-\beta}} \log\left(\frac{1}{|x-y|}\right)
\]
\[
f_\varepsilon(y)f_\varepsilon(z) + O\left(\frac{1}{\varepsilon}\right)
\]
\[
= \frac{1}{2^3} C^2(\beta) \beta p_1(0) 2\pi^{d/2} \frac{1}{2} \frac{\log\left(\frac{1}{\varepsilon}\right)}{\Gamma(d/2)} + O\left(\frac{1}{\varepsilon}\right),
\]
where \( C(\beta) \) is the coefficient of \( 1/X^{d-\beta} \) in Eq. (9.1). Putting all this together and using \( d = 2\beta \), we find that
\[
\wedge(d) = 2 \cdot 2^3 C^2(\beta) \beta p_1(0) 2\pi^{d/2} \frac{1}{\Gamma(d/2)} \cdot \frac{1}{2}
\]
\[
= 2^3 \frac{1}{(2\pi)^{2\beta}} \frac{1}{\beta^{2-1} \pi^{d/2}} \frac{1}{\Gamma(d/2)} \frac{2\pi^{d/2}}{\Gamma(d/2)}
\]
\[
= \frac{2^{5-2d}}{\pi^d} \frac{1}{\Gamma^2(d/2)}.
\]
We note that for \( \beta = 2, d = 4 \) this gives
\[
\wedge(d) = 2^{5-2d} \frac{1}{\pi^4} = \frac{1}{2^3 \pi^4},
\]
consistent with our normalization as described at the beginning of this section, and the \( 2/\pi^4 \) which appears in Theorem 2. \( \square \)
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References


