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Logarithmic averages for the local times of recurrent random walks and Levy processes

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1. Introduction

Logarithmic averages have received a great deal of attention in probability theory since the recent discovery that logarithmic averaging enables one to go from weak convergence in the classical central limit theorem to an almost sure limit theorem. More precisely, if Y_j denotes a sequence of i.i.d. random variables with mean zero and variance σ^2, and $S_k = \sum_{j=1}^{k} Y_j$ is the sum of the first k terms, then we have

$$\lim_{n\to\infty} \frac{1}{\log n} \sum_{k=1}^{n} \frac{1_{\{S_k \leqslant x\sigma\sqrt{k}\}}}{k} = \Phi(x) \qquad \text{a.s.,}$$

where $\Phi(x)$ is the standard normal distribution function (see Brosamler, 1988; Schatte, 1988; Fisher, to appear; Lacey and Philipp, 1990; Berkes and Dehling, 1993).

The effect of logarithmic averaging on local time asymptotics has been studied by Csaki, Foldes and Revesz (to appear) for certain random walks. If X_n is a random walk in $\mathbb{Z}^d$ we let $L_n = L_n^0$, where $L_n^x = \{\text{number of } j \mid X_j = x,\ 1 \leqslant j \leqslant n\}$. They show that

$$\lim_{n\to\infty} \frac{1}{\log n} \sum_{k=1}^{n} \frac{L_k}{k^{3/2}} = \frac{2\sqrt{2}}{\sigma} \qquad \text{a.s.,} \tag{1.1}$$

for one-dimensional random walks with mean zero and incremental variance σ^2, and

$$\lim_{n\to\infty} \frac{1}{\log\log n} \sum_{k=2}^{n} \frac{L_k}{k \log^2 k} = \frac{1}{\pi} \qquad \text{a.s.,} \tag{1.2}$$

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for the simple random walk in the plane. Furthermore, they establish a second-order limit law for the simple random walk on the line

$$\lim_{n\to\infty} \frac{1}{\sqrt{\log n}} \sum_{k=1}^{n} \frac{L_k^1 - L_k^0}{k^{5/4}} \overset{w}{=} \frac{2^{9/4}}{\pi^{1/4}} N(0,1), \tag{1.3}$$

where $\overset{w}{=}$ denotes weak convergence and $N(0,1)$ is a standard normal random variable.

In this paper we shall show how results such as (1.1), (1.2) and (1.3) for a very wide class of random walks and Levy processes follow from the laws of the iterated logarithm for local times which we obtained in Marcus and Rosen (to appear).

Let X_n be a recurrent random walk in $\mathbb{Z}^d$ in the domain of attraction of a non-degenerate strictly stable random variable U of index β. Thus

$$\frac{X_n}{b(n)} \to U$$

in law where $b(n)$ is regularly varying of order $1/\beta$. For X_n to be recurrent, we must have either $d = 1$, $1 \leqslant \beta \leqslant 2$ or $d = 2$, $\beta = 2$. We assume for simplicity that X_n is strongly aperiodic. We can always take $b(x)$ to be a continuous and monotone increasing function from $\mathbb{R}_+$ to $\mathbb{R}$ with $b(0) = 0$. With the notation $p_n(x) = P(X_n = x)$ we then have that

$$p_n(0) \sim \frac{1}{b^d(n)} \tag{1.4}$$

is regularly varying of order $-d/\beta$, and

$$g(n) = \sum_{j=1}^{n} p_j(0) \sim \int_1^n \frac{1}{b^d(t)}\,\mathrm{d}t \tag{1.5}$$

is regularly varying of order $1 - d/\beta$. We use the notation $r(n) \sim s(n)$ to mean $\lim_{n\to\infty} r(n)/s(n) = 1$. The recurrence of X_n is equivalent to the fact that $\lim_{n\to\infty} g(n) = \infty$.

We have shown in Marcus and Rosen (to appear) that the following law of the iterated logarithm holds for L_n:

$$\limsup_{n\to\infty} \frac{L_n}{g(n/\log\log g(n))\log\log g(n)} = a_0 \quad \text{a.s.} \tag{1.6}$$

where a_0 is a computable constant. We will see that by taking logarithmic averages we can go from the lim sup in (1.6) to a true limit theorem, generalizing (1.1) and (1.2).

Theorem 1. *With the above hypotheses on X_n we have*

$$\lim_{N\to\infty} \frac{1}{\log g(N)} \sum_{n=1}^{N} \frac{p_n(0)}{g^2(n)} L_n = 1 \quad \text{a.s.} \tag{1.7}$$

Here is our generalization of the second order theorem for local time differences (1.3).

Theorem 2. *With the above hypotheses on X_n and assuming in addition that X_n is symmetric we have*

$$\lim_{N\to\infty} \frac{1}{\sqrt{\log g(N)}} \sum_{n=1}^{N} \frac{p_n(0)}{g^{3/2}(n)}(L_n^0 - L_n^x) \stackrel{w}{=} 4a(x)N(0,1) \tag{1.8}$$

where $\stackrel{w}{=}$ *denotes the weak limit,* $N(0,1)$ *denotes a standard normal random variable, and*

$$a^2(x) = \sum_{n=1}^{\infty} p_n(0) - p_n(x). \tag{1.9}$$

Besides logarithmic averages we can consider 'f-averages', in which the $\log g(N)$ of Theorem 1 is replaced by $f(g(N))$, and $1/g^2(n)$ is replaced by $-f''(g(n))$. More precisely, let $f(x)$ be an ultimately C^2, monotonically increasing, concave function of slow variation, such that $f(x) \uparrow \infty$, and $f'(x) = O(1/x)$. For example, we can take $f(x) = \log\log x$, or more generally $f(x) = \log_p x$, the pth iterated logarithm of x.

Theorem 3. *With the above hypotheses on X_n and $f(x)$, if*

$$\frac{f'(x)}{f(x)} = O\left(\frac{1}{x\log\log x}\right) \tag{1.10}$$

then

$$\lim_{N\to\infty} \frac{1}{f(g(N))} \sum_{n=1}^{N} (-f'')(g(n))\, p_n(0) L_n = 1 \qquad \text{a.s.} \tag{1.11}$$

Furthermore, if X_n is symmetric and

$$\frac{\sqrt{f'(x)}}{f(x)} = O\left(\frac{1}{\sqrt{x}\log\log x}\right) \tag{1.12}$$

then

$$\lim_{N\to\infty} \frac{1}{\sqrt{f(g(N))}} \sum_{n=1}^{N} (-\sqrt{f'})'(g(n))\, p_n(0)(L_n^0 - L_n^x) \stackrel{w}{=} 2a(x)N(0,1). \tag{1.13}$$

For example, taking $f(x) = \log\log x$ we obtain the following

Theorem 4. *With the above hypotheses on X_n we have*

$$\lim_{N\to\infty} \frac{1}{\log\log g(N)} \sum_{n=1}^{N} \frac{p_n(0)}{g^2(n)\log g(n)} L_n = 1 \qquad \text{a.s.} \tag{1.14}$$

Furthermore, if X_n is symmetric we have

$$\lim_{N\to\infty} \frac{1}{\sqrt{\log\log g(N)}} \sum_{n=1}^{N} \frac{p_n(0)}{g^{3/2}(n)\sqrt{\log g(n)}}(L_n^0 - L_n^x) \stackrel{w}{=} 4a(x)N(0,1). \tag{1.15}$$

We next consider local time asymptotics for Levy processes. Logarithmic averaging for the local times of Brownian motion was first considered by Brosamler (1973), and later by Csaki and Foldes (to appear). Let X_t be a recurrent Levy process in $\mathbb{R}^1$ and set

$$E(\exp(\mathrm{i}\lambda X_t)) = \exp(-t\psi(\lambda)).$$

X_t has a local time if and only if $(1+\psi(\lambda))^{-1} \in L^1(\mathbb{R}^+)$. We shall assume that in fact

$$\int_0^\infty \frac{\log(1+\lambda)}{1+\psi(\lambda)}\,\mathrm{d}\lambda < \infty.$$

We denote the local time of X by L_t^x, which we normalize by setting

$$E^x\left(\int_0^\infty \mathrm{e}^{-t}\,\mathrm{d}L_t^y\right) = \int_0^\infty \mathrm{e}^{-t}p_t(x-y)\,\mathrm{d}t$$

where $p_t(x-y) = p_t(x,y)$ is the transition density function for X. We shall assume that $p_t(0)$ is regularly varying at infinity of order $-1/\beta$ for some $1 \leqslant \beta \leqslant 2$. Set

$$g(t) = \int_0^t p_s(0)\,\mathrm{d}s.$$

If $\beta = 1$, so that $g(t)$ is slowly varying at infinity, we shall also require that at least one of the following three mild regularity conditions holds:

$$\lim_{t\to\infty} \frac{g(t/\log\log g(t))}{g(t)} = 1, \tag{1.16}$$

$$\limsup_{t\to\infty} \frac{g(t/\log g(t))}{g(t)} < 1, \tag{1.17}$$

$$\frac{1}{g(t)}\int_1^t \frac{g(s)}{s}\,\mathrm{d}s \leqslant C\frac{g(t)}{tg'(t)} \quad \text{for all } t \text{ sufficiently large.} \tag{1.18}$$

We can now state our generalization of Theorem 1 for Levy processes.

Theorem 5. *With the above hypotheses on X_t we have*

$$\lim_{t\to\infty} \frac{1}{\log g(t)}\int_2^t \frac{p_s(0)}{g^2(s)}L_s^0\,\mathrm{d}s = 1 \quad \text{a.s.} \tag{1.19}$$

Here is our generalization of Theorem 2.

Theorem 6. *With the above hypotheses on X_t and assuming in addition that X_t is symmetric we have*

$$\lim_{t\to\infty} \frac{1}{\sqrt{\log g(t)}}\int_2^t \frac{p_s(0)}{g^{3/2}(s)}(L_s^0 - L_s^x)\,\mathrm{d}s \overset{w}{=} 4a(x)N(0,1) \tag{1.20}$$

where $\overset{w}{=}$ denotes the weak limit, $N(0,1)$ denotes a standard normal random variable, and

$$a^2(x) = \int_0^\infty (p_s(0) - p_s(x))\,\mathrm{d}s. \tag{1.21}$$

2. Proofs

Proof of Theorem 1. This will follow almost immediately from our law of the iterated logarithm (1.6) which implies that

$$\frac{L_N}{g(N)\log(g(N))} \to 0 \quad \text{a.s} \tag{2.1}$$

and Theorem 6 of Chung and Erdős (1951):

$$\lim_{N\to\infty} \frac{1}{\log g(N)} \sum_{j=1}^{N} \frac{\delta_0(X_j)}{g(j)} = 1 \quad \text{a.s.} \tag{2.2}$$

Although their theorem was formulated for integer valued random walks, the proof of (2.2) is valid for random walks in $\mathbb{Z}^d$ for any d.

We first note that from (1.4) and (1.5)

$$\sum_{n=j}^{N} \frac{p_n(0)}{g^2(n)} \sim \left(\frac{1}{g(j)} - \frac{1}{g(N)}\right). \tag{2.3}$$

We now rewrite

$$\begin{aligned}\sum_{n=1}^{N} \frac{p_n(0)L_n}{g^2(n)} &= \sum_{n=1}^{N} \frac{p_n(0)}{g^2(n)} \sum_{j=1}^{n} \delta_0(X_j) \\ &= \sum_{j=1}^{N} \delta_0(X_j) \sum_{n=j}^{N} \frac{p_n(0)}{g^2(n)} \\ &\sim \sum_{j=1}^{N} \delta_0(X_j) \left(\frac{1}{g(j)} - \frac{1}{g(N)}\right) \\ &\sim \sum_{j=1}^{N} \frac{\delta_0(X_j)}{g(j)} - \frac{L_N}{g(N)},\end{aligned} \tag{2.4}$$

and Theorem 1 follows using (2.2) and (2.1).

Proof of Theorem 2. We first recall the second-order law of the iterated logarithm for local times of Marcus and Rosen (to appear):

$$\limsup_{n\to\infty} \frac{L_n^0 - L_n^x}{g^{1/2}(n/\log\log g(n))\log\log g(n)} = a_1 a^2(x) \quad \text{a.s.} \tag{2.5}$$

where a_1 is a computable constant.

Using this and resumming as in (2.4) we see that it suffices to prove that

$$\lim_{n\to\infty} \frac{1}{\sqrt{\log g(n)}} \sum_{j=1}^{n} \frac{\delta_0(X_j) - \delta_x(X_j)}{g^{1/2}(j)} \stackrel{w}{=} 2a(x)N(0,1) \tag{2.6}$$

and to this end it suffices to show that for all $k \in \mathbb{Z}_+$ we have

$$E\left(\left\{\frac{1}{\sqrt{\log g(n)}}\sum_{j=1}^{n}\frac{\delta_0(X_j)-\delta_x(X_j)}{g^{1/2}(j)}\right\}^{2k}\right)\to\frac{(2k)!}{2^k k!}(2a(x))^{2k} \tag{2.7}$$

and

$$E\left(\left\{\frac{1}{\sqrt{\log g(n)}}\sum_{j=1}^{n}\frac{\delta_0(X_j)-\delta_x(X_j)}{g^{1/2}(j)}\right\}^{2k-1}\right)\to 0 \tag{2.8}$$

as $n \to \infty$, see e.g. Durret (1991, Chapter 2, Section 3).

We have the basic identity from Rosen (1993):

$$E\left(\left\{\sum_{j=1}^{n}\frac{\delta_0(X_j)-\delta_x(X_j)}{g^{1/2}(j)}\right\}^{2k}\right) \tag{2.9}$$

$$= (2k)! \sum_{1\leqslant j_1\leqslant j_2\ldots\leqslant j_{2k}\leqslant n}\frac{\prod_{r=1}^{2k}(p_{j_r-j_{r-1}}(0)-(-1)^r p_{j_r-j_{r-1}}(x))}{g^{1/2}(j_1)g^{1/2}(j_2)\cdots g^{1/2}(j_{2k})},$$

with the convention $p_0(x) = 1_{\{x=0\}}$. We will see below that asymptotically, for each r odd, we can replace the factor $(p_{j_r-j_{r-1}}(0)-(-1)^r p_{j_r-j_{r-1}}(x))$ by $2p_{j_r-j_{r-1}}(0)$. Hence to prove (2.7) it suffices to show that

$$\sum_{1\leqslant j_1\leqslant j_2\ldots\leqslant j_{2k}\leqslant n}\frac{\prod_{r=1}^{k} p_{j_{2r-1}-j_{2r-2}}(0)(p_{j_{2r}-j_{2r-1}}(0)-p_{j_{2r}-j_{2r-1}}(x))}{g^{1/2}(j_1)g^{1/2}(j_2)\cdots g^{1/2}(j_{2k})}$$

$$\sim \frac{(a^2(x)\log g(n))^k}{k!} \tag{2.10}$$

We will establish (2.10) by evaluating in turn the sum over j_s, $s = 1,\ldots,2k$. The result will depend on whether s is odd or even. We will show inductively that for the sum over j_s, $s = 1,\ldots,2i-1$ for any $1\leqslant i\leqslant k$ we have that

$$\sum_{1\leqslant j_1\leqslant j_2\ldots\leqslant j_{2i-1}\leqslant n}\frac{\prod_{r=1}^{i} p_{j_{2r-1}-j_{2r-2}}(0)(p_{j_{2r}-j_{2r-1}}(0)-p_{j_{2r}-j_{2r-1}}(x))}{g^{1/2}(j_1)g^{1/2}(j_2)\cdots g^{1/2}(j_{2i-1})}$$

$$\sim \frac{a^{2i}(x)\{\log g(j_{2i})\}^{i-1}p_{j_{2i}}(0)}{(i-1)!g^{1/2}(j_{2i})}, \tag{2.11}$$

and for the sum over j_s, $s = 1,\ldots,2i$ for any $1\leqslant i\leqslant k-1$ we have that

$$\sum_{1\leqslant j_1\leqslant j_2\ldots\leqslant j_{2i}\leqslant n}\frac{\prod_{r=1}^{i+1} p_{j_{2r-1}-j_{2r-2}}(0)\prod_{r=1}^{i}(p_{j_{2r}-j_{2r-1}}(0)-p_{j_{2r}-j_{2r-1}}(x))}{g^{1/2}(j_1)g^{1/2}(j_2)\cdots g^{1/2}(j_{2i})}$$

$$\sim \frac{a^{2i}(x)\{\log g(j_{2i+1})\}^{i}p_{j_{2i+1}}(0)}{i!}. \tag{2.12}$$

At the final stage we use

$$\sum_{j_{2k}=1}^{n} \frac{\{\log(g(j_{2k}))\}^{k-1} p_{j_{2k}}(0)}{g(j_{2k})} \sim \frac{1}{k}\{\log(g(n))\}^{k}, \tag{2.13}$$

which follows from (1.4) and (1.5) and we will arrive at (2.10).

It is easy to prove (2.11) and (2.12) inductively using the following two asymptotic relations which, as we will show, hold for all $1 \leqslant r \leqslant k$:

$$\sum_{j_{2r-1}=1}^{j_{2r}} \frac{\{\log g(j_{2r-1})\}^{r-1} p_{j_{2r-1}}(0)}{g^{1/2}(j_{2r-1})}(p_{j_{2r}-j_{2r-1}}(0) - p_{j_{2r}-j_{2r-1}}(x)) \sim a^2(x)\frac{\{\log g(j_{2r})\}^{r-1} p_{j_{2r}}(0)}{g^{1/2}(j_{2r})}. \tag{2.14}$$

and

$$\sum_{j_{2r}=1}^{j_{2r+1}} \frac{\{\log(g(j_{2r}))\}^{r-1} p_{j_{2r}}(0) p_{j_{2r+1}-j_{2r}}(0)}{g(j_{2r})} \sim \frac{1}{r}\{\log g(j_{2r+1})\}^{r} p_{j_{2r+1}}(0). \tag{2.15}$$

To prove (2.14) we first note that

$$\sum_{j=n(1-\varepsilon)}^{n} \frac{\{\log g(j)\}^{r-1} p_j(0)}{g^{1/2}(j)}(p_{n-j}(0) - p_{n-j}(x)) \sim a^2(x)\frac{\{\log g(n)\}^{r-1} p_n(0)}{g^{1/2}(n)} \tag{2.16}$$

using

$$\sum_{j=n(1-\varepsilon)}^{n} (p_{n-j}(0) - p_{n-j}(x)) \sim a^2(x),$$

and the regular variation of

$$\{\log g(j)\}^{r-1} p_j(0)/g^{1/2}(j). \tag{2.17}$$

We will show that

$$\sum_{j=1}^{n(1-\varepsilon)-1} \frac{\{\log g(j)\}^{r-1} p_j(0)}{g^{1/2}(j)}(p_{n-j}(0) - p_{n-j}(x)) = \mathrm{o}\left(\frac{\{\log g(n)\}^{r-1} p_n(0)}{g^{1/2}(n)}\right) \tag{2.18}$$

for any x, and this together with (2.16) proves our claim (2.14). To see (2.18), note first that

$$\sum_{j=1}^{n(1-\varepsilon)-1} \frac{\{\log g(j)\}^{r-1} p_j(0)}{g^{1/2}(j)}(p_{n-j}(0) - p_{n-j}(x)) \leqslant \left(\sum_{j=1}^{n(1-\varepsilon)-1} \frac{\{\log g(j)\}^{r-1} p_j(0)}{g^{1/2}(j)}\right) \sup_{n\varepsilon \leqslant k \leqslant n} (p_k(0) - p_k(x)) \tag{2.19}$$

so that to prove (2.18) it suffices to show that

$$\sup_{n\varepsilon \leqslant k \leqslant n} (p_k(0) - p_k(x)) = o(\Lambda(n)) \tag{2.20}$$

where

$$\Lambda(n) = \left(\frac{\{\log g(n)\}^{r-1} p_n(0)}{g^{1/2}(n)}\right) \Big/ \left(\sum_{j=1}^{n(1-\varepsilon)-1} \frac{\{\log g(j)\}^{r-1} p_j(0)}{g^{1/2}(j)}\right).$$

On the one hand, since (2.17) is regularly varying of order $-1/2(1+d/\beta) \geqslant -1$, we see that $\Lambda(n)$ is regularly varying of order -1. On the other hand, if ϕ denotes the characteristic function of X_1 then since X_1 is symmetric

$$\begin{aligned} p_n(0) - p_n(x) &= \frac{1}{(2\pi)^d} \int (1 - \cos(vx)) \phi^n(v)\, \mathrm{d}v \\ &\leqslant \frac{x^2}{(2\pi)^d} \int v^2 \phi^n(v)\, \mathrm{d}v \\ &\leqslant \frac{C}{(b(n))^{2+d}} \end{aligned} \tag{2.21}$$

where the last inequality follows as in the proof of Proposition 2.4 of LeGall and Rosen (1991). Since $(b(n))^{2+d}$ is regularly varying of order $(2+d)/\beta \geqslant 3/2$ this completes the proof of (2.20).

To prove (2.15) we first note that by the monotonicity of $p_j(0)$, and using (2.13) we have

$$\begin{aligned} \sum_{j=1}^{n} \frac{\{\log(g(j))\}^{r-1} p_j(0) p_{n-j}(0)}{g(j)} &\geqslant \sum_{j=1}^{n} \frac{\{\log(g(j))\}^{r-1} p_j(0)}{g(j)} p_n(0) \\ &\sim \frac{1}{r} \{\log(g(n))\}^r p_n(0). \end{aligned} \tag{2.22}$$

On the other hand, using the regular variation of $p_n(0)$, for any $0 < \varepsilon < 1$ we have

$$\begin{aligned} \sum_{j=1}^{n\varepsilon} \frac{\{\log(g(j))\}^{r-1} p_j(0) p_{n-j}(0)}{g(j)} &\leqslant \sum_{j=1}^{n} \frac{\{\log(g(j))\}^{r-1} p_j(0)}{g(j)} p_{n(1-\varepsilon)}(0) \\ &\sim \frac{1}{r} \{\log(g(n))\}^r p_n(0) (1-\varepsilon)^{-d/\beta}, \end{aligned} \tag{2.23}$$

while

$$\begin{aligned} \sum_{j=n\varepsilon}^{n} \frac{\{\log(g(j))\}^{r-1} p_j(0) p_{n-j}(0)}{g(j)} &\leqslant \{\log(g(n))\}^{r-1} \frac{p_{n\varepsilon}(0)}{g(n\varepsilon)} \sum_{j=n\varepsilon}^{n} p_{n-j}(0) \\ &\leqslant \{\log(g(n))\}^{r-1} \frac{p_{n\varepsilon}(0) g(n)}{g(n\varepsilon)} \\ &\sim \{\log(g(n))\}^{r-1} p_n(0) \varepsilon^{-1}. \end{aligned} \tag{2.24}$$

Thus (2.22), (2.23), and (2.24) establish (2.15).

Our arguments make it clear that indeed asymptotically, for each r odd we can replace the factor $(p_{j_r-j_{r-1}}(0) - (-1)^r p_{j_r-j_{r-1}}(x)) = (p_{j_r-j_{r-1}}(0) + p_{j_r-j_{r-1}}(x))$ in (2.9) by $2p_{j_r-j_{r-1}}(0)$. More precisely, the error term introduced involves the difference $(p_{j_r-j_{r-1}}(0) - p_{j_r-j_{r-1}}(x))$ and we see on comparing (2.14) with (2.15) that this error term is $O(\{\log(g(n))\}^{k-1})$. This completes the proof of (2.7). These same calculations immediately show (2.8), completing the proof of Theorem 2. □

Proof of Theorem 3. We note the analogues of (2.3) and (2.13):

$$\sum_{n=j}^{N}(-f'')(g(n))p_n(0) \sim f'(g(j)) - f'(g(N)) \tag{2.25}$$

and

$$\sum_{j=1}^{n} f^{r-1}(g(j))f'(g(j))p_j(0) \sim \frac{1}{r}f^r(g(n)). \tag{2.26}$$

Consider first the proof of (1.11). Using (2.25) to resum as in (2.4), and using our hypothesis (1.10) together with the law of the iterated logarithm (1.6) to ignore the error term, we find that we are reduced to showing

$$\lim_{N\to\infty} \frac{1}{f(g(N))} \sum_{j=1}^{N} f'(g(j))\delta_0(X_j) = 1 \qquad \text{a.s.} \tag{2.27}$$

A straightforward modification of the proof of Theorem 6 in Chung and Erdős (1951) establishes (2.27).

To prove (1.13) we proceed as in the proof of Theorem 2, using (2.26) in place of (2.13).

The proofs of Theorems 4–6 are similar and left to the reader.

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