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INTERSECTION LOCAL TIMES OF ALL ORDERS FOR BROWNIAN AND STABLE DENSITY PROCESSES—CONSTRUCTION, RENORMALISATION AND LIMIT LAWS

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The Brownian and stable density processes are distribution valued processes that arise both via limiting operations on infinite collections of Brownian motions and stable Lévy processes and as the solutions of certain stochastic partial differential equations. Their (self-) intersection local times (ILT's) of various orders can be defined in a manner somewhat akin to that used to define the self-intersection local times of simple $\mathfrak{R}^d$-valued processes; that is, via a limiting operation involving approximate delta functions. We obtain a full characterisation of this limiting procedure, determining precisely in which cases we have convergence and deriving the appropriate renormalisation for obtaining weak convergence when only this is available. We also obtain results of a fluctuation nature, that describe the rate of convergence in the former case. Our results cover all dimensions and all levels of self-intersection.

1. Introduction.

(*a*) *Density processes*. Perhaps the best way to describe and understand density processes is via a particle picture. Thus, let $X_t^1, X_t^2, \dots,\ t \ge 0$, be a sequence of independent, $\mathfrak{R}^d$-valued Markov processes, with stationary transition density either

$$p_t(x,y) = p_t(x-y) = \frac{1}{(2\pi t)^{d/2}} e^{-\|x-y\|^2/2t} \tag{1.1}$$

(i.e., d-dimensional Brownian motions with infinitesimal generator $A = \frac{1}{2}\Delta$, where Δ is the d-dimensional Laplacian) or

$$\begin{aligned} p_t^\alpha(x,y) &= p_t^\alpha(x-y) \\ &= \frac{1}{(2\pi)^d}\int_{\mathfrak{R}^d} \exp(-ip\cdot(x-y) - t2^{-\alpha/2}\|p\|^\alpha)\,dp \end{aligned} \tag{1.2}$$

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[i.e., d-dimensional symmetric Lévy processes of index $\alpha \in (0,2)$, with infinitesimal generator A given by the fractional half Laplacian $\Delta_\alpha = -(-\frac{1}{2}\Delta)^{\alpha/2}$]. Both transition densities are taken with respect to Lebesgue measure, which serves, in both cases, as an invariant measure. Without further comment, we shall denote the Brownian transition density by both p_t, as above, or p_t^2.

Let Π^λ be a Poisson point process on $\Re^d$ with intensity measure $\lambda\, dx$ for $\lambda \geq 0$; that is, the number of points of Π^λ in a Borel set $A \subset \Re^d$ is a Poisson random variable with parameter λ times the d-dimensional volume of A and the numbers in disjoint sets are independent. Since the probability that any two points of Π^λ lie exactly the same distance from the origin is zero, we can order them by magnitude, and shall denote them by $X_0^1, X_0^2, \dots,$ which we now take as the initial values of the X^i.

For $\phi \in \mathscr{S}_d = \mathscr{S}(\Re^d)$, the Schwartz space of infinitely differentiable functions on $\Re^d$ decreasing rapidly at infinity, define the following two $\mathscr{S}_d'$-valued random processes:

$$\mu_t^\lambda(\phi) = \lambda^{-1} \sum_{i=1}^{\infty} \phi(X_t^i), \tag{1.3}$$

$$\eta_t^\lambda(\phi) = \lambda^{-1/2} \sum_{i=1}^{\infty} \sigma^i \phi(X_t^i), \tag{1.4}$$

where $\sigma^1, \sigma^2, \dots$ is a sequence of independent Rademacher random variables. ($P\{\sigma^i = +1\} = P\{\sigma^i = -1\} = \frac{1}{2}$.) The two sequences $\{X^i\}$, $\{\sigma^i\}$ and Π^λ are all assumed independent of one another except for the fact that Π^λ determines the initial values of the X^i.

The density process associated with the Markov process X is the weak limit, in the Skorohod space $D([0,1], \mathscr{S}_d')$, of η_t^λ as $\lambda \to \infty$. It is also closely related to the weak limit of $\lambda^{1/2}(\mu_t^\lambda - E\mu_t^\lambda)$, which was studied in some detail by Martin-Löf (1976).

As is clear from the form of (1.4), η_t is a mean zero Gaussian process, with covariance functional given, for $\phi_1, \phi_2 \in \mathscr{S}_d$ and $t > s$, by

$$E\{\eta_t(\phi_1) \cdot \eta_s(\phi_2)\} = \iint \phi_1(x) p_{t-s}^\alpha(x,y) \phi_2(y)\, dx\, dy. \tag{1.5}$$

When $s = t$ we have

$$E\{\eta_t(\phi_1) \cdot \eta_t(\phi_2)\} = \int \phi_1(x)\phi_2(x)\, dx. \tag{1.6}$$

Looked at from this point of view, η_t is a particularly simple process. In fact, it is also easily seen to be Markovian, and to satisfy the following stochastic partial differential equation (SPDE) for $\alpha \in (0,2]$:

$$\left(\frac{\partial \eta}{\partial t}\right)(\phi) = \eta(\Delta_\alpha \phi) + \sqrt{2}\, W_t(\Delta_{\alpha/2}\phi), \tag{1.7}$$

$$\eta_0 = N, \tag{1.8}$$

where we have adopted the convention that $\Delta_2 \equiv \Delta$, where N is a Gaussian white noise on $\Re^d$; that is, $N(B)$ is mean zero Gaussian for each Borel $B \subset \Re^d$ with $E\{N(B_1)N(B_2)\} = \text{Leb}(B_1 \cap B_2)$ and where W is a mean zero, space-time Gaussian process with covariance functional

$$(1.9)\quad EW(\phi_1 \times \psi_1)W(\phi_2 \times \psi_2) = \int_0^\infty \phi_1(t)\phi_2(t)\,dt \int_{\Re^d} \psi_1(x)\psi_2(x)\,dx,$$

with $\phi_i \in \mathscr{S}_1$ and $\psi_i \in \mathscr{S}_d$. The SPDE (1.7) should be understood in the weak form of Walsh (1986), namely,

$$\eta_t(\phi) = \int_0^t \eta_s(\Delta_\alpha \phi)\,ds + \sqrt{2}\int_0^t \int_{\Re^d} \Delta_{\alpha/2}\phi(x) W(ds, dx),$$

so that $W_t(\phi) = \int_0^t \int_{\Re^d} \phi(x) W(ds, dx)$.

The weak limit of $\lambda^{1/2}(\mu_t^\lambda - E\mu_t^\lambda)$ mentioned above satisfies the same SPDE, but with a different initial condition.

While the above SPDE is also valid in the Brownian case $\alpha = 2$, there is a simpler representation in this case, for then the square root Δ_1 of the Laplacian can be replaced by a first order differential operator. In particular, we have

$$(1.10)\qquad \frac{\partial \eta}{\partial t} = \tfrac{1}{2}\Delta\eta + \sqrt{2}\,\nabla \cdot \dot{W},$$

where we have now used W to denote a $\Re^d$-valued Gaussian white noise in space-time based on Lebesgue measure, and the last term in (1.10), after integration in the time variable, should be understood as

$$(1.11)\qquad \int_0^t \int_{\Re^d} \langle \nabla\phi(x), W(dx, ds)\rangle.$$

The equivalences between these various representations of the basic SPDE will be established in Section 3, where we shall also be more precise about the correct formulation of the SPDE. [Note that, when $\alpha \neq 2$, $\phi \in \mathscr{S}_d$ does not necessarily imply that $\Delta_\alpha \phi \in \mathscr{S}_d$, so that, without proper interpretation, the RHS of (1.7) is not well defined.]

(*b*) *Intersection local times.* Let $\delta(x) = \delta_d(x)$ denote the Dirac delta function on $\Re^d$. If it would make sense, we would like to use the following expression to define a new $\mathscr{S}'$-valued stochastic process, which we would call the *intersection local time process* corresponding to the density process η_t:

$$(1.12)\qquad \int_0^t du \int_0^t dv \int_{\Re^d}\int_{\Re^d} (\eta_u \times \eta_v)(\delta(x-y))\phi(x)\,dx\,dy,$$

where $\eta_u \times \eta_v$ is the usual product of distributions.

When $d = 1$ and $\alpha > 1$ this is quite simple, for then the distribution $\int_0^t \eta_s \, ds$ has function form. That is, there exists a function $L_t(x)$, which can be interpreted as a kind of local time for η, for which $\int_0^t \eta_s(\phi)\, ds = \int_{\Re} L_t(x)\phi(x)\, dx$ for all ϕ and t. [This follows, for example, from the construction of density processes in Adler and Epstein (1987) and the existence of a local time for real-valued Brownian motion and stable processes on $\Re^1$ when $\alpha > 1$.] In these cases it is therefore not hard to show that an appropriate interpretation of (1.12) is given by $\int_{\Re}(L_t(x))^2\phi(x)\, dx$. Since these cases are simple and require none of the delicateness of the forthcoming analysis, we shall make no further reference to them. In general, however, one cannot make mathematical sense out of (1.12) without introducing a certain renormalisation. For this we require some notation.

Let $G_1, \dots, G_k$ be a sequence of zero mean, but otherwise completely general, Gaussian variables. We define the *Wick ordering* of their product as

$$:G_1 \cdots G_k: = \sum_A (-1)^{|A|} \prod_{(i,j)\in A} E\{G_iG_j\} \prod_{l \in A^c} G_l, \tag{1.13}$$

where the sum runs over all collections A of pairs of integers from $\{1, \dots, k\}$ (including the empty pair), $|A|$ denotes the number of elements (i.e., pairs of integers) in A and A^c comprises those integers not in A. For example, if $k = 2$, then $:G_1G_2: = G_1G_2 - EG_1G_2$.

Now equip $\mathscr{S}_d$ with the usual topology and let $\mathscr{A}_{kd}$ be the dense subset of $\mathscr{S}_{kd}$ made up of functions of the form

$$\phi_N(x_1, \dots, x_k) = \sum_{i=1}^N \phi_i^{(1)}(x_1) \cdots \phi_i^{(k)}(x_k), \tag{1.14}$$

where $\phi_i^{(j)}(x) \in \mathscr{S}_d$ for all $1 \le j \le k$. If $\eta_1, \dots, \eta_k$ are Gaussian distributions on $\mathscr{S}_d$, then we define their centered, or *Wick ordered*, product $:\eta_1 \times \cdots \times \eta_k:$ on $\mathscr{S}_{kd}$ by setting

$$(:\eta_1 \times \cdots \times \eta_k:)(\phi_N) = \sum_{i=1}^N :\eta_1(\phi_i^{(1)}) \cdots \eta_k(\phi_i^{(k)}): \tag{1.15}$$

for test functions of the form (1.14) in $\mathscr{A}_{kd}$ and then extending to all of $\mathscr{S}_{kd}$. That this is legitimate is standard fare in the theory of Gaussian distributions. [See, for example, Chapter 6 of Glimm and Jaffe (1987).]

We are now in a position not only to make sense out of (1.12), but also to extend the order of self-intersection above two. Recall that C_o^∞ denotes the space of C^∞ functions of bounded support.

1.1 Definition. Let $f \in C_o^\infty \subset \mathscr{S}_d$ be symmetric and satisfy $\int_{\Re^d} f(x)\, dx = 1$. For $\varepsilon > 0$, set $f_\varepsilon(x) = \varepsilon^{-d} f(x/\varepsilon)$, and for $k \ge 2$, $\phi \in \mathscr{S}_d$ and $t \ge 0$, set

$$\begin{aligned}\gamma_{k,\varepsilon}(t,\phi) &= \gamma_{k,\varepsilon}(f: t, \phi) \\ &= \int_0^t dt_1 \cdots \int_0^t dt_k (:\eta_{t_1} \times \cdots \times \eta_{t_k}:)\left(\phi(x_1) \prod_{i=2}^k f_\varepsilon(x_i - x_1)\right).\end{aligned} \tag{1.16}$$

If $\gamma_{k,\varepsilon}$, which a priori depends on f, converges as $\varepsilon \to 0$ (in $\mathscr{L}^2$, for example) to a limit, then the limit process is denoted by $\gamma_k(t)$ and is called a k-fold (self-) intersection local time (ILT) process of η_t. It is, of course, an $\mathscr{S}_d'$ distribution valued process, and may, or may not, depend on the choice of f.

Since f_ε approaches a delta function as $\varepsilon \to 0$, it is clear that we have now found a way in which to give meaning to (1.12). The following results will show that the renormalisation inherent in the Wick ordering of the product distribution is at least one way to handle the divergences inherent in (1.12). [To appreciate the need for renormalisation, one should note that without this reordering a regular product of distributions in (1.16) would lead to $\gamma_{k,\varepsilon}$'s that diverge in $\mathscr{L}^2$ in cases where the renormalised variant converges.]

The central question that will interest us is the convergence or nonconvergence of $\gamma_{k,\varepsilon}$. We shall find conditions for convergence and study the rate of divergence in those cases where $\gamma_{k,\varepsilon}$ diverges as $\varepsilon \to 0$. We shall also obtain fluctuation results, which give even finer information on rates of convergence. We shall do this via two routes: In the case of the simple ILT, that is, when $k = 2$, we shall adopt an approach via stochastic analysis that gives detailed insight into why our results hold, as well as establishing, en passant, stochastic evolution equations for the ILT. This approach does not seem to be generalisable to higher orders of intersection, however, and so we shall also present an approach, valid for all d and k, based on moment calculations.

ASIDE. The reader who may want to compare the results of this paper with earlier results in Adler, Feldman and Lewin (1990) (hereafter referred to as AFL) should note that the definition of $\gamma_{k,\varepsilon}$ that we have employed involves time integration over all of $[0,t]^k$ and not just over an ordered wedge in which $t_1 \le \cdots \le t_k$, as was done there, so that results between the two papers differ by factors of 2^{k-1}.

(*c*) *The approximate ILT process.* To state our first result we need to introduce the Green's function

$$G_\alpha^\lambda(x,y) = \int_0^\infty e^{-\lambda t} p_t^\alpha(x,y)\,dt, \tag{1.17}$$

corresponding to the transition probabilities (1.1) and (1.2). It is clear from (1.1) and (1.2) that $G_\alpha^\lambda(x,y) = G_\alpha^\lambda(y,x) = G_\alpha^\lambda(\|x-y\|)$. We shall write $G_\alpha^\lambda * f$ to denote the convolution $(G_\alpha^\lambda * f)(x) = \int_{\Re^d} G_\alpha^\lambda(x,y) f(y)\,dy$. Note that $G_\alpha^\lambda(x,y)$ is always well defined for $\lambda > 0$, but may be identically infinite if $\lambda = 0$.

A particularly useful result, in the case of simple ILT, is the following, which allows us to express the simple $\gamma_{2,\varepsilon}$ via an evolution equation involving η_t and a space-time white noise. We bring it in now in order to better understand Theorem 1.4 below.

1.2 THEOREM. *Let $\lambda > 0$. The approximate ILT of order $k = 2$ is given by the following evolution equation, in which W is a space-time Gaussian white* *noise with covariance functional* (1.9):

$$\begin{aligned}\gamma_{2,\varepsilon}(t,\phi) &= 2\lambda\int_0^t du\int_0^u dv(:\eta_u\times\eta_v:)\big(G_\alpha^\lambda * f_\varepsilon(x-y)\phi(y)\big)\\ &\quad - 2\int_0^t du(:\eta_t\times\eta_u:)\big(G_\alpha^\lambda * f_\varepsilon(x-y)\phi(y)\big)\\ &\quad + 2\int_0^t du(:\eta_u\times\eta_u:)\big(G_\alpha^\lambda * f_\varepsilon(x-y)\phi(y)\big)\\ &\quad + 2\sqrt{2}\int_0^t\int_0^s\int_{\Re^d}\eta_u\big(\Delta_{\alpha/2}G_\alpha^\lambda * f_\varepsilon(x-\cdot)\phi(\cdot)\big)\,du\,W(dx,ds).\end{aligned}\tag{1.18}$$

The stochastic integral is of the type studied by Walsh (1986).

We shall establish this result in the following section, after we have set up an Itô formula for density processes. It is worth noting at this point, however, that in the Brownian case we can replace the last term above by

$$2\int_0^t\int_{\Re^d}\left\langle\int_0^s\nabla\eta_u\big(G_2^\lambda * f_\varepsilon(x-\cdot)\phi(\cdot)\big)\,du,W(dx,ds)\right\rangle,$$

where, as in (1.10) and (1.11), we have now moved to a vector-valued white noise. Although this representation is more natural in the Brownian case, and, indeed, is at the basis of the results in AFL, we shall not use it again here, so as to avoid having to write out special arguments for the Brownian situation.

(*d*) *The $\mathscr{L}^2$ convergent case.* This is actually the easiest of all cases and is covered by two results. The first is a basic existence theorem.

1.3 THEOREM. *Let $\alpha > d(k-1)/k$. Then, for each $f\in C_o^\infty$ with $\int_{\Re^d}f(x)\,dx = 1$, $\phi\in\mathscr{S}_d$ and $t>0$, the approximate ILT's $\gamma_{k,\varepsilon}(f\colon t,\phi) = \gamma_{k,\varepsilon}(t,\phi)$ converge in $\mathscr{L}^2$ as $\varepsilon\to 0$. Furthermore, the limit random process $\gamma_k(t,\phi)$ is independent of the function f and is called the kth order ILT process for η_t.*

The second result is a generalisation to the stable case of a result first established for the simple ($k=2$) ILT of the Brownian density process in AFL by completely different methods.

1.4 THEOREM. *Let $d<2\alpha$. Then, for all $\lambda>0$, the pairwise ILT γ_2 satisfies the evolution equation* (1.18) *with $\gamma_{2,\varepsilon}$ replaced by γ_2 and $G_\alpha^\lambda * f_\varepsilon$ replaced by G_α^λ throughout the right-hand side.*

It is clear that this is simply the evolution equation for $\gamma_{k,\varepsilon}$ with f_ε replaced by a delta function. The proof, of course, requires a little more detail.

[To compare the above equation to the representation given in AFL for the ILT of the Brownian density process on $\Re^2$, note that there we take $\lambda=\frac{1}{2}$ and replace the integral operator $\Delta_{\alpha/2}$ by the differential operator $2^{-1/2}\nabla$. This, of course, is only possible in the Brownian case, $\alpha = 2$.]

(*e*) *The $\mathscr{L}^2$ divergent case and renormalisation.* Although it is not explicitly stated in Theorem 1.3, if $\alpha \le d(k-1)/k$, then $E\gamma_{k,\varepsilon}^2$ actually diverges as $\varepsilon \to 0$, and so it is not possible to define an ILT in these cases. Nevertheless, it is possible to obtain quite precise information on the rate of divergence.

In order to formulate this precisely, we need to decide on what spaces our processes live. Let $C([0,T], \mathscr{S}_d')$ denote the space of continuous $\mathscr{S}_d'$-valued processes. That is, $\eta \in C([0,T], \mathscr{S}_d')$ if $\eta_t(\phi)$ is continuous on $[0,T]$ in the usual sense for every $\phi \in \mathscr{S}_d$. We use $\Rightarrow$ to denote weak convergence on $C([0,1], \mathscr{S}_d')$. [See Walsh (1986) for details.]

We also require some notation: Thus, for $d \ge 1$ and $\alpha \in (0,2]$, set

$$s_d = \frac{2\pi^{d/2}}{\Gamma(d/2)}, \qquad c_{d,\alpha} = \frac{\Gamma((d-\alpha)/2)}{\pi^{d/2}\Gamma(\alpha/2)}. \tag{1.19}$$

Furthermore, let $W_t^{(k)}$ be the $\mathscr{S}_d'$-valued Gaussian process with zero mean and covariance functional

$$EW_s^{(k)}(\phi)W_t^{(k)}(\psi) = (s \wedge t)^k \int_{\Re^d} \phi(x)\psi(x)\,dx. \tag{1.20}$$

If $k = 1$, then $W^{(k)}$ is just the time-integrated form of the Gaussian white noise defined by (1.9).

1.5 THEOREM. *Fix $f \in C_o^\infty$ such that $\int_{\Re^d} f(x)\,dx = 1$.*

(a) *If $\alpha = d(k-1)/k$, then*

$$\frac{\gamma_{k,\varepsilon}(f)}{\sqrt{\ln(1/\varepsilon)}} \Rightarrow \theta W^{(k)} \tag{1.21}$$

as $\varepsilon \to 0$, where

$$\theta^2 = \theta^2(d,k,\alpha) = 2^k c_{d,\alpha}^k s_d k!.$$

(b) *If $\alpha < d(k-1)/k$, then*

$$\frac{\gamma_{k,\varepsilon}(f)}{\varepsilon^{(d-(d-\alpha)k)/2}} \Rightarrow B_f W^{(k)} \tag{1.22}$$

as $\varepsilon \to 0$, where

$$\begin{aligned} B_f^2 &= B^2(f\colon d,k,\alpha) \\ &= 2^k(k-1)!\int_{\Re^d}\big((G_\alpha^0 * f * f)(x)\big)^{k-1} G_\alpha^0(x)\,dx \\ &\quad + (k-1)(k-1)!\int_{\Re^d}\big((G_\alpha^0 * f * f)(x)\big)^{k-2}\big((G_\alpha^0 * f)(x)\big)^2\,dx. \end{aligned}$$

Note that in (1.22) the exponent of ε is negative, so that the denominator diverges as $\varepsilon \to 0$ and the result therefore captures the rate of divergence of $\gamma_{k,\varepsilon}$ mentioned at the beginning of this subsection. Perhaps the most interesting aspect of this result is that in part (b), for the first time, the function f affects the limiting distribution for $\gamma_{k,\varepsilon}$. Note that this was not the case in either part (a) or in the $\mathscr{L}^2$ convergent situation of Theorem 1.4.

(f) *Fluctuation theory for* ILT. We now return to the $\mathscr{L}^2$ convergent situation of Theorem 1.3 and obtain a result of a fluctuation nature that describes the rate at which the $\mathscr{L}^2$ convergence occurs. To make the notation a little clearer, we shall denote a Euclidean inner product, which up until now has been $x \cdot y$, by $\langle x, y \rangle$.

1.6 THEOREM. *As usual, fix an* $f \in C_o^\infty$, *but assume that* $\int_{\mathfrak{R}^d} f(x)\,dx = 0$. *Let* $\alpha > d(k-1)/k$. *If* $\alpha = d = 2$ *and* $k = 2$, *then*

$$\frac{\gamma_{k,\varepsilon}(f)}{\varepsilon \ln(1/\varepsilon)} \Rightarrow C_f W^{(k)} \tag{1.23}$$

as $\varepsilon \to 0$, *where* $C_f^2 = 4\pi \int \langle x, y \rangle f(x) f(y)\,dx\,dy$. *If* $\alpha = d = 2$ *and* $k \geq 3$, *then*

$$\frac{\gamma_{k,\varepsilon}(f)}{\varepsilon \sqrt{\ln(1/\varepsilon)}} \Rightarrow D_f W^{(k)} \tag{1.24}$$

as $\varepsilon \to 0$. *If* $\alpha = d = 1$ *and* $k \geq 2$, *then*

$$\frac{\gamma_{k,\varepsilon}(f)}{\sqrt{\varepsilon \ln(1/\varepsilon)}} \Rightarrow E_f W^{(k)} \tag{1.25}$$

as $\varepsilon \to 0$. *In all other cases*,

$$\frac{\gamma_{k,\varepsilon}(f)}{\varepsilon^{(d-(d-\alpha)k)/2}} \Rightarrow F_f W^{(k)} \tag{1.26}$$

as $\varepsilon \to 0$. *Here* D_f, E_f *and* F_f *are rather involved constants whose definition we defer until Section* 5 [*equations* (5.23), (5.24) *and* (5.7), *respectively*] *when we shall have set up the appropriate notation.*

Note that under the assumed condition $\alpha > d(k-1)/k$, the exponent of ε in (1.26) is positive, as one would expect for a fluctuation result.

(g) *About the proofs.* By way of an introduction to the following sections, and for the reader who will leave us before the hard work really starts, we have two main comments to make about the proofs of the various convergence results.

The first point to note is that the limit processes in Theorems 1.4–1.6 are always multiples of the Gaussian $W^{(k)}$, which is determined by its moments. Thus, if one were to concentrate only on a proof of convergence of finite-dimensional distributions of the variously renormalised versions of $\gamma_{k,\varepsilon}$ to their limits, it would suffice to calculate the asymptotic moments. All that is needed, in principle, to do this is the moment formula of Theorems 2.5 and 2.6 in the following section. (In fact, even the tightness calculation required for full weak convergence requires little more.) This, in fact, is how our most general proofs will proceed, and these will form the content of Sections 4 and 5 of the paper.

The second point is that moment proofs do not generally explain why a result may be true. In particular, it is not immediately clear from moment calculations what the ubiquitous process $W^{(k)}$ has to do with $\gamma_{k,\varepsilon}$. This becomes clearer when, for the case $k = 2$, a proof is built on the basis of the evolution equation (1.18). Roughly speaking, $G_\alpha^\lambda * f_\varepsilon$ is much better behaved than $\Delta_{\alpha/2} G_\alpha^\lambda * f_\varepsilon$ as $\varepsilon \to 0$, so that when $\gamma_{k,\varepsilon}$ diverges, it is because of the last term in (1.18). That is, after the renormalisations of Theorems 1.5–1.6, this is the only term left in the evolution equation. It is now not hard to convince oneself that $W^{(2)}$ has something to do with the limit process after renormalisation.

Since this proof seems to us the more attractive, we shall treat it first in Section 3 of the paper. Unfortunately, however, it only works for the case $k = 2$, for which we have an explicit evolution equation for $\gamma_{2,\varepsilon}$. The following section contains a number of technical results required throughout the paper. We close this section with some comments.

(h) *Historical notes.* Density processes seem to have been originally introduced into the probability literature by Martin-Löf (1976) and studied in depth, for the case $d = 1$, by Itô (1983). An approach based on stochastic partial differential equations can be found in the notes of Walsh (1986). Results in two and three dimensions, for the Brownian density process and its simple ILT can be found in AFL, where it is also shown how the ILT of the Brownian density process can be related to the ILT's of the initial Brownian motions X_t^i of (1.4) and how this relationship can be exploited to derive a special case of Theorem 1.4.

Although density processes are of intrinsic interest, it is worth noting that they are also of additional interest in that they share many qualitative properties with measure valued diffusions which are, ab initio, much more difficult to study. Thus density processes are a good test case for their more complicated cousins. Material on the existence and representation of ILT for superprocesses can be found, for example, in Dawson, Iscoe and Perkins (1989), Dynkin (1988) and Perkins (1990), while Rosen (1990) treats renormalisation and weak convergence results for the case $k = 2$. Tanaka-like evolution equations for the simple local time are given in Adler and Lewin (1992), while ILT ($k = 2$ only) is treated in Adler and Lewin (1991). Further results on ILT for superprocesses can be found in Adler (1993).

2. Some technicalities. In this section we shall collect a number of facts that will be required in the proofs of the results of the Introduction. From the point of view of continuity, you can move directly to the following section, returning here only when you need a specific result.

(*a*) *Fractional Laplacians.* Details and proofs related to the claims we are about to make can be found, for example, in Stein (1970) and Yoshida (1980).

The fractional Laplacian Δ_α has a natural definition via Fourier transforms. In particular, for a Schwartz function ϕ on $\Re^d$ we have

$$\widehat{(\Delta_\alpha \phi)}(p) = -2^{-\alpha/2}\|p\|^\alpha \hat{\phi}(p). \tag{2.1}$$

An immediate consequence of this representation, plus Parseval's equality for inner products, is the following integration by parts formula, which will be crucial in a number of the calculations to follow:

$$\int_{\Re^d} |\Delta_{\alpha/2}\phi(x)|^2\, dx = -\int_{\Re^d} \phi(x)\Delta_\alpha \phi(x)\, dx. \tag{2.2}$$

A first principles calculation shows that the Green's function G_α^0 of (1.17) satisfies $\widehat{G_\alpha^0}(p) = 2^{\alpha/2}\|p\|^{-\alpha}$ for all d and for all $\alpha \in (0,2]$, so that an immediate consequence of (2.1) is that G_α^0 is the fundamental solution, in the distribution sense, of the equation $-\Delta_\alpha u = \delta$, where δ is the Dirac delta function. That is, for every $\phi \in \mathscr{S}_d$,

$$\int_{\Re^d} (-\Delta_\alpha G_\alpha^0)(x)\phi(x)\, dx = \phi(0). \tag{2.3}$$

Fourier inversion [see, e.g., Gelfand and Shilov (1964) for details] then gives us the following formula:

2.1 FORMULA. *Assume that $\alpha = 2$, $d \geq 3$, or $\alpha \in (0,2)$, $d \geq 2$, or $\alpha \in (0,1)$, $d = 1$. Then the Green's function G_α^0 is a distributional solution of the equation $-\Delta_\alpha u = \delta$, and is given explicitly by*

$$G_\alpha^0(x) = c_{d,\alpha}\|x\|^{\alpha-d},$$

where the constants $c_{d,\alpha}$ are as given at (1.19). *Note that in the cases $\alpha = d = 2$ and $d = 1$, $\alpha \geq 1$, the definition* (1.17) *leads to $G_\alpha^0 \equiv \infty$.*

At this stage we note the following useful fact, whose proof is an immediate consequence of either carrying out the convolution integral or, more readily, looking at Fourier transforms:

2.2 USEFUL FACT. *Suppose $d > \alpha + \beta$. Then $G_\alpha^0 * G_\beta^0(x) = G_{\alpha+\beta}^0(x)$.*

Another Fourier argument gives that for *all* $\lambda > 0$, the equation

$$(-\Delta_\alpha + \lambda)u = \delta \tag{2.4}$$

is solved by G_α^λ. In this case, however, there are no restrictions on the values of d and α. For general λ, the Green's function is somewhat more complicated than for the case $\lambda = 0$, and it is not generally possible to give its explicit form.

Given the general structure of Green's functions, the following lemma will be of considerable importance to us in obtaining many of the bounds we shall require later. The proof is straightforward. As usual, C is a generic constant that may change from line to line.

2.3 LEMMA. *Let $U(x)$: $\Re^d \to \Re$ satisfy $|U(x)| \le C\|x\|^{-\gamma}$ for some $\gamma < d$. Let f and f_ε be as in Definition* 1.1. *Then*

$$\|U * f_\varepsilon(x)\| \le \begin{cases} C\|x\|^{-\gamma}, & \text{if } \|x\| \ge 2\varepsilon, \\ C\varepsilon^{-\gamma}, & \text{if } \|x\| \le 2\varepsilon. \end{cases}$$

(*b*) *Density processes*. In this section we shall show that the two representations of density processes—as a Gaussian process on $\Re_+ \times \mathscr{S}_d$ and as a solution of an SPDE—are consistent, as well as tidying up an important technical point for the latter. At the same time we shall make certain that all factors of $\sqrt{2}$, which are inconsistent between various presentations, are consistent here.

The technical point, for which we are grateful to a careful referee and Associate Editor, relates to the test function $\phi \in \mathscr{S}_d$ appearing in the SPDE (1.7). As pointed out in Dawson and Gorostiza (1990) [see also Dawson, Fleischmann and Gorostiza (1989)], unless $\alpha = 2$ it is not generally true that $\phi \in \mathscr{S}_d$ implies that $\Delta_\alpha \phi \in \mathscr{S}_d$. (The problem is with decay at infinity.) Thus the terms on the RHS of (1.7) are not well defined. A way around this problem involves introducing the spaces

$$C_{p,0}(\Re^d) := \left\{\phi \in C(\Re^d): \lim_{\|x\| \to \infty} \phi(x)(1 + \|x\|^2)^p = 0\right\}, \tag{2.5}$$

$p > 0$, with norms

$$\|\!|\phi|\!\|_p := \sup_{x \in \Re^d} |\phi(x)(1 + \|x\|^2)^p|. \tag{2.6}$$

For us, one of the main properties of these spaces is that if $p > d/2$ and, in the case $\alpha < 2$, if $p < (d + \alpha)/2$, then $\phi \in \mathscr{S}_d$ implies $\Delta_\alpha \phi \in C_{p,0}$. Denoting the dual of $C_{p,0}$ by $C'_{p,0}$, and equipping it with the dual norm designated by $\|\!| \cdot |\!\|_{-p}$, we have, under the same conditions on p, that

$$\mathscr{S}_d \subset C_{p,0}(\Re^d) \subset \mathscr{L}^2(\Re^d) \subset C'_{p,0}(\Re^d) \subset \mathscr{S}'_d.$$

Now note that the Gaussian processes η_t and W_t, are defined, respectively, by the covariance functions (1.5)–(1.6) and (1.9), as $\mathscr{S}'_d$-valued random variables. However, since $\mathscr{S}_d$ is continuously, densely embeddable into $C_{p,0}(\Re^d)$, it is an easy $\mathscr{L}^2$ calculation to check that for the values of p given above, both η_t and W_t have unique extensions to $C_{p,0}(\Re^d)$. Thus, in view of the above paragraph, both $\eta_t(\Delta_\alpha \phi)$ and $W_t(\Delta_{\alpha/2}\phi)$ are well defined for $\phi \in \mathscr{S}_d$ for

$p > d/2$ and, additionally, $p < (d + \alpha)/2$ if $\alpha < 2$. These facts can be used to make sense of the RHS of (1.7) and similar equations.

This approach has been formalised by Dawson and Gorostiza (1990) (Definition 3.1), who have used it to introduce the notion of a *generalised solution* of the SPDE (1.7). We refer the reader to their paper for details. What is important for us is that we can now formulate the following result, which, when $\alpha = 2$, and when the SPDE is replaced by (1.10), is essentially Theorem 5 of Martin-Löf (1976).

2.4 THEOREM. *The zero mean Gaussian process $\eta_t(\phi)$, $t \geq 0$, $\phi \in \mathscr{S}_d$, with covariance functional given by (1.5) and (1.6), is a generalised solution in $C([0,\infty), \mathscr{S}_d')$ of the weak SPDE $\dot{\eta}(\phi) = \eta(\Delta_\alpha \phi) + \sqrt{2}\, W(\Delta_{\alpha/2}\phi)$, with η_0 a Gaussian white noise on $\Re^d$, and W a Gaussian white noise in space-time.*

PROOF. In view of the covariance structure of W given by (1.9), all we need show to prove the theorem is that for $\psi_1, \psi_2 \in \mathscr{S}_d$,

$$E\left[\left(\frac{\partial \eta_t}{\partial t}\right)(\psi_1) - \eta_t(\Delta_\alpha \psi_1)\right]\left[\left(\frac{\partial \eta_s}{\partial t}\right)(\psi_2) - \eta_s(\Delta_\alpha \psi_2)\right]$$

$$= 2(s \wedge t)\int_{\Re^d}\big(\Delta_{\alpha/2}\psi_1(x)\big)\big(\Delta_{\alpha/2}\psi_2(x)\big)\, dx, \tag{2.7}$$

where the terms involving fractional derivatives should be understood in terms of the extension described above, with p chosen appropriately.

The easiest way to do this is to extend η to a $(\mathscr{S}_1 \times \mathscr{S}_d)'$-valued distribution by setting

$$\eta(\phi \times \psi) \equiv \int_0^\infty \eta_t(\psi)\phi(t)\, dt.$$

Denoting the (time) derivative of $\phi \in \mathscr{S}_d$ by $\dot{\phi}$, the SPDE in question then becomes, in terms of distributions, $\eta(-\dot{\phi} \times \psi) = \eta(\phi \times \Delta_\alpha \psi) + \sqrt{2}\, W(\phi \times \Delta_{\alpha/2}\psi)$, while (2.7) is also appropriately changed. For ease of notation, consider only the case $\psi_1 = \psi_2$, $\phi_1 = \phi_2$. By (1.5) the left-hand side of (2.7) thus becomes

$$E\big[\eta\big(\dot{\phi} \times \psi + \phi \times \Delta_\alpha \psi\big)\big]^2$$

$$= 2\int\big[\eta\big(\dot{\phi}(s) \times \psi(x) + \phi(s) \times \Delta_\alpha \psi(x)\big)\big]\, dx\, ds \tag{2.8}$$

$$\times \int_{s \leq t} p_{t-s}^\alpha(x, dy)\big[\eta\big(\dot{\phi}(t) \times \psi(y) + \phi(t) \times \Delta_\alpha \psi(y)\big)\big]\, dt.$$

Recall that for a transition semigroup S_t with generator A and $\psi \in \mathrm{Dom}(A)$, we have $(S_t - I)\psi = \int_0^t S_s A\psi\, ds$. Hence, for $\Psi, \Phi \in \mathscr{S}_1 \times \mathscr{S}_d$,

$$\begin{aligned}
\int_{s\le t} & dx\, \Psi(s,x) S^{(\alpha)}_{t-s} \Delta_\alpha \Phi(t,x)\, ds\, dt \\
&= -\int_{s\le t\le u} dx\, \Psi(s,x) S^{(\alpha)}_{t-s}\Delta_\alpha \frac{\partial \Phi(u,x)}{\partial u}\, ds\, dt\, du \\
&= -\int_{s\le u} dx\, \Psi(s,x)\left(\int_s^u S^{(\alpha)}_{t-s}\Delta_\alpha \frac{\partial \Phi(u,x)}{\partial u}\, dt\right) ds\, du \\
&= -\int_{s\le u} dx\, \Psi(s,x)\big(S^{(\alpha)}_{u-s} - I\big)\frac{\partial \Phi(u,x)}{\partial u}\, ds\, du,
\end{aligned}$$

which, on rearranging, becomes

$$\int_{s\le t} dx\, \Psi(s,x) S^{(\alpha)}_{t-s}\left(\Delta_\alpha \Phi(t,x) + \frac{\partial \Phi(t,x)}{\partial t}\right) ds\, dt = -\int \Psi(t,x)\Phi(t,x)\, dt\, dx.$$

Putting $\Phi(t,x) = \phi(t) \times \psi(x)$, $\Psi(t,x) = \Delta_\alpha \Phi(t,x) + (\partial\Phi(t,x))/\partial t$ and substituting the above into (2.8), we obtain that

$$\begin{aligned}
E\big[\eta\big(\dot\phi \times \psi + \phi \times \Delta_\alpha \psi\big)\big]^2 &= -2\int\left(\Delta_\alpha \Phi(t,x) + \frac{\partial \Phi(t,x)}{\partial t}\right)\Phi(t,x)\, dt\, dx \\
&= -2\int \phi^2(t)\psi(x)\Delta_\alpha\psi(x)\, dt\, dx.
\end{aligned}$$

Now apply the integration by parts formula (2.2) to see that this is precisely the variance of $\sqrt{2}\,W$, regarded as a space-time white noise, evaluated at $\phi \times \Delta_{\alpha/2}\psi$.

(*c*) *Moment formulae.* The moment information of this subsection will be crucial in virtually all our proofs, including those that are of an essentially stochastic analysis nature. Despite the fact that at first sight they seem to be rather complicated, their compact form is in fact a further indication that the Wick renormalisation inherent in the definition of $\gamma_{k,\varepsilon}$ actually has a substantial simplifying effect.

We need a little notation to start. Fix $k \ge 1$, $m \ge 2$, and consider a (Feynman) graph based on m distinct vertices, with k distinct legs growing out of each vertex, and legs paired in such a way that no two legs from the same vertex are paired together. The totality of such graphs is denoted by $\mathscr{G}_{m,k}$ and each graph can be described by sets of links of the form $\{L = ((v_1, l_1), (v_2, l_2))\}$ which describe the vertices v_i and legs l_i that form the joins of the graph. We can now state:

2.5 FORMULA. *Let the approximate ILT*, $\gamma_{k,\varepsilon}(t,\phi)$, *be defined as in Definition* 1.1 *for* $k \ge 2$ *and* $t \ge 0$. *Then* $\gamma_{k,\varepsilon} \in C([0,T], \mathscr{S}_d')$ *for every* $\varepsilon > 0$ *and*

$0 < T < \infty$. *Furthermore, for every* $\phi \in \mathscr{S}_d$ *and even* $mk \geq 2$,

$$E(\gamma_{k,\varepsilon}(t,\phi))^m = \sum_{G \in \mathscr{G}_{m,k}} I(G), \tag{2.9}$$

where

$$\begin{aligned} I(G) = \int \Bigg\{ \prod_{L \in G} p^{\alpha}_{|t_{v_1,l_1} - t_{v_2,l_2}|}(x_{v_1,l_1} - x_{v_2,l_2}) \Bigg\} \\ \times \prod_{i=1}^{m} \left\{ \phi(x_{i,1}) \prod_{j=2}^{k} f_\varepsilon(x_{i,j} - x_{i,1}) \right\}, \end{aligned} \tag{2.10}$$

and the integral is to be taken over all the mk t*'s in the range* $[0,t]$ *and the* mk x*'s in* $\Re^d$. *If* mk *is odd, then* $E(\gamma_{k,\varepsilon}(t,\phi))^m = 0$.

Note that the x's in (2.10) are at first indexed via the links and later by a simpler ordering that is independent of the graph. (In both cases, the first of the two indices describes a vertex, and the second a leg growing from that vertex.) Although at first confusing, this will turn out to be a useful notation later on. The proof of (2.9) comes from Glimm and Jaffe (1987). The continuity follows directly from the continuity of η_t and the simple form of $\gamma_{k,\varepsilon}$ for ε strictly positive.

Whereas Formula 2.5 will generally suffice for proving convergence of finite dimensional distributions, we shall also require the following for tightness calculations.

2.6 FORMULA. *For every* $\phi, \psi \in \mathscr{S}_d$ *and* $m, n \geq 1$ *such that* $(m+n)k \geq 2$ *is even*,

$$E(\gamma_{k,\varepsilon}(t,\phi))^m(\gamma_{k,\delta}(s,\psi))^n = \sum_{G \in \mathscr{G}_{m+n,k}} \tilde{I}(G), \tag{2.11}$$

where

$$\begin{aligned} \tilde{I}(G) = \int \Bigg\{ \prod_{L \in G} p^{\alpha}_{|t_{v_1,l_1} - t_{v_2,l_2}|}(x_{v_1,l_1} - x_{v_2,l_2}) \Bigg\} \\ \times \prod_{i=1}^{m} \left\{ \phi(x_{i,1}) \prod_{j=2}^{k} f_\varepsilon(x_{i,j} - x_{i,1}) \right\} \\ \times \prod_{i'=1}^{n} \left\{ \phi(x_{i',1}) \prod_{j'=2}^{k} f_\delta(x_{i',j'} - x_{i',1}) \right\}, \end{aligned} \tag{2.12}$$

and the integral is to be taken over $[0,t]$ *for* t_v *if* $v \in [1, mk]$ *and* $[0,s]$ *for* t_v *with* $v \in [mk+1, (m+n)k]$. *Again, if* $(m+n)k$ *is odd, then the mixed moment vanishes, and, as in* (2.10), *there are two indexing systems operating at once.*

3. The stochastic analysis approach for $k = 2$. The main aim of this section is to show that many of the main results of the Introduction result, in a rather natural fashion, from a basic evolution equation describing the pairwise ILT $\gamma_{2,\varepsilon}$. Although this equation makes the proofs somewhat easier in this case, there is no free lunch, and some of the moment calculations required for the general case appear here as well. So as not to make the paper overlong, and unnecessarily repeat calculations, often in this section you will be referred to the next one for details of a specific calculation. Nevertheless, we still feel that the evolution equation based approach has much to offer in terms of adding insight into the results that in general arise from an approach based purely on rather formal moment calculations.

Our first task is to establish the evolution equation (1.18) (Theorem 1.2) for the approximate ILT $\gamma_{k,\varepsilon}$ in the case $k = 2$. Once we have this, we shall establish the evolution equation for the 2-fold ILT, γ_2, when this exists, along with a fluctuation result and the renormalisation results of Theorem 1.5 when it does not.

(*a*) *Proof of Theorem* 1.2. By Theorem 2.4 and the SPDE representation (1.7), the real-valued process $\eta_t(\phi)$ is a continuous semimartingale for every $\phi \in \mathscr{S}_d$. Either a straightforward calculation or a reliance on the notes of Walsh (1986), shows that the associated increasing process is given by $\langle \eta(\phi)\rangle_t = 2\|\Delta_{\alpha/2}\phi\|^2 t = 2t\int_{\Re^d}(\Delta_{\alpha/2}\phi(x))^2\,dx$. It therefore follows from Itô's formula for continuous semimartingales that for $\Psi = \Psi(t,x) \in C^2(\Re_+ \times \Re^d)$ and $\phi \in \mathscr{S}_d$,

$$
\begin{aligned}
\Psi(t,\eta_t(\phi)) = \Psi(0,\eta_0(\phi)) &+ \int_0^t \Psi_t(s,\eta_s(\phi))\,ds + \int_0^t \Psi_x(s,\eta_s(\phi))\eta_s(\Delta_\alpha\phi)\,ds \\
&+ \sqrt{2}\int_0^t\int_{\Re^d}\Psi_x(s,\eta_s(\phi))\Delta_{\alpha/2}\phi(y)W(ds,dy) \qquad (3.1)\\
&+ \|\Delta_{\alpha/2}\phi\|^2\int_0^t \Psi_{xx}(s,\eta_s(\phi))\,ds,
\end{aligned}
$$

where the subscripts on Ψ refer to the obvious partial derivatives. Fix $\psi \in \mathscr{S}_d$ and define the nonanticipating functional $\Psi\colon \Re_+ \times \Re^d \to \Re$ by

$$
\Psi(t,x) = x\int_0^t \eta_s(\psi)\,ds. \qquad (3.2)
$$

Now fix $\phi \in \mathscr{S}_d$ and apply (3.1) to obtain

$$
\begin{aligned}
\int_0^t \eta_t(\phi)\eta_s(\psi)\,ds = \int_0^t \eta_s(\phi)\eta_s(\psi)\,ds &+ \int_0^t\int_0^s \eta_u(\psi)\eta_s(\Delta_\alpha\phi)\,du\,ds \qquad (3.3)\\
&+ \sqrt{2}\int_0^t\int_0^s\int_{\Re^d}\eta_u(\psi)\cdot\Delta_{\alpha/2}\phi(y)\,du\,W(ds,dy).
\end{aligned}
$$

Extend (3.3), by linearity, to $\eta \times \eta$ acting on functions of the form $\sum_{i=1}^N \phi_i \times \psi_i$ in $\mathscr{S}_{2d}$. A standard approximation argument (see, for example, the proof of Theorem 2.1 in AFL for a similar argument) then extends (3.3) to all of $\mathscr{S}_{2d}$. Apply this to functions of the form $\phi(x)\psi(x-y) \in \mathscr{S}_{2d}$, $\phi, \psi \in \mathscr{S}_d$ and ψ symmetric, to obtain, after rearrangement,

$$
\begin{aligned}
&\int_0^t ds \int_0^s du \int_{\Re^d \times \Re^d} \phi(x) \Delta_\alpha \psi(x-y)(\eta_u \times \eta_s)(dx, dy) \\
&\quad = \int_0^t ds \int_{\Re^d \times \Re^d} \phi(x)\psi(x-y)(\eta_t \times \eta_s)(dx, dy) \\
(3.4)\qquad &\quad\quad - \int_0^t ds \int_{\Re^d \times \Re^d} \phi(x)\psi(x-y)(\eta_s \times \eta_s)(dx, dy) \\
&\quad\quad - \sqrt{2} \int_0^t \int_0^s \int_{\Re^d \times \Re^d} \phi(x) \Delta_{\alpha/2} \psi(x-y) \eta_s(dx)\, du\, W(ds, dy),
\end{aligned}
$$

where we have allowed ourselves the luxury of writing distributions in measure notation.

Subtract from each term in (3.4) its expectation. This has the effect of replacing each of the $\eta \times \eta$ terms in (3.4) by their Wick order $:\eta \times \eta:$. The stochastic integral term, however, has expectation zero, so that it remains unchanged.

Finally, replace ψ in the resulting equation by $G_\alpha^\lambda * f_\varepsilon$. [To justify this, we must first check that $G_\alpha^\lambda * f_\varepsilon \in C_{p,0}$ for an appropriate p, so that all terms in (3.4) are well defined. It is not hard to convince oneself that, in fact, $G_\alpha^\lambda(x)$ is at most $O(\|x\|^{-d-\alpha/2})$ for large $\|x\|$. [It is also not too hard to prove. The argument requires the kind of bounding that comes out of the integration by parts arguments used in Section 5—cf. the argument leading to (5.8). If $\alpha > 1$, integrate by parts $d+1$ times, as in the argument leading to (5.8), while if $\alpha < 1$, after d such integrations work separately on the regions where p_1 is less than or greater than $1/\|x\|$. The case $\alpha = 1$ can be handled via explicit formulae. It thus follows that if we choose $p \in (d/2, (d+\alpha/2)/2)$, we are justifying in replacing ψ by $G_\alpha^\lambda * f_\varepsilon$.]

Now use the relationship $(-\Delta_\alpha + \lambda)G_\alpha^\lambda * f_\varepsilon = f_\varepsilon$ [cf. (2.4)] and the definition (1.16) of $\gamma_{2,\varepsilon}$ to obtain (1.18) and so the proof of the theorem. □

(*b*) *Theorem* 1.3—*existence of the* ILT: $k = 2$ *and* $d < 2\alpha$. We now wish to establish Theorem 1.2 for the case $k = 2$; that is, that $\gamma_{2,\varepsilon}(f: t, \phi)$ has a well defined $\mathscr{L}^2$ limit as $\varepsilon \to 0$ and that the limit is independent of f. It clearly suffices to show that

$$
\lim_{\varepsilon \to 0,\, \delta \to 0} E\{\gamma_{2,\varepsilon}(f: t, \phi)\gamma_{2,\delta}(f: t, \phi)\} = \text{constant},
$$

and that if $f \neq f'$, then $\lim_{\varepsilon \to 0} E|\gamma_{2,\varepsilon}(f: t, \phi) - \gamma_{2,\varepsilon}(f': t, \phi)|^2 = 0$ for every ϕ and t. We shall show only the first of these, since the second follows via similar calculations.

Using the symmetry properties of p^α, it follows from Formula 2.5 that the expectation above is given by

$$\begin{aligned}(3.5)\quad &\int_{[0,t]^4}\int_{\Re^{4d}} p^\alpha_{|s-u|}(x,z)p^\alpha_{|r-v|}(y,w)\,f_\varepsilon(x-y)\,f_\delta(z-w)\phi(x)\phi(z)\\ &+\int_{[0,t]^4}\int_{\Re^{4d}} p^\alpha_{|s-v|}(x,w)p^\alpha_{|r-u|}(y,z)\,f_\varepsilon(x-y)\,f_\delta(z-w)\phi(x)\phi(z),\end{aligned}$$

where we have left out the eight differentials to save space.

To show that each of these integrals has a finite $\varepsilon, \delta \to 0$ limit, we can use a dominated convergence argument. Consider the second integral. The first is similar, but a little more complicated. Let

$$\begin{aligned}(3.6)\quad D_{\varepsilon,\delta}(x,z) = \int_{[0,t]^4} dr\,ds\,du\,dv \int_{\Re^{2d}} p^\alpha_{|s-v|}(x,w)p^\alpha_{|r-u|}(y,z)\\ \times f_\varepsilon(x-y)\,f_\delta(z-w)\,dy\,dw.\end{aligned}$$

Since for all $x \neq z$ the function $D_{\varepsilon,\delta}(x,z)$ converges to a well-defined limit when $\varepsilon, \delta \to 0$, we need only show that $D_{\varepsilon,\delta}(x,z)$ is bounded by a function which, when multiplied by $\phi(x)\phi(z)$, is integrable. It then follows that the second integral in (3.5) has a limit, as required. To do this, extend the range of the time integrals from $[0,t]^4$ to $\Re^4_+$ and then perform these integrations. Then for some constant C,

$$\begin{aligned}D_{\varepsilon,\delta}(x,z) &\le C\int_{\Re^{2d}} G^0_\alpha(x,w)G^0_\alpha(z,y)\,f_\varepsilon(x-y)\,f_\delta(z-w)\,dy\,dw\\ (3.7)\quad &= C\int_{\Re^d} G^0_\alpha(x,w)\,f_\delta(z-w)\,dw\int_{\Re^d} G^0_\alpha(z,y)\,f_\varepsilon(x-y)\,dy\\ &\le C\|x-z\|^{2(\alpha-d)},\end{aligned}$$

where the last line follows from Formula 2.1 and Lemma 2.3 as long as $\alpha \neq d$. [For the $\alpha = d$ case, see the next section.] Since the integrability problem for $\|x-z\|^{2(\alpha-d)}\phi(x)\phi(z)$ is at the origin, it is now immediate that $2\alpha > d$ is sufficient to ensure integrability. □

In fact, one can now go somewhat further and show that the $\mathscr{L}^2$ limit, established independently for each test function ϕ, can be extended to define γ_2 as a proper distribution. The proof of this fact is based on a result of Martin-Löf (1976) (Lemma 4) for Gaussian distributions and follows almost exactly as in the proof of the last part of Theorem 2.2 in AFL.

(c) *Theorem* 1.4—*the evolution equation for ILT*: $k = 2$ *and* $d < 2\alpha$. We now wish to show that if $d < 2\alpha$, we can send $\varepsilon \to 0$ in every term in (1.18) and so obtain the evolution equation described in the statement of the theorem. The simplest way to do this is to show that each term has a well defined $\mathscr{L}^2$ limit as $\varepsilon \to 0$. We have already treated $\gamma_{2,\varepsilon}(t,\phi)$ itself in the previous subsection. We now claim that each of the first three terms on the right-hand side of (1.18) can be handled similarly and that this is standard enough to be left to the reader. [To see that this claim is justified, you should either look at the corresponding argument (albeit for a special case) in AFL or wait until you have mastered the moment calculations of the following section, after which this one will seem easy.]

Since four of the five terms of (1.18) are now $\mathscr{L}^2$ convergent, the same must be true of the fifth; that is, the stochastic integral term. (A priori, this is a harder term to handle than the others, but this observation makes a separate analysis of it unnecessary.) This is enough to prove the result. □

(*d*) *Theorem* 1.5—*renormalisation*: $k = 2$ *and* $d \geq 2\alpha$. Consider the evolution equation (1.18) describing $\gamma_{2,\varepsilon}$ and the power and logarithmic renormalisations of Theorem 1.5. The first claim that we make is that if we divide any of the first three terms on the right-hand side of (1.18) by the renormalisation demanded in Theorem 1.5, then its $\mathscr{L}^2$ limit, as $\varepsilon \to 0$, is 0. We leave this to the reader.

This being the case, we need only concentrate on the stochastic integral term, which we denote by

$$I_t(\varepsilon) =: 2\sqrt{2}\int_0^t\int_0^s\int_{\Re^d}\eta_u\big(\Delta_{\alpha/2}G_\alpha^\lambda * f_\varepsilon(x - \cdot)\phi(\cdot)\big)\,du\,W(dx,ds). \tag{3.8}$$

Note that $I_t(\varepsilon)$ depends on both f and ϕ, as well as λ and α, although we do not explicitly display this in our notation. It is, of course, a martingale.

Now let

$$\begin{aligned} A_t(\varepsilon) &= \langle I_t(\varepsilon)\rangle_t \\ &= 8\int_0^t ds\int_0^s\int_0^s\int_{\Re^d}\eta_{u_1}\big(\Delta_{\alpha/2}G_\alpha^\lambda * f_\varepsilon(x - \cdot)\phi(\cdot)\big) \\ &\qquad \times \eta_{u_2}\big(\Delta_{\alpha/2}G_\alpha^\lambda * f_\varepsilon(x - \cdot)\phi(\cdot)\big)\,du_1\,du_2\,dx \end{aligned} \tag{3.9}$$

be the predictable, increasing process associated with $I_t(\varepsilon)$ and note that the increasing process associated with the $W_t^{(k)}(\phi)$ of (1.20) is $t^k\int_{\Re^d}\phi^2(x)\,dx$. Then Theorem 8.3.11 of Jacod and Shiryaev (1987) implies that in order to prove Theorem 1.5, it suffices to show that when $d = 2\alpha$,

$$\frac{A_t(\varepsilon)}{\ln(1/\varepsilon)} \to_P t^2\theta^2\int_{\Re^d}\phi^2(x)\,dx \tag{3.10}$$

as $\varepsilon \to 0$, where $\theta^2 = 8c_{d,\alpha}^2 s_d$, and when $d > 2\alpha$,

$$E\left\{\frac{A_t(\varepsilon)}{\varepsilon^{2\alpha-d}}\right\} \to_P t^2B_f^2\int_{\Re^d}\phi^2(x)\,dx \tag{3.11}$$

as $\varepsilon \to 0$, where $B_f^2 = 8\int_{\Re^d}(G_\alpha^0 * f * f)(x)G_\alpha^0(x)\,dx$.

In order to establish (3.10) and (3.11) we shall show that the expectations of the left-hand sides converge to the expressions on the right and then that the variances of the left-hand sides converge to zero. This will be enough.

We start by calculating the expectation of the increasing process A_t. With f fixed and with the properties required in the statement of the theorem, set

$$F(x) = f * f(x) = \int_{\Re^d} f(x-y) f(y)\, dy \tag{3.12}$$

and $F_\varepsilon(x) = f_\varepsilon * f_\varepsilon(x) = \varepsilon^{-d} F(x/\varepsilon)$. It then follows from (3.9) and (1.5) that

$$EA_t(\varepsilon) = 16 \int_0^t ds \int_0^s du_1 \int_0^{u_1} du_2 \int_{\Re^{3d}} \Delta_{\alpha/2} G_\alpha^\lambda * f_\varepsilon(x - y_1)\phi(y_1)$$
$$\times p_{u_1 - u_2}(y_1 - y_2) \Delta_{\alpha/2} G_\alpha^\lambda * f_\varepsilon(x - y_2)\phi(y_2)\, dx\, dy_1\, dy_2.$$

Simplify this using the integration by parts formula (2.2) and note that G_α^λ solves the equation $(-\Delta_\alpha + \lambda)u = \delta$, to obtain

$$EA_t(\varepsilon) = 16 \int_0^t ds \int_0^s du_1 \int_0^{u_1} du_2 \int_{\Re^{2d}} p_{u_1 - u_2}(y_1 - y_2)\phi(y_1)\phi(y_2)$$
$$\times \left[G_\alpha^\lambda * F_\varepsilon(y_1 - y_2) - \lambda G_\alpha^\lambda * G_\alpha^\lambda * F_\varepsilon(y_1 - y_2) \right] dy_1\, dy_2.$$

Since this is true for all $\lambda > 0$, it should also be true for $\lambda = 0$, as long as the limit is obtained in a continuous fashion. We claim, and leave it to the reader to check, that this is in fact the case. Thus, setting $\lambda = 0$ in the above we obtain

$$EA_t(\varepsilon) = 16 \int_0^t ds \int_0^s du \int_0^{s-u} dv \int_{\Re^{2d}} p_v(y_1) G_\alpha^0 * F_\varepsilon(y_1)$$
$$\times \phi(y_1 + y_2)\phi(y_2)\, dy_1\, dy_2$$
$$= 16 \int_0^t ds \int_0^s du \int_0^{s-u} dv \int_{\Re^d} \Phi(y) p_v(y) G_\alpha^0 * F_\varepsilon(y)\, dy,$$

where $\Phi(y) = \int_{\Re^d} \phi(y + x)\phi(x)\, dx$. Note that

$$\int_0^{s-u} p_v(y)\, dv = G_\alpha^0(y) - \int_{s-u}^\infty p_v(y)\, dv, \tag{3.13}$$

and substitute into the last line above, obtaining two terms. The first of these is

$$16 \int_0^t ds \int_0^s du \int_{\Re^d} \Phi(y) G_\alpha^0(y) G_\alpha^0 * F_\varepsilon(y)\, dy$$
$$= 8t^2 \int_{\Re^d} \Phi(y) G_\alpha^0(y) G_\alpha^0 * F_\varepsilon(y)\, dy \tag{3.14}$$
$$= 8t^2 \varepsilon^{2\alpha - d} \int_{\Re^d} \Phi(\varepsilon x) G_\alpha^0(x) G_\alpha^0 * F(x)\, dx,$$

where the last line follows from the scaling relationship inherent in Formula 2.1.

CASE 1. Assume now that $d > 2\alpha$. Since, by definition, $\Phi(0) = \int \phi^2(x)\,dx$ and Φ is Hölder-continuous of any order $\beta \in [0, 1]$, it is easy to check that the last integral in (3.14) converges to $B_f^2 \int \phi^2(x)\,dx$ as $\varepsilon \to 0$.

Furthermore, we claim that the second term arising from the difference (3.13) is $o(\varepsilon^{2\alpha-d})$ and therefore not important. Again, the details are similar to calculations you will find in the following section. This, of course, implies that when $d > 2\alpha$,

$$E\left\{ \frac{A_t(\varepsilon)}{\varepsilon^{2\alpha-d}} \right\} \to t^2 B_f^2 \int_{\Re^d} \phi^2(x)\,dx$$

as $\varepsilon \to 0$, which is almost (3.11). To complete the proof in the case $d > 2\alpha$, we need only show that the variance of the above ratio converges to zero.

The same considerations that lead to Formulae 2.5 and 2.6 show that the variance of $A_t(\varepsilon)$ is given by the sum of two terms, each one of which is bounded by an expression of the form

$$C \int_{[0,t]^4} \int_{\Re^{4d}} p_{|t_1 - s_1|}(x_1 - y_1) p_{|t_2 - s_2|}(x_2 - y_2) \phi(x_1)\phi(x_2)\phi(y_1)\phi(y_2) \\ \times G_\alpha^0 * F_\varepsilon(x_1 - x_2) G_\alpha^0 * F_\varepsilon(y_1 - y_2)\, ds_1\, ds_2\, dt_1\, dt_2\, dx_1\, dx_2\, dy_1\, dy_2,$$

where once again we have integrated by parts, used (2.4) and glibly taken $\lambda = 0$. To simplify this, add a factor of $\exp(-|t_1 - s_1| - |t_2 - s_2|)$ to the integrand, extend the time integrals from $[0, t]^4$ to $\Re_+^4$, and perform the time integrations to obtain a bound of the form

$$\begin{aligned} & C \int_{\Re^{4d}} G_\alpha^1(x_1 - y_1) G_\alpha^1(x_2 - y_2) G_\alpha^0 * F_\varepsilon(x_1 - x_2) G_\alpha^0 * F_\varepsilon(y_1 - y_2) \\ & \qquad \times \phi(x_1)\phi(x_2)\phi(y_1)\phi(y_2)\, dx_1\, dx_2\, dy_1\, dy_2 \\ & \le C \int_{\Re^2} \left(G_\alpha^1 * G_\alpha^0 * F_\varepsilon(y - x)\right)^2 \phi(y)\, dx\, dy \\ & = C G_\alpha^1 * G_\alpha^1 * G_\alpha^0 * G_\alpha^0 * F_\varepsilon * F_\varepsilon(0), \end{aligned} \tag{3.15}$$

where the last line follows from the fact that for any symmetric function $g(x)$, $\int g^2(x)\,dx = g * g(0)$ and the constant now includes $\sup_y |\phi(y)|$ and $\int_{\Re^d} \phi(y)\,dy$.

Set $H_\varepsilon(x) = F_\varepsilon * F_\varepsilon(x)$ and note that H_ε satisfies the usual scaling relation: That is, $H_\varepsilon(x) = \varepsilon^{-d} H(x/\varepsilon)$, with $H \equiv H_1$. Again applying the equivalence between $\int g^2(x)\,dx$ and $g * g(0)$, rewrite the last line of (3.15) as

$$\begin{aligned} & \int_{\Re^d} G_\alpha^1 * G_\alpha^1(x) G_\alpha^0 * G_\alpha^0 * H_\varepsilon(x)\,dx \\ & \qquad = \int_{|x| \le 1} G_\alpha^1 * G_\alpha^1(x) G_\alpha^0 * G_\alpha^0 * H_\varepsilon(x)\,dx + O(1), \end{aligned}$$

where the $O(1)$ comes from the fact that H_ε has bounded support.

Recall the Useful Fact 2.2, and the fact that $G_\alpha^1(x) \le G_\alpha^0(x)$ for all x implies that $C\|x\|^{2\alpha-d}$ serves as an upper bound for $G_\alpha^0 * G_\alpha^0(x)$. These bounds, along with scaling, yield that the first term above is bounded above by

$$\varepsilon^{4\alpha-d}\int_{\|x\|\le\varepsilon^{-1}}\|x\|^{2\alpha-d}\left(\int_{\Re^d}\|x+y\|^{2\alpha-d}H(y)\,dy\right). \tag{3.16}$$

Once again, we must split the argument, this time into three separate cases. First, assume that $d > 4\alpha$. Then the integral in (3.16) is convergent and thus

$$\operatorname{var}\left\{\frac{A_t(\varepsilon)}{\varepsilon^{2\alpha-d}}\right\} \le C\varepsilon^d,$$

which obviously goes to zero as $\varepsilon \to 0$. If $d < 4\alpha$, then the integral in (3.18) is of the order of

$$\int_{1\le\|x\|\le 1/\varepsilon}\|x\|^{4\alpha-2d}\,dx = O(\varepsilon^{d-4\alpha}),$$

so that in this case,

$$\operatorname{var}\left\{\frac{A_t(\varepsilon)}{\varepsilon^{2\alpha-d}}\right\} \le C\varepsilon^{2(d-2\alpha)},$$

which, since we are still in the case $d > 2\alpha$, also goes to zero as $\varepsilon \to 0$. If $d = 4\alpha$, then the integral in (3.16) is of order $\log(1/\varepsilon)$ and the same conclusion holds. This concludes the case $d > 2\alpha$.

CASE 2. It remains to treat the case $d = 2\alpha$, in order to complete the proof of Theorem 1.5. The primary difference in this case is, obviously, in the treatment of the last term of (3.14). Since $d = 2\alpha$, there is no power of ε before the integral. What one has to show is that as $\varepsilon \to 0$, the integral itself has a logarithmic singularity. The second part of the argument, that is, showing that the variance term converges to zero, is as in the previous case. Details are left to the reader. □

(*e*) *Theorem* 1.6—*fluctuation theory.* We claim that the fluctuation result of Theorem 1.6 can, for the case $k = 2$, also be proven from the evolution equation of Theorem 1.4, much along the lines of the previous proof. To be honest, however, we should note that we have not carried out the detailed calculations necessary to justify our claim. Since the previous proof, treating the divergent case, serves the didactic role we required of it, and an alternative proof for Theorem 1.6, valid in greater generality, follows in Section 5 below, there did not seem to be sufficient justification to do so.

4. Proof by moment analysis. In this section we shall prove our results relating to the $\mathscr{L}^2$ convergence of the ILT when this occurs, and the weak convergence of the renormalised ILT when $\mathscr{L}^2$ convergence fails, via the method of moments. The following section will treat the fluctuation result—Theorem 1.6—for the former case.

(*a*) *Proof of Theorem* 1.3. We assume that $\alpha > d(k-1)/k$ and have to prove the $\mathscr{L}^2$ convergence of the approximate ILT $\gamma_{k,\varepsilon}$. It suffices, of course, to show that $E\gamma_{k,\varepsilon}\gamma_{k,\eta}$ converges to a finite constant as $\varepsilon, \eta \to 0$.

Let $\pi = (\pi_1, \dots, \pi_k)$ represent a permutation of $(1, \dots, k)$ and let $\mathscr{P}_k$ be the collection of all such permutations. Then from the definition (1.16) of $\gamma_{k,\varepsilon} = \gamma_{k,\varepsilon}(f\colon t, \phi)$ and the moment formula (2.11) we have that

$$E\gamma_{k,\varepsilon}\gamma_{k,\eta} = \sum_{\pi \in \mathscr{P}_k} \int_{[0,t]^{2k}} \int_{\Re^{2k}} \phi(x_1)\phi(y_1) \prod_{i=1}^{k} p^{\alpha}_{|t_i - s_i|}(x_i - y_{\pi_i}) \times \prod_{i=2}^{k} f_\varepsilon(x_i - x_1) f_\eta(y_i - y_1) \prod_{i=1}^{k} ds_i\, dt_i\, dx_i\, dy_i.$$

(Note that here, as in similar formulae below, the subscripts following the various products are not purely dummy subscripts, that is, the ith subscript of each product must match with the ith subscript of the others.)

Change variables in the above as follows: $\bar{x}_i = x_i - x_1$, $\bar{y}_i = y_i - y_1$, $i = 1, \dots, k$, $x = x_1$ and $z = x_1 - y_1$. Note that $\bar{x}_1$ and $\bar{y}_1$ are both identically zero and so their appearance below is as dummies. Nevertheless, they serve to keep the formulae relatively tidy. Then $E\gamma_{k,\varepsilon}\gamma_{k,\eta}$ is equal to

$$\sum_{\pi \in \mathscr{P}_k} \int_{[0,t]^{2k}} \int_{\Re^{2k}} \phi(x)\phi(x-z) \prod_{i=1}^{k} p^{\alpha}_{|t_i - s_i|}(z + \bar{x}_i - \bar{y}_{\pi_i}) \times \prod_{i=2}^{k} f_\varepsilon(\bar{x}_i) f_\eta(\bar{y}_i)\, dx\, dz \prod_{i=2}^{k} d\bar{x}_i\, d\bar{y}_i \prod_{i=1}^{k} ds_i\, dt_i.$$

Setting $\Phi(z) = \phi * \phi(z) = \int \phi(x)\phi(x+z)\, dx$ and

$$G^{(t)}_{\alpha}(x) = \tfrac{1}{2}\int_0^t dr \int_0^t ds\, p^{\alpha}_{|r-s|}(x) = \int_0^t dr \int_0^r p^{\alpha}_s(x)\, ds, \tag{4.1}$$

we obtain

$$E\gamma_{k,\varepsilon}\gamma_{k,\eta} = \sum_{\pi \in \mathscr{P}_k} 2^k \int_{\Re^{2k}} \Phi(z) \prod_{i=1}^{k} G^{(t)}_{\alpha}(z + \bar{x}_i - \bar{y}_{\pi_i}) \times \prod_{i=2}^{k} f_\varepsilon(\bar{x}_i) f_\eta(\bar{y}_i)\, dz \prod_{i=2}^{k} d\bar{x}_i\, d\bar{y}_i. \tag{4.2}$$

Note that so far we have not used the assumption $d < k\alpha/(k-1)$, so that (4.2) is valid in general. Now, however, we shall use it. Note first that, for each $i = 1, \dots, k$,

$$\int_{\Re^{2k}} G^{(t)}_{\alpha}(z + \bar{x}_i - \bar{y}_{\pi_i}) f_\varepsilon(\bar{x}_i) f_\eta(\bar{y}_i)\, d\bar{x}_i\, d\bar{y}_i \to G^{(t)}_{\alpha}(z)$$

as $\varepsilon, \eta \to 0$ and that

$$(4.3)\quad G_\alpha^{(t)}(z) \le te^t G_\alpha^1(z) \le \begin{cases} \text{const} \cdot \|z\|^{\alpha-d}, & \text{for } \alpha < d, \\ \text{const} \cdot \log(1/\|z\|), & \text{for } \alpha = d = 1, 2, \end{cases}$$

where the last inequality is a consequence of Formula 2.1 for $\alpha < d$, Formulae 2.3 and well-known properties of Green's functions for the cases $\alpha = d = 2$ and $\alpha = d = 1$. Dominated convergence then gives us that, under the condition $d < k\alpha/(k-1)$,

$$(4.4)\qquad \lim_{\varepsilon, \eta \to 0} E\gamma_{k,\varepsilon}\gamma_{k,\eta} = k!2^k \int_{\Re^d} \left(G_\alpha^{(t)}(z)\right)^k \Phi(z)\, dz.$$

This proves the $\mathscr{L}^2$ convergence of Theorem 1.3. What remains to be proved is that the limit of $\gamma_{k,\varepsilon}(f\colon t, \phi)$ is actually independent of f. This is left to the reader and can be shown by noting that if f and f' are two functions in C_0^∞ satisfying the conditions of the theorem, then

$$\lim_{\varepsilon \to 0} E|\gamma_{k,\varepsilon}(f\colon t, \phi) - \gamma_{k,\varepsilon}(f'\colon t, \phi)|^2 = 0$$

for every ϕ and t. The calculations are similar to those we have just carried out. □

(*b*) *Second moment calculations for the* $\mathscr{L}^2$ *divergent case.* In the following subsection we shall prove the convergence of the univariate distributions required to establish Theorem 1.5. The argument will proceed by first showing convergence of all moment sequences and then noting that the limit moments are precisely those that characterise the limit process of the theorem. Since the limit process is Gaussian, the second moments play a central role in the argument.

Our starting point will be (4.2) with $\varepsilon \equiv \eta$, so that we need to study the terms appearing there a little more carefully. We start with the useful inequality [cf. (1.9)]

$$(4.5)\qquad \begin{aligned} p_t^\alpha(x) &= \frac{1}{(2\pi)^d} \int_{\Re^d} \exp(-ipx - t2^{-\alpha/2}\|p\|^\alpha)\, dp \\ &\le \text{const} \int_{\Re^d} \exp(-t2^{-\alpha/2}\|p\|^\alpha)\, dp \\ &= \text{const}\, t^{-d/\alpha}, \end{aligned}$$

the last line following from a scaling argument.

Note that since we are now interested only in the case $\alpha \le d(k-1)/k$, we have that $d > \alpha$, and so the Green's function G_α^λ is well defined for *all* $\lambda \ge 0$ and when $\lambda = 0$, it has the explicit form $G_\alpha^0(x) = c_{d,\alpha}\|x\|^{\alpha-d}$. We can therefore write

$$(4.6)\qquad \int_0^t p_s^\alpha(x)\, ds = G_\alpha^0(x) - \int_t^\infty p_s^\alpha(x)\, ds.$$

Note that (4.5) implies

$$\int_t^\infty p_s^\alpha(x)\,ds \le \text{const}\, t^{1-d/\alpha}. \tag{4.7}$$

Since it is also trivially true that $\int_t^\infty p_s^\alpha(x)\,ds$ is also bounded above by G_α^0, it follows that for any $\gamma \in [0,1]$, we have

$$\int_t^\infty p_s^\alpha(x)\,ds \le \text{const}\,\frac{\left(G_\alpha^0(x)\right)^\gamma}{t^{(d/\alpha-1)(1-\gamma)}}. \tag{4.8}$$

Fix $\delta > 0$, and use the definition (4.1) of $G_\alpha^{(t)}$, (4.6) and (4.8) to see that, for any $\gamma \in [0,1]$,

$$\begin{aligned} G_\alpha^{(t)}(x) &= \left(\int_0^\delta dr + \int_\delta^t dr\right)\left(\int_0^r p_s^\alpha(x)\,ds\right) \\ &= O(\delta)G_\alpha^0(x) + O(\delta^{-(1-\gamma)(d/\alpha-1)})\left(G_\alpha^0(x)\right)^\gamma \\ &\quad + (t-\delta)G_\alpha^0(x). \end{aligned} \tag{4.9}$$

We are now in a position to tackle the following version of (4.2):

$$\begin{aligned} E\gamma_{k,\varepsilon}^2 = \sum_{\pi\in\mathscr{P}_k} 2^k \int \Phi(z) \prod_{i=1}^k G_\alpha^{(t)}\left(z + \bar{x}_i - \bar{y}_{\pi_i}\right) \\ \times \prod_{i=2}^k f_\varepsilon(\bar{x}_i) f_\varepsilon(\bar{y}_i)\,dz \prod_{i=2}^k d\bar{x}_i\,d\bar{y}_i. \end{aligned} \tag{4.10}$$

We shall show that if we replace $G_\alpha^{(t)}$ in this (divergent as $\varepsilon \to 0$) expression by (4.9), then the main part of the divergence comes from the last term in (4.9). Furthermore, we shall identify the rate of divergence. To do this precisely, consider the following term, one of many that come from such a substitution:

$$\begin{aligned} \sum_{\pi\in\mathscr{P}_k} (t-\delta)^k 2^k \int_{\Re^k} \Phi(z) \prod_{i=1}^k G_\alpha^0\left(z + \bar{x}_i - \bar{y}_{\pi_i}\right) \\ \times \prod_{i=2}^k f_\varepsilon(\bar{x}_i) f_\varepsilon(\bar{y}_i)\,dz \prod_{i=2}^k d\bar{x}_i\,d\bar{y}_i. \end{aligned} \tag{4.11}$$

We shall treat this by treating the two cases $\alpha = d(k-1)/k$ and $\alpha < d(k-1)/k$ separately, and then in each case breaking the z integral in (4.11) into the two integrals corresponding to $\|z\| \le 4\varepsilon$ and $\|z\| > 4\varepsilon$. We start with the case $\alpha = d(k-1)/k$ and commence by recalling from Formula 2.1 and Lemma 2.3 that

$$\int f_\varepsilon(u) G_\alpha^0(z+u)\,du \le \begin{cases} C\|z\|^{\alpha-d}, & \text{if } \|z\| \ge 4\varepsilon, \\ C\varepsilon^{\alpha-d}, & \text{if } \|z\| \le 4\varepsilon. \end{cases}$$

Since we have $\alpha = d(k-1)/k$, it is easy to see that doing the full $\bar{x}_i$, $\bar{y}_i$

integrals in (4.11) and integrating z over the sphere $\|z\| \le 4\varepsilon$ yields a result that is $O(1)$.

In the region $\|z\| > 4\varepsilon$, we replace $G_\alpha^0(z + \bar{x}_i - \bar{y}_{\pi_i})$ by $G_\alpha^0(z)$ for an error of

$$\big|\|z + \bar{x}_i - \bar{y}_{\pi_i}\|^{\alpha-d} - \|z\|^{\alpha-d}\big| \le C\frac{\|\bar{x}_i - \bar{y}_{\pi_i}\|}{\|z\|^{d-\alpha+1}},$$

which, after integration, also yields a factor of $O(1)$. (Note the importance here of the fact that $\Phi \in \mathscr{S}_d$, which ensures the convergence of the integral.) Therefore it remains to consider what happens after this replacement. Since the $\bar{x}_i$ and $\bar{y}_{\pi_i}$ integrals each give a unit contribution, all that we need to consider is

$$\int_{\|z\|>4\varepsilon} \Phi(z)\big(G_\alpha^0(z)\big)^k\,dz.$$

Recall that Φ is rapidly decreasing at infinity and that it is C^∞. To make our lives a little easier, we shall also assume that it has compact support, which we shall denote by A. This assumption makes no intrinsic difference to the following argument, but makes the notation a little easier. Thus, throughout its support we have $\Phi(z) = \Phi(0) + D(z)$, where $D(z) \le C\|x\|$ for some finite C. Consequently, the above integral is equal to

$$\Phi(0)\int_{\|z\|>4\varepsilon} I_A(z)\big(G_\alpha^0(z)\big)^k\,dz + \int_{\|z\|>4\varepsilon} I_A(z)D(z)\big(G_\alpha^0(z)\big)^k\,dz.$$

Since G_α^0 is given by Formula 2.1, it is immediate from the fact that $\alpha = d(k-1)/k$, that is, $k = d/(d-\alpha)$, that the second integral is $O(1)$, while the first is equal to

$$\Phi(0)c_{d,\alpha}^k\int_{\|z\|>4\varepsilon}\frac{I_A(z)}{\|z\|^d}\,dz = \Phi(0)C_{d,\alpha}^k s_d\log\left(\frac{1}{\varepsilon}\right) + O(1).$$

[The $O(1)$ term comes from the boundedness of A. Recall that s_d is given by (1.19).]

To complete the calculation of $E\gamma_{k,\varepsilon}^2$ for this case we need to consider the effects that the other terms in (4.9) have on (4.10). It suffices, for the moment, to take $\gamma = 0$ in (4.9). It is then easy to see, relying on the above calculations where necessary, and summing over the $k!$ permutations of (4.10), that

$$E\gamma_{k,\varepsilon}^2 = 2^k k!(t-\delta)^k\Phi(0)c_{d,\alpha}^k s_d\log(1/\varepsilon) + O(\delta\log(1/\varepsilon)) + O(\delta^{k(1-d/\alpha)}).$$

By sending first $\varepsilon \to 0$ and then $\delta \to 0$, we thus obtain

$$\text{(4.12)} \qquad \lim_{\varepsilon\to 0}\frac{E\gamma_{k,\varepsilon}^2}{\log(1/\varepsilon)} = 2^k k!t^k\Phi(0)c_{d,\alpha}^k s_d.$$

This completes the case $\alpha = d(k-1)/k$. We therefore now assume that $\alpha < d(k-1)/k$ and continue, as before, from (4.11). Consider the integral there, and note that since it is over a finite region, the domain can be taken to be $\|z\| \le 1$, with the introduction of an $O(1)$ error only. Now transform

$\bar{x}_i/\varepsilon \to x_i$, $\bar{y}_i/\varepsilon \to y_i$ and $z \to \varepsilon z$. Then use the scaling inherent in Formula 2.1 to see that for fixed x_i and y_4, the z integral is equivalent to

$$\varepsilon^{d-k(d-\alpha)} \int_{\|z\| \le \varepsilon^{-1}} \Phi(\varepsilon z) \prod_{i=1}^{k} G_\alpha^0(z + x_i - y_{\pi_i}) \times \prod_{i=2}^{k} f(x_i) f(y_i)\, dz \prod_{i=2}^{k} dx_i\, dy_i. \tag{4.13}$$

Introduce the function $K: (\Re^d)^k \times (\Re^d)^k \to \Re^1$ by

$$K(x_1, \dots, x_k, y_1, \dots, y_k) = \sum_{\pi \in \mathscr{P}_k} \int_{\Re^d} \prod_{i=1}^{k} G_0^\alpha(z + x_i + y_{\pi_i})\, dz. \tag{4.14}$$

Note that the integral here is well defined for almost every x_i and y_i, and, because of the symmetry properties of G_α^0, is separately symmetric in the x_i's and y_i's. Recall, also, that in (4.13), x_1 and y_1 are actually zero, and have appeared in all of the above as dummy variables.

Now, following the lines of the previous argument, replace $\Phi(\varepsilon z)$ in (4.13) by $\Phi(0) + O(\varepsilon z)$, again relying on the assumed compactness of the support of Φ and the fact that it is C^∞. Then, since $\alpha < d(k-1)/k$, the integral there (i.e., without the initial factor of ε) converges for all $\varepsilon \ge 0$ and so we have that (4.13) is equal to

$$\varepsilon^{d-k(d-\alpha)} \int_{\|z\| \le \varepsilon^{-1}} \Phi(0) K(0, x_2, \dots, x_k, 0, y_2, \dots, y_k) \times \prod_{i=2}^{k} f(x_i) f(y_i)\, dz \prod_{i=2}^{k} dx_i\, dy_i + o(\varepsilon^{d-k(d-\alpha)}).$$

Substitute this into (4.11) and also consider the other terms that arise out of (4.10) from the substitution of (4.9). This time, however, take γ close to 1 in (4.9). Then it is easy to see that

$$\lim_{\varepsilon \to 0} \frac{E\gamma_{k,\varepsilon}^2}{\varepsilon^{d-k(d-\alpha)}} = 2^k t^k \Phi(0) \int K(0, x_2, \dots, x_k, 0, y_2, \dots, y_k) \times \prod_{i=2}^{k} f(x_i) f(y_i)\, dz \prod_{i=2}^{k} dx_i\, dy_i. \tag{4.15}$$

This completes the second moment calculation for the case $\alpha < d(k-1)/k$. □

Before moving on to higher moments, it is worth noting that (4.12) and (4.15) give the appropriate variances for (1.21) and (1.22). In the former case this is immediate, once one takes into account (1.20) and the fact that $\Phi(0) = \int \phi^2(x)\, dx$. In the latter, note that there are $(k-1)!$ permutations in the $\mathscr{P}_k$ of (4.10) with $y_{\pi_i} = y_1$ to give the first summand in the B_f of (1.22), while the other $(k-1)(k-1)!$ terms lead to the second summand.

(*c*) *Proof of Theorem* 1.5—*univariate distributions*. In this subsection we shall tackle the hardest part of the proof of Theorem 1.5, in that we shall show the convergence of univariate distributions in (1.21) and (1.22). Multivariate distributions will follow in the following subsection and tightness after that. These will complete the proof of the theorem [cf. Walsh (1986)].

In view of the fact that we claim a Gaussian limit, it clearly suffices to show that for every even $n = 2m$,

$$E\gamma_{k,\varepsilon}^n = \frac{(2m)!}{2^m m!}\left(E\gamma_{k,\varepsilon}^2\right)^m + o\Big(\left(E\gamma_{k,\varepsilon}^2\right)^m\Big), \tag{4.16}$$

where o refers to $\varepsilon \to 0$, while for odd $n = 2m+1$,

$$\lim_{\varepsilon\to 0} E\left(\frac{\gamma_{k,\varepsilon}}{\sqrt{\log(1/\varepsilon)}}\right)^n = 0, \tag{4.17}$$

if $\alpha = d(k-1)/k$ or

$$\lim_{\varepsilon\to 0} E\left(\frac{\gamma_{k,\varepsilon}}{\varepsilon^{(d-k(d-\alpha))/2}}\right)^n = 0, \tag{4.18}$$

if $\alpha < d(k-1)/k$. We start with (4.16).

To evaluate the left-hand side of (4.16), we start with Formula 2.5, which, in the notation of Section 2(c), means summing over all graphs $G \in \mathscr{G}_{n,k}$, expressions of the form

$$\begin{aligned} I_\varepsilon(G) = 2^{nk/2} \int \prod_{v\in G} \phi(z_v) \prod_{L\in G} G_\alpha^{(t)}(z_\nu - z_{\bar\nu} + x_{\nu,l} - x_{\bar\nu,\bar l}) \\ \times \prod_v \prod_{l=2}^k f_\varepsilon(x_{v,l}), \end{aligned} \tag{4.19}$$

where the passage from Formula 2.5 to this result involves integrating out the time variables as in (4.1) and performing the same type of substitution required to get to (4.2). The indices v, ν and $\bar\nu$ relate to vertices of the graph G, while l and $\bar l$ refer to legs. Thus, for example, $L = ((v,l),(\bar v,\bar l))$ is a link from the lth leg on the vth vertex to the $\bar l$th leg on the $\bar v$th vertex.

It is immediate from the form of (4.19) that if G is made up of M disconnected components $G_1,\dots,G_M$, then

$$I_\varepsilon(G) = \prod_{i=1}^M I_\varepsilon(G_i). \tag{4.20}$$

The following important stage of our proof is an immediate consequence of this product formula, (4.19) and (4.10) and counting graphs:

4.1 Lemma. *Let $n = 2m$ and suppose that the graph G has m disjoint components, each with two vertices. Then*

$$I_\varepsilon(G) = \left(E\gamma_{k,\varepsilon}^2\right)^m. \tag{4.21}$$

Furthermore, there are $(2m)!/(2^m m!)$ such graphs G in $\mathscr{G}_{m,k}$.

This lemma shows where the first term on the right-hand side of (4.16) comes from. What remains is to show that all the remaining graphs in $\mathscr{G}_{m,k}$ contribute the remaining, lower order, term. In fact, it will be sufficient to show that for any connected graph G with at least three distinct vertices,

$$I_\varepsilon(G) = o\Big(\big(E\gamma_{k,\varepsilon}^2\big)^{m/2}\Big). \tag{4.22}$$

To start showing this, for $z \in \Re^d$, let

$$u_{\gamma,\varepsilon}(z) := (\max(\|z\|, \varepsilon))^{-\gamma}, \tag{4.23}$$

so that

$$u_{d-\alpha,\varepsilon}(z) = c_{d,\alpha}^{-1} \min\big(c_{d,\alpha}\varepsilon^{d-\alpha}, G_{d-\alpha}^0(z)\big), \tag{4.24}$$

the last line following from the fact that we are still in the case $\alpha < d(k-1)/k$ so that $d > \alpha$. Then we have:

4.2 Lemma. *Let G be a connected graph with at least three distinct vertices. Then*

$$I_\varepsilon(G) \le C\int \prod_{v\in G}\phi(z_v)\prod_{L\in G} u_{d-\alpha,\varepsilon}(z_\nu - z_{\bar\nu}). \tag{4.25}$$

Proof. Note first that, by (4.3), $I_\varepsilon(G)$ is bounded above by a k- and n-dependent, but ultimately unimportant, constant times

$$J_\varepsilon(G) := \int \prod_{v\in G}\phi(z_v)\prod_{L\in G} G_\alpha^0(z_\nu - z_{\bar\nu} + x_{\nu,l} - x_{\bar\nu,\bar l})\prod_v \prod_{l=2}^k f_\varepsilon(x_{v,l}). \tag{4.26}$$

Note also that by Lemma 2.4, we have

$$G_\alpha^0 * f_\varepsilon(z) \le Cu_{d-\alpha,\varepsilon}(z), \qquad G_\alpha^0 * f_\varepsilon * f_\varepsilon(z) \le Cu_{d-\alpha,\varepsilon}(z). \tag{4.27}$$

We shall use these two inequalities and (4.26) to prove the lemma. Comparison of (4.26) and (4.27) indicates how the proof must proceed: We need to integrate out the x variables in (4.26) by putting them into convolution with G_α^0 and then bound this by (4.27). The main technical problem arises from an essential asymmetry in the variables in (4.26), for, although everything looks symmetric, it is important to remember that for the first leg out of each vertex [i.e., a leg of the form $(v, 1)$], we have $x_{v,l} \equiv 0$.

Suppose that in the graph G there are no links made up between first legs of different vertices. Then one can do the x integrals in (4.26) as just described and so (4.25) follows from (4.26).

On the other hand, suppose that there is a one link between the two first legs of vertices ν and $\bar\nu$. Then one of the terms in (4.26) is actually of the form $G_\alpha^0(z_\nu - z_{\bar\nu})$ and there are no corresponding x terms. All other terms in the product, after doing all the x integrals, are of the form $G * f_\varepsilon(z_v - z_{\bar v})$ or $G * f_\varepsilon * f_\varepsilon(z_v - z_{\bar v})$ (including the possibility that one of our specific vertices ν

and $\bar{\nu}$ appears) and they can be nicely bounded, in preparation for (4.25), by (4.27). We now have to provide a similar bound for the term involving $G_\alpha^0(z_\nu - z_{\bar{\nu}})$.

Note, however, that over that part of the integral for which $\|z_\nu - z_{\bar{\nu}}\| > \varepsilon$, we can replace $G_\alpha^0(z_\nu - z_{\bar{\nu}})$ by $u_{d-\alpha,\varepsilon}(z_\nu - z_{\bar{\nu}})$ for the cost only of a multiplicative constant. Therefore, we restrict attention to $\|z_\nu - z_{\bar{\nu}}\| \le \varepsilon$. Consider fixed z_ν for the moment and bound all the u factors involving $z_{\bar{\nu}}$ by

$$u_{d-\alpha,\varepsilon}(z_v - z_{\bar{\nu}}) \le C u_{d-\alpha,\varepsilon}(z_v - z_\nu). \tag{4.28}$$

Once this has been done, the only place $z_{\bar{\nu}}$ still appears is in $G_\alpha^0(z_\nu - z_{\bar{\nu}})$. However,

$$\begin{aligned}\int G_\alpha^0(z_\nu - z_{\bar{\nu}})\, dz_\nu\, dz_{\bar{\nu}} &= C\int_{\|z\|<\varepsilon} \|z\|^{\alpha-d}\, dz \\ &= C\varepsilon^{d-(d-\alpha)} \\ &= \int u_{d-\alpha}(z_\nu - z_{\bar{\nu}})\, dz_\nu\, dz_{\bar{\nu}}.\end{aligned}$$

Thus, $u_{d-\alpha}(z_\nu - z_{\bar{\nu}})$ can be used to replace $G_\alpha^0(z_\nu - z_{\bar{\nu}})$ throughout the integral. Now use (4.28) once again, this time to replace the $z_{\bar{\nu}}$ factors lost above, and the lemma is proven also for this case. A similar argument holds if more than one set of first legs match up and this completes the proof in general. □

The first step towards exploiting Lemma 4.2 is the following lemma. A collection of vertices and legs, $\{v_i, l_{i,1}, l_{i,2}\}_{i=1}^N$ in G is called a *chain* of length N if all the links $L = ((v_i, l_{i,2}), (v_{i+1}, l_{i+1,1}))$, $i = 1, \dots, N-1$, appear in G. The same collection, but without either one or both of the legs $l_{1,1}$, $l_{N,2}$, is also a chain. If the link $((v_1, l_{1,1}), (v_N, l_{N,2}))$ also appears in G, the collection is called a *cycle* of length N. The numbering system used in this definition, is, of course, incidental to the definition.

4.3 LEMMA. *All connected graphs G containing at least three distinct vertices also contain a cycle of length at least three.*

PROOF. Denote the vertices of G by $v_1, \dots, v_N$, $N \ge 3$. Let v_1 be linked to v_2. Since G is connected and there is at least one more vertex, not all legs issuing from v_2 can be connected to legs of v_1. Let one of the legs of v_2 be linked to a leg of v_3. If v_3 is linked to v_1, we are done. If not, the same argument shows that not all legs issuing from v_3 can be linked to those of v_2 and so there must be a link between v_3 and some v_4. Continuing in this fashion, we find that the finiteness of the graph implies there must be a cycle of length at least three. □

We shall also require the following technical result.

4.4 LEMMA. *Let* $n \geq 1$, $\beta > 0$, $\gamma = n\beta - (n-1)d$ *and let* $u_{\gamma,\varepsilon}$ *be as defined at* (4.23). *Let* $\phi_1, \dots, \phi_n$ *be a sequence of functions from* $\mathscr{S}_d$. *Then*

$$\int_{\Re^{nd}} \prod_{i=1}^{n} \phi_i(z_i) u_{\beta,\varepsilon}(z_i - z_{i-1})\, dz_i \leq C \begin{cases} 1, & \text{if } \gamma < 0, \\ (\log(1/\varepsilon))^{n-1}, & \text{if } \gamma = 0, \\ \varepsilon^{-\gamma}, & \text{if } \gamma > 0, \end{cases} \tag{4.29}$$

where the constant C *depends on the* ϕ_i *and we set* $z_0 \equiv z_n$.

PROOF. We commence by showing that we can replace each of the ϕ_i of the lemma by the indicator function of the unit sphere.

Let $M = (m_1, \dots, m_d)$ be an integer lattice point in $\Re^d$, with each component m_i even and $M_1, \dots, M_n$ be n such points. Set $C_1 := \{z = (z_1, \dots, z_d) \in \Re^d\colon |z_i| \leq 1\}$ and note that $\Re^d = \bigcup_M (M + C_1)$, where $+$ is the usual set translation. Then the integral in (4.29), which we shall denote by I_ε^n, is clearly less than

$$\begin{aligned} &\sup_{M_1, \dots, M_n} \left(\prod_{i=1}^{n} \int_{z_i \in M_i + C_1} u_{\beta,\varepsilon}(z_i - z_{i-1})\, dz_i \right) \times \sum_{M_1, \dots, M_n} \left(\sup_{z_i \in M_i + C_1} \phi_i(z_i) \right)^n \\ &\quad \leq C \sup_{M_1, \dots, M_n} \left(\prod_{i=1}^{n} \int_{z_i \in M_i + C_1} u_{\beta,\varepsilon}(z_i - z_{i-1})\, dz_i \right) \\ &\quad \leq C \prod_{i=1}^{n} \int_{\|z_i - z_{i-1}\| \leq 1} u_{\beta,\varepsilon}(z_i - z_{i-1})\, dz_i, \end{aligned}$$

which is what we wanted to show. Now set $x_i = z_i - z_n$, $i = 1, \dots, n$, $x_n = z_n$, to see that the above (after integrating out x_n) is bounded by

$$\begin{aligned} &C \int_{\|x_i\| \leq 2} u_{\beta,\varepsilon}(x_1) u_{\beta,\varepsilon}(x_2 - z_1) \\ &\qquad \times \cdots u_{\beta,\varepsilon}(x_{n-1} - x_{n-2}) u_{\beta,\varepsilon}(x_{n-1})\, dx_1 \cdots dx_{n-1} \\ &\qquad \leq C\varepsilon^{-\gamma} \int_{\|x_i\| \leq 2/\varepsilon} u_{\beta,1}(x_1) u_{\beta,1}(x_2 - z_1) \\ &\qquad\qquad \times \cdots u_{\beta,1}(x_{n-1} - x_{n-2}) u_{\beta,1}(x_{n-1})\, dx_1 \cdots dx_{n-1}. \end{aligned}$$

For ease of notation, set $x_0 \equiv x_n \equiv 0$ and rewrite the integrand above as

$$\prod_{i=1}^{n} u_{\beta,1}(x_i - x_{i-1}) = \prod_{i=1}^{n} \left(\prod_{j \neq i} \left(u_{\beta,1}(x_i - x_{i-1}) \right)^{1/(n-1)} \right).$$

A generalised Hölder inequality then gives that

$$I_\varepsilon^n \leq C\varepsilon^{-\gamma} \prod_{i=1}^{n} \left(\int_{\|x_i\| \leq 2/\varepsilon} \prod_{j \neq i} \left(u_{\beta,1}(x_i - x_{i-1}) \right)^{n/(n-1)} \right)^{1/n}.$$

Each of the integrals here can be written as

$$\int_{\|u_i\|\le 4/\varepsilon}\prod_{i=1}^{n-1} u_{\beta,1}^{n/(n-1)}(u_i)\,du_i = \left(\int_{\|u\|\le 4/\varepsilon} u_{\beta,1}^{n/(n-1)}(u)\,du\right)^{n-1}$$

$$\le C\left(1+\int_1^{4/\varepsilon}\|u\|^{-\beta n/(n-1)}\,du\right)^{n-1}$$

$$\le \begin{cases} C\big(\max(1,\varepsilon^{-(d-\beta n/(n-1))})\big)^{n-1}, & \text{if } \gamma\ne 0,\\ C\big(\max(1,\log(1/\varepsilon))\big)^{n-1}, & \text{if } \gamma = 0,\end{cases}$$

$$\le \begin{cases} C\max(1,\varepsilon^{\gamma}), & \text{if } \gamma\ne 0,\\ C\big(\log(1/\varepsilon)\big)^{n-1}, & \text{if } \gamma = 0,\end{cases}$$

which is precisely what we have to prove. □

We can now finally return to the proof of (4.22). Before establishing this inequality in general, however, we shall first do so under restrictive but simplifying assumptions that should make the ideas behind the general proof somewhat clearer.

We shall assume that the order of intersection is even and that G can be expressed as the union of $k/2$ subgraphs, $G_1,\dots,G_{k/2}$, each one of which is the union of disjoint cycles. The subgraphs are not necessarily disjoint, in that they may have vertices, but not legs, in common. Assume also that one of the subgraphs, say G_1, contains a cycle of length at least three. We can bound the contribution to the moments of $\gamma_{k,\varepsilon}$ from each one of these cycles by Lemmas 4.2 and 4.4.

Note first that by Lemma 4.2 and the product formula (4.20), we have

$$I_\varepsilon(G) \le C\int\prod_{v\in G}\phi(z_v)\prod_{L\in G}u_{d-\alpha,\varepsilon}(z_\nu - z_{\bar\nu})$$

$$= C\int\prod_{j=1}^{k/2}\left(\prod_{L\in G_j}(\phi(z_\nu))^{2/k}u_{d-\alpha,\varepsilon}(z_\nu - z_{\bar\nu})\right) \tag{4.30}$$

$$\le C\prod_{j=1}^{k/2}\left(\int\prod_{L\in G_j}\phi(z_\nu)u_{d-\alpha,\varepsilon}^{k/2}(z_\nu - z_{\bar\nu})\right)^{2/k},$$

where the last inequality follows from a generalised Hölder inequality.

To estimate the integrals here, use Lemma 4.4 with $\beta = k(d-\alpha)/2$. Cycles G_i of length $n=2$ give $\gamma = k(d-\alpha)-d$ in (4.29), in which case we have that the upper bound that arises for each cycle is precisely of the order (in ε) of

$E\gamma_{k,\varepsilon}^2$. If $n \geq 3$, however, we have

$$\gamma = \tfrac{1}{2}nk(d-\alpha) - d(n-1) < \tfrac{1}{2}n(k(d-\alpha)-d),$$

since $\frac{1}{2}n < n-1$ for $n \geq 3$ and these are bounds of a lower order. Now add up all the exponents of ε to see that we have established the crucial (4.22) for this special case.

We now turn to the somewhat more involved general case, which will complete our task for even moments. Our aim is to divide the graph G into subgraphs which, if they are not cycles, are at least close to cycles in some sense. We shall then obtain moment bounds on each of the simpler subgraphs, as we did for the special case above in which the subgraphs were all cycles. The construction we are about to describe will in fact divide G into subgraphs which are cycles with extra chains dangling from some of the vertices.

Let G be a general graph on $n > 2$ vertices, with k legs issuing from each vertex. Let C_1 be a cycle in G containing the maximum number of vertices available to cycles. (By Lemma 4.3 it must contain at least three vertices.) Let $\hat{G}_1$ be the subgraph of G obtained by removing from G all the vertices of C_1 and all legs issuing from such vertices. Now, if possible, perform the same operations on $\hat{G}_1$ that were just performed on G, the only difference being that the cycle chosen out of $\hat{G}_1$ may no longer have more than two vertices. Keep doing this until you run out of cycles. If all the vertices have been used up this way, call the collection of cycles thus obtained G_1. If some vertices remain, then link these to one another and the cycles in some fashion and call *this* collection of cycles and chains G_1. Form G_2 in an identical fashion from $G \setminus G_1$ (except for the fact that G_2 may not contain cycles of length at least 3) and continue inductively, until all the vertices and legs in G have been exhausted. Let the subgraphs so obtained be denoted by $G_1, \dots, G_M$. Note that in G_1 we are assured of the existence of a cycle of length at least 3 and that $M \leq \frac{1}{2}k$ if k is even, while $M \leq [\frac{1}{2}k] + 1$ if k is odd.

To use this construction, consider first the case of even k. Then we can partition out the term $\phi(z_v)$ to the various subgraphs as we did in the first line of (4.30) and then continue as there with the generalised Hölder inequality. Thus,

$$I_\varepsilon(G) \leq C \prod_{j=1}^{k/2} \left(\int \prod_{L \in G_j} \phi(z_\nu) u_{d-\alpha,\varepsilon}^{k/2}(z_\nu - z_{\bar{\nu}}) \right)^{2/k}. \tag{4.31}$$

(If $M < \frac{1}{2}k$, then we interpret the empty products here as being equal to 1.) Now, however, we can no longer use Lemma 4.4 to complete the argument in a simple fashion, since the G_j are no longer necessarily cycles. We shall return to (4.31) in a moment, but shall first derive a corresponding inequality for the case of odd k.

Here there are two subcases. If $M \leq \frac{1}{2}k$, then the same argument as above remains valid and (4.31) still holds. If $M = [\frac{1}{2}k] + 1$, however, then in G_M there can be no more than one leg issuing from each vertex and so a slightly different application of the generalised Hölder inequality leads to

$$
(4.32)\qquad \begin{aligned} I_\varepsilon(G) \le C \prod_{j=1}^{[k/2]} \left(\int \prod_{L \in G_j} \phi(z_\nu) u_{d-\alpha,\varepsilon}^{k/2}(z_\nu - z_{\bar\nu}) \right)^{2/k} \\ \times \left(\int \prod_{L \in G_M} \phi(z_\nu) u_{d-\alpha,\varepsilon}^{k}(z_\nu - z_{\bar\nu}) \right)^{1/k}. \end{aligned}
$$

Since we are trying to prove (4.22), which is an order of magnitude relationship, we lose no rigour, but gain considerably in notational simplicity, if we ignore the specific choice of ϕ in (4.31) and (4.32). In fact, we shall adopt the rather unusual convention in what follows of letting ϕ denote a generic function from $\mathscr{S}_d$ that may change from line to line. [We cannot, of course, completely ignore the effect of the ϕ_i in (4.29), since they play a major role in ensuring that certain integrals converge at infinity.] We thus claim that if we could prove that

$$
(4.33)\qquad \int \prod_{v \in G} \phi(z_v) \prod_{L \in G_j} u_{(1/2)k(d-\alpha),\varepsilon}(z_\nu - z_{\bar\nu}) = \begin{cases} O\left(E\gamma_{k,\varepsilon}^2\right)^{N_j/2}, & \text{for } j \ne 1, \\ o\left(E\gamma_{k,\varepsilon}^2\right)^{N_1/2}, & \text{for } j = 1, \end{cases}
$$

where N_j is the number of distinct links in G_j, then we would be done. The reasoning behind this is as for the special case considered previously, applied to the bounds (4.31) and (4.32) and using the fact that $\Sigma N_j = nk/2$. Thus it remains to prove (4.33).

Consider then one of the subgraphs G_j and recall that it is made up of a number (perhaps 0) of disjoint cycles joined by chains with perhaps further chains dangling from some of the cycles. We shall establish (4.33) by considering one subgraph at a time, and removing the various chains sequentially, while carefully bounding their contributions until only cycles are left. Their contributions can then be bounded by Lemma 4.4.

Let $z_0, \dots, z_n$ be the vertices of a chain in G_j that joins two cycles. (The cases of a chain dangling from one cycle, or a free chain in a subgraph without cycles, can be handled in a similar, but simpler, fashion.) Note that both z_0 and z_n also appear as vertices of a cycle. Rewrite the integral over a subgraph in (4.33) as

$$
(4.34)\qquad \int \prod_{i=1}^{n} \phi(z_i) u_{(1/2)k(d-\alpha),\varepsilon}(z_i - z_{i-1}) F,
$$

where we use F to denote all the terms that no longer explicitly appear. (F also includes a ϕ for each vertex appearing in it.) Noting that

$$
\begin{aligned} &\int \prod_{i=1}^{n} \phi(z_i) u_{(1/2)k(d-\alpha),\varepsilon}(z_i - z_{i-1}) F \\ &\quad = \int \prod_{i=1}^{n} \left(\prod_{j \ne i} \left(\phi(z_j) u_{(1/2)k(d-\alpha),\varepsilon}(z_j - z_{j-1})\right)^{1/(n-1)} F^{1/n} \right), \end{aligned}
$$

we can again use a generalised Hölder inequality to bound (4.34) by

$$\prod_{i=1}^{n}\left(\int\prod_{j\neq i}\phi(z_j)\left(u_{(1/2)k(d-\alpha),\varepsilon}(z_j-z_{j-1})\right)^{n/(n-1)}F\right)^{1/n}. \tag{4.35}$$

Consider one of the inner products here and integrate out $z_1,\dots,z_{n-1}$ in the order $z_{i+1},\dots,z_{n-1}$ and then $z_{i-1},\dots,z_1$. Since none of these variables appear in F, the translation invariance of $u_{(1/2)k(d-\alpha),\varepsilon}$ gives us that (4.35) is equal to

$$\begin{aligned}
&\prod_{i=1}^{n}\left(\int\phi(z)\left(u_{(1/2)k(d-\alpha),\varepsilon}(z)\right)^{n/(n-1)}\right)^{(n-1)/n}\cdot\int F\\
&=\left(\int\phi(z)\left(u_{(1/2)k(d-\alpha),\varepsilon}(z)\right)^{n/(n-1)}\right)^{n-1}\cdot\int F\\
&=O\left(\varepsilon^{d-nk(d-\alpha)/2(n-1)}\right)^{n-1}\cdot\int F\\
&=\begin{cases} O\left(\left(E\gamma_{k,\varepsilon}^2\right)^{n/2}\right)\cdot\int F, & \text{for } n=2,\\ o\left(\left(E\gamma_{k,\varepsilon}^2\right)^{n/2}\right)\cdot\int F, & \text{for } n\geq 3.\end{cases}
\end{aligned}$$

If we now substitute this back into (4.33), then what has happened is that nothing has changed, other than the fact that the $n-1$ internal vertices ($=2n$ legs or n links) of our chain have disappeared from one of the graphs and there is a factor of $O((E\gamma_{k,\varepsilon}^2)^{n/2})$ appearing before the integral. We can continue in this fashion, removing the chains that either link or dangle from cycles, as well as free chains, until only cycles are left. Simple counting and the lower order contribution of cycles of length at least 3 (as in the special case above) now complete the proof of (4.33).

This almost completes the proof of the convergence of the univariate distributions, since the asymptotic result (4.16) for the even moments of $\gamma_{k,\varepsilon}$ has now been established. It remains only to show that the odd moments, after normalisation, converge to 0 with ε. By Formula 2.5, if mk is odd, then we automatically have $E\gamma_{k,\varepsilon}^m=0$ for all ε. If mk is even, we argue as follows:

Note that the argument for the even moment case hinged on the fact that the only graphs that made an asymptotic contribution to $E\gamma_{k,\varepsilon}^m$ were those in which there were no cycles of length 3 or more. However, it is easy to see (along the lines of the proof of Lemma 4.3) that in the case of odd moments, which means an odd number of vertices in the graph, *every* graph must contain at least one such cycle. This fact, together with the bounds above, establishes the asymptotic negligibility of the moments in this case. □

(*d*) *Proof of Theorem* 1.5—*Fidi distributions and tightness*. As is customary, we shall not give a full proof of the convergence of the finite-dimensional distributions to the right limit, but shall merely indicate why an argument based on the Cramér–Wold device works.

There are two free parameters that we have to worry about—the time t and the test function ϕ. However, the linearity of $\gamma_{k,\varepsilon}$ and the limit process $W^{(k)}$ as functionals on $\mathscr{S}_d$ implies that we need only worry seriously about the time parameter. Since the limit process, being based on a time changed Brownian motion, has independent increments, what we now need to prove is that this is also approximately true of $\gamma_{k,\varepsilon}(t,\phi)$ for each fixed ϕ. Indeed, we claim that if we could show that (for example)

$$\lim_{\varepsilon\to 0} \frac{E(\gamma_{k,\varepsilon}(t,\phi)\gamma_{k,\varepsilon}(s,\psi))}{\varepsilon^{d-(d-\alpha)k}} = \lim_{\varepsilon\to 0} \frac{E(\gamma_{k,\varepsilon}(s,\phi)\gamma_{k,\varepsilon}(s,\psi))}{\varepsilon^{d-(d-\alpha)k}} \tag{4.36}$$

for the case $\alpha < d(k-1)/k$ and $s \le t$, along with an analogous result for the case $\alpha = d(k-1)/k$ and similar expressions for higher, more complicated moments, convergence of the finite-dimensional distributions to the appropriate limit would follow easily. Details of this are left to the reader and we shall concentrate on establishing (4.36). As a first step, let $s \le t$ and write

$$\int_0^t du \int_0^s dv\, p^{\alpha}_{|u-v|}(x) = G^{(s)}_{\alpha}(x) + \int_s^t du \int_0^s dv\, p^{\alpha}_{|u-v|}(x), \tag{4.37}$$

where $G^{(s)}_{\alpha}$ was defined at (4.1) and, by Formula 2.6, is the function underlying the calculation of (4.36). Consider the rightmost term here, which we shall now show is of lower order, for small $\|x\|$, than $G^{(t)}_{\alpha}(x)$, and which we rewrite as

$$\begin{aligned}
&\int_0^{t-s} du \int_0^s dv\, p^{\alpha}_{|t+u-v|}(x) \\
&\quad = \int_0^{t-s} du \int_0^t dv\, p^{\alpha}_{|u+v|}(x) \\
&\quad \le C\int_0^{\infty} du \int_0^{\infty} dv \int_{\Re^d} dy\, e^{-\lambda(u+v)} p^{\alpha}_u(x-y) p^{\alpha}_v(y) \\
&\quad \le C G^{\lambda}_{\alpha} * G^{\lambda}_{\alpha}(x),
\end{aligned} \tag{4.38}$$

where the second line follows from Chapman–Kolmogorov and the last is true for any $\lambda \ge 0$.

When $2\alpha < d$, it follows from the Useful Fact 2.2 that $G^0_{\alpha} * G^0_{\alpha}(x) = C\|x\|^{2\alpha-d}$, so that this serves as an upper bound for (4.38). When $2\alpha > d$, it is easy to check that $G^1_{\alpha} \in \mathscr{L}^2$ (Lebesgue) and so by Cauchy–Schwarz, $G^0_{\alpha} * G^0_{\alpha}(x)$ is uniformly bounded in x, as is (4.38). When $2\alpha = d$, we return to the first line of (4.38) to see that

$$\int_0^{t-s} du \int_0^s dv\, p^{\alpha}_{|t+u-v|}(x) \le C\int_0^t re^{-r} p^{\alpha}_r(x)\, dr.$$

Now use the scaling relation $p^{\alpha}_t(x) = t^{-d/\alpha} p^{\alpha}_1(xt^{-1/\alpha})$ and the fact that

$d/\alpha = 2$ to see that the last expression is equivalent to

$$\begin{aligned}\int_0^t r^{-1} p_1^\alpha(x r^{-1/\alpha})\,dr &= \int_0^{t/\|x\|^\alpha} r^{-1} p_1^\alpha(1)\,dr \\ &= \int_0^1 r^{-1} p_1^\alpha(1)\,dr + \int_1^{t/\|x\|^\alpha} r^{-1} p_1^\alpha(1)\,dr \\ &\le CG_\alpha^0(1) + \log(t\|x\|^{-\alpha}),\end{aligned}$$

the last line again following from a scaling argument.

Our earlier calculations show that the first term on the right-hand side of (4.37), $G_\alpha^{(s)}(x)$, is, near the origin, of order $O(\|x\|^{\alpha-d})$, which by the above calculations is larger than the order of the second term. Now recall from the proof of the above subsection that it is the divergence at zero of terms like (4.37) that determines the asymptotic normalisation in ε, and so it is only this first term that is asymptotically important. But since this is $G_\alpha^{(s)}(x)$, which is the core expression for the calculation of the right-hand side of (4.36), the equality there is now clear. A similar argument works for higher mixed moments, and so the proof of the convergence of the finite-dimensional distributions is complete. □

We turn, finally, to the question of tightness. We shall consider only the case $\alpha < d(k-1)/k$, leaving the similar case of $\alpha = d(k-1)/k$ to the reader. All we need to show is the tightness of $\{\varepsilon^{(k(d-\alpha)-d)/2}\gamma_{k,\varepsilon}(\cdot,\phi)\}_\varepsilon$ for each fixed $\phi \in \mathscr{S}_d$. This, in turn, will follow if we can show that some $m \ge 1$ and all $0 \le s \le t$,

$$E(\gamma_{k,\varepsilon}(t,\phi) - \gamma_{k,\varepsilon}(s,\phi))^{2m} \le C\varepsilon^{m(k(d-\alpha)-d)}(t-s)^m. \tag{4.39}$$

To see this, note, from the original integral form of $\gamma_{k,\varepsilon}$ [cf. (1.16)] and Formulae 2.5 and 2.6, that when we write the expectation in (4.39) as a sum over graphs, each graph will contain a total of m factors and each one of these will contain, along with other multiplicative factors and integrals over the space dimensions, either a term of the form

$$\int_s^t\int_s^t p_{|u-v|}^\alpha(x)\,du\,dv = \int_0^{t-s}\int_0^{t-s} p_{|u-v|}^\alpha(x)\,du\,dv \le C(t-s)G_\alpha^0(x),$$

or of the form

$$\int_s^t\int_0^s p_{|u-v|}^\alpha(x)\,du\,dv.$$

Since the latter term is what appears in (4.37), we can argue as above to show that it, too, is bounded by $C(t-s)G_\alpha^0(x)$. This fact, together with the observations made above about order of magnitude relationships, is enough to prove (4.39), and we are done. □

5. Proof of the fluctuation result—Theorem 1.6. In this section we prove the fluctuation result, Theorem 1.6, for the ILT when $\alpha > d(k-1)/k$, that is, when there is $\mathscr{L}^2$ convergence. In principle, the proof follows the lines of the previous section. We start by calculating second moments and then show that the higher moments relate, asymptotically, to the second moments exactly as Gaussian higher moments relate to a variance. This gives convergence of the one-dimensional distributions as the first part of a weak convergence argument. The extension to general finite-dimensional distributions and the problem of tightness are both left to the reader, since the treatment here is no more difficult than that of the previous case.

Unfortunately, however, the higher moment calculations in the present case are somewhat more involved than those of the previous section, in that there are a number of special cases to consider. In particular, we shall have to differentiate between the two cases $\alpha = d = 2$ and $\alpha = d = 1$ and all other permissible cases. These two special cases, in which the underlying processes of (1.3) and (1.4) are neighbourhood recurrent, did not arise in the proof of Theorem 1.5, as these parameter choices can never meet the conditions of that theorem. Thus this section will be divided into a large number of subsections, primarily in order to treat all special cases in a readable fashion.

We start with second moment calculations. Here, as throughout most of this section, we shall make the simplifying assumption that the test functions are not only in $\mathscr{S}_d$, but are supported in the unit ball. In the last subsection we shall indicate how to remove this condition, although most of the work will be left to the reader. If you do not want to do this work, then you can simply add the condition of compact support to the statement of Theorem 1.6.

(*a*) *Second moments for the general case.* Throughout this and the following section, we shall assume that $d \neq \alpha$. Our aim, therefore, is to prove (1.26). We start by recalling some finite difference notation. For a function ψ: $\Re^d \to \Re$ and $a, b, z \in \Re^d$, set

$$\Delta_a \psi(z) = \psi(z+a) - \psi(z),$$

$$\Delta^2_{a,b}\psi(z) = \psi(z+a-b) - \psi(z+a) - \psi(z-b) + \psi(z).$$

We now turn to $E\gamma^2_{k,\varepsilon}$, which, as before, is given by (4.10). However, we can make use of the condition $\int f(x)\,dx = 0$ to add and subtract terms to (4.10), each with total integral zero, to see that $E\gamma^2_{k,\varepsilon}$ is given by

$$\sum_{\pi \in \mathscr{P}_k:\, \pi_1 = 1} 2^k \int \Phi(z) G^{(t)}_\alpha(z) \prod_{i=2}^{k} \Delta^2_{x_i, y_{\pi_i}} G^{(t)}_\alpha(z)\, f_\varepsilon(x_i)\, f_\varepsilon(y_i)\, dz \prod_{i=2}^{k} dx_i\, dy_i$$

$$\begin{aligned} (5.1) \qquad + \sum_{\pi \in \mathscr{P}_k:\, \pi_1 \neq 1} & 2^k \int \Phi(z) \Delta_{y_{\pi_1}} G^{(t)}_\alpha(z) \Delta_{x_{\pi^{-1}(1)}} G^{(t)}_\alpha(z) \\ & \times \prod_{i=2,\, \pi_i \neq 1}^{k} \Delta^2_{x_i, y_{\pi_i}} G^{(t)}_\alpha(z) \prod_{i=2}^{k} f_\varepsilon(x_i)\, f_\varepsilon(y_i)\, dz \prod_{i=2}^{k} dx_i\, dy_i. \end{aligned}$$

The point of this change of form is to ensure the convergence of certain integrals [such as (5.5) below] that will appear later.

We shall now show how to obtain the $\varepsilon \to 0$ limit of the first integral in (5.1). All the other terms can be handled similarly. To do this, note that for each $\varepsilon \in (0, 1/5)$, we can break up the z integral into the regions $0 \le \|z\| \le 5\varepsilon$ and $5\varepsilon < \|z\| \le 1$. In the first case, we split $G_\alpha^{(t)}$ into the sum of three parts, obtaining a result analogous to (4.9), and then apply Formula 2.1 and Lemma 2.3, as before, to $G_\alpha^0 * f_\varepsilon(z)$, to see that the integral over $0 \le \|z\| \le 5\varepsilon$ is of the form

$$(5.2)\qquad \begin{aligned} &(t-\delta)^k \int_{\|z\| \le 5\varepsilon} \Phi(z) G_\alpha^0(z) \prod_{i=2}^k \Delta^2_{x_i, y_i} G_\alpha^0(z) f_\varepsilon(x_i) f_\varepsilon(y_i)\, dz\, dx_i\, dy_i \\ &\quad + O(\delta) \int_{\|z\| \le 5\varepsilon} \varepsilon^{k(\alpha-d)}\, dz \\ &\quad + O(\delta^{-(1-\gamma)(d/\alpha - 1)}) \int_{\|z\| \le 5\varepsilon} \|z\|^{\gamma(\alpha-d)} \varepsilon^{(\alpha-d)(k-1)}\, dz \end{aligned}$$

for any $\gamma \in [0, 1]$. Since the last two terms here are clearly

$$O(\delta)\varepsilon^{d+k(\alpha-d)} + O(\delta^{-(1-\gamma)(d/\alpha-1)}) o(\varepsilon^{d+k(\alpha-d)}),$$

it follows from the normalisation in the denominator of (1.26) that these terms converge to 0 in the $\varepsilon \to 0$, $\delta \to 0$, limit, and so can be ignored.

We shall show below that for $5\varepsilon < \|z\| \le 1$ and $\|x_i\|$, $\|y_i\| \le \varepsilon$,

$$(5.3)\qquad \begin{aligned} \Delta^2_{x_i, y_i} G_\alpha^{(t)}(z) &= (t-\delta) \Delta^2_{x_i, y_i} G_\alpha^0(z) + O(\delta^{2-d/\alpha}) \varepsilon^2 \|z\|^{-2} G_\alpha^0(z) \\ &\quad + O(\delta^{(1-\gamma)(1-d/\alpha)}) \left(\varepsilon^2 \|z\|^{-2} G_\alpha^0(z)\right)^\gamma. \end{aligned}$$

Apply this and (5.2) to the first integral in (5.1) and scale all variables, to find that the integral is of the form

$$(5.4)\qquad \begin{aligned} &\varepsilon^{d-k(d-\alpha)} (t-\delta)^k \int \Phi(\varepsilon z) G_\alpha^0(z) \prod_{i=2}^k \Delta^2_{x_i, y_i} G_\alpha^0(z) f(x_i) f(y_i)\, dx_i\, dy_i\, dz \\ &\quad + O(\delta^{2-d/\alpha}) \varepsilon^{d-k(d-\alpha)} + O(\delta^{(1-\gamma)(1-d/\alpha)}) o(\varepsilon^{d-k(d-\alpha)}). \end{aligned}$$

The integral is easily seen to converge as $\varepsilon \to 0$ and the last two terms are, again, unimportant in the $\delta \to 0$ limit, since $\alpha > d(k-1)/k$ implies that $d/\alpha < (k-1)/k < 1$. Thus, all that remains is

$$(5.5)\qquad \varepsilon^{d-k(d-\alpha)} t^k \Phi(0) \int G_\alpha^0(z) \prod_{i=2}^k \Delta^2_{x_i, y_i} G_\alpha^0(z) f(x_i) f(y_i)\, dx_i\, dy_i\, dz.$$

By analysing the other terms in (5.1) in a similar fashion, we thus obtain that

$$\frac{E\gamma_{k,\varepsilon}^2}{\varepsilon^{(d-(d-\alpha)k)}} \to t^k \Phi(0) F_f^2$$

as $\varepsilon \to 0$, where

$$(5.6) \qquad F_f^2 := 2^k \int \overline{K}(x_2, \ldots, x_k, y_2, \ldots, y_k) \prod_{i=2}^{k} f(x_i) f(y_i)\, dx_i\, dy_i$$

and

$$\begin{aligned}
&\overline{K}(x_2, \ldots, x_k, y_2, \ldots, y_k) \\
&\quad := \sum_{\pi \in \mathscr{P}_k:\, \pi_1 = 1} \int G_\alpha^0(z) \prod_{i=2}^{k} \Delta^2_{x_i, y_{\pi_i}} G_\alpha^0(z)\, dz \\
&\qquad + \sum_{\pi \in \mathscr{P}_k:\, \pi_1 \neq 1} \int \Delta_{y_{\pi_1}} G_\alpha^0(z) \Delta_{x_{\pi^{-1}(1)}} G_\alpha^0(z) \prod_{i=2,\, \pi_i \neq 1}^{k} \Delta^2_{x_i, y_{\pi_i}} G_\alpha^0(z)\, dz \\
&\quad = (k-1)! \int G_\alpha^0(z) \prod_{i=2}^{k} \Delta^2_{x_i, y_i} G_\alpha^0(z)\, dz \\
&\qquad + (k-1)(k-1)! \int \Delta_{x_2} G_\alpha^0(z) \Delta_{y_2} G_\alpha^0(z) \prod_{i=3}^{k} \Delta^2_{x_i, y_i} G_\alpha^0(z)\, dz.
\end{aligned}$$

Noting the symmetries in these sums we obtain

$$\begin{aligned}
F_f^2 &:= 2^k (k-1)! \int G_\alpha^0(z) \left(\int \Delta^2_{x,y} G_\alpha^0 f(x) f(y)\, dx\, dy \right)^{k-2} dz \\
(5.7) \qquad &\quad + (k-1)(k-1)! \int \left(\int \Delta_x G_\alpha^0(z) f(x)\, dx \right)^2 \\
&\quad \times \left(\int \Delta^2_{x,y} G_\alpha^0(z) f(x) f(y)\, dx\, dy \right)^{k-2} dz.
\end{aligned}$$

This gives the constant F_f of (1.26).

Thus, in order to complete the derivation of the second moment, all that remains is to establish (5.3). We start this by obtaining some new estimates for the transition density $p_t^\alpha(x)$.

Assume that $\min(|x_1|, \ldots, |x_d|) > 0$ and, without any loss of generality, that $|x_1| \geq \max(|x_2|, \ldots, |x_d|)$. Then differentiate the expression (1.2) twice to obtain

$$\begin{aligned}
\frac{\partial^2 p_t^\alpha(x)}{\partial x_i\, \partial x_j} &= \frac{-1}{(2\pi)^d} \int_{\Re^d} p_i p_j e^{-ip\cdot x - t2^{-\alpha/2}\|p\|^\alpha}\, dp \\
&= \frac{1}{(2\pi)^d x_1^2} \int_{\Re^d} e^{-ip\cdot x} \frac{\partial^2}{\partial p_1^2} \left(p_i p_j e^{-t2^{-\alpha/2}\|p\|^\alpha} \right) dp,
\end{aligned}$$

the last line a result of integrating by parts, twice, with respect to p_1. Thus

$$\begin{aligned}\left|\frac{\partial^2 p_t^\alpha(x)}{\partial x_i\,\partial x_j}\right| &\le \frac{1}{(2\pi)^d x_1^2}\int_{\mathfrak{R}^d}\left|\frac{\partial^2}{\partial p_1^2}\left(p_i p_j e^{-t2^{-\alpha/2}\|p\|^\alpha}\right)\right| dp,\\ &\le \frac{d}{(2\pi)^d\|x\|^2}\int_{\mathfrak{R}^d}\left|\left(1+c_1t\|p\|^\alpha+c_2\left(t\|p\|^\alpha\right)^2\right)e^{-t2^{-\alpha/2}\|p\|^\alpha}\right| dp\\ &\le C\|x\|^{-2}t^{-d/\alpha}.\end{aligned}$$

Thus, if we now set

$$h(s,x) := \int_s^\infty p_t^\alpha(x)\,dt,$$

then we have

$$\begin{aligned}\left|\frac{\partial^2 h(s,x)}{\partial x_i\,\partial x_j}\right| &\le \int_s^\infty\left|\frac{\partial^2}{\partial x_i\,\partial x_j}p_t^\alpha(x)\right| dt\\ &\le C\|x\|^{-2}t^{1-d/\alpha}.\end{aligned}$$

The mean value theorem then implies that, if $4(|a|\vee|b|)\le\|x\|$, then

$$|\Delta^2_{a,b}h(s,x)| \le C|a|\cdot|b|\cdot\|x\|^{-2}t^{1-d/\alpha}. \tag{5.8}$$

We can now turn to proving (5.3). Using the definition (4.1) of $G_\alpha^{(t)}$ and noting (4.6), we have

$$\begin{aligned}\Delta^2_{x_i,y_i}G_\alpha^{(t)}(z) &= \Delta^2_{x_i,y_i}\int_0^t ds\int_0^s p_r^\alpha(z)\,dr\\ &= t\Delta^2_{x_i,y_i}G_\alpha^0(z) - \Delta^2_{x_i,y_i}\int_0^t h(s,z)\,ds\\ &= t\Delta^2_{x_i,y_i}G_\alpha^0(z) - \int_0^\delta\Delta^2_{x_i,y_i}h(s,z)\,ds - \int_\delta^t\Delta^2_{x_i,y_i}h(s,z)\,ds.\end{aligned} \tag{5.9}$$

Recall that we only have to prove (5.3) under the assumptions $5\varepsilon<\|z\|\le 1$ and $\|x_i\|,\ \|y_i\|\le\varepsilon$. Thus we can apply (5.8) to see that the first integral in the last line of (5.9) can be bounded above by

$$\begin{aligned}\int_0^\delta\Delta^2_{x_i,y_i}h(s,z)\,ds &\le C\varepsilon^2\delta^{2-d/\alpha}\|z\|^{-2}\\ &\le C\varepsilon^2\delta^{2-d/\alpha}\|z\|^{-2}G_\alpha^0(z).\end{aligned} \tag{5.10}$$

The addition of the seemingly redundant factor of $G_\alpha^0(z)$ is permissible since we are assuming that $\|z\|\le 1$ and can change the constant in (5.10) at will.

A similar argument gives a bound of $C\varepsilon^2\|z\|^{-2}G_\alpha^0(z)$ for the second integral in (5.9), and noting the bound inherent in (4.7), we get an interpolated bound of the form

$$O\left(\delta^{(1-\gamma)(1-d/\alpha)}\right)\left(\varepsilon^2\|z\|^{-2}G_\alpha^0(z)\right)^\gamma \tag{5.11}$$

for any $\gamma \in [0,1]$. The requisite (5.3) now follows from (5.9)–(5.11) and we are done. □

REMARK. The last few lines of the above proof seem to contain bounds that are somewhat forced, in the sense that factors [such as $G_\alpha^0(z)$ in (5.10)] seem to have been introduced without need and thus equation (5.3) has actually been made more complex than is necessary. The reason for this will become clear when we come to subsection (d) below, in which we shall indicate how to lift the above proof from test functions of compact support to test functions in $\mathscr{S}_d$. The extra term of G_α^0 then gives us extra integrability, required to keep certain integrals finite.

(*b*) *Higher moments for the general case.* The calculation of the higher moments for the case $\alpha > d(k-1)/k$ is, in principle, much the same as in the previous section for the case $\alpha \le d(k-1)/k$ and relies on the summing over all graphs $G \in \mathscr{G}_{n,k}$ expressions analogous to (4.19). As we did in calculating the second moment, however, we change the $G_\alpha^{(t)}$ terms appearing in (4.19) to $\Delta G_\alpha^{(t)}$ or $\Delta^2 G_\alpha^{(t)}$, if one or both (respectively) of the x variables is not identically zero. Again, the fact that $\int f(x)\,dx = 0$ means that the overall expectation has not been changed.

We can then carry out the graph theoretic counting as before, until we reach (4.30) and the later similar equations. Once again, it is easy to see that the graphs G made up only of cycles of length $n = 2$ are precisely those required to give moments of $\gamma_{k,\varepsilon}(f)\varepsilon^{-(1/2)(d-(d-\alpha)k)}$ that are asymptotically Gaussian. This will, therefore, be all that we require, once we have proven that graphs containing longer cycles make asymptotically negligible contributions to the moments.

It follows much as before that each cycle of length $n \ge 3$ makes a contribution of the form

$$(5.12)\qquad I_\varepsilon(n) = \varepsilon^{d(n-1)-nk(d-\alpha)/2} \int_{\|z_i\| \le 1/\varepsilon} \prod_{i=1}^{n} V_{j_i}(z_i - z_{i-1})\,dz_i,$$

where, for $j = 0, 1, 2$,

$$(5.13)\qquad V_j(z) := u_{k(d-\alpha+j)/2,1}(z) = (\max(\|z\|, 1))^{-k(d-\alpha+j)/2}.$$

To see this, there are four facts that must be taken into account. First, the factors of $G_\alpha^{(t)}$ that appear, unchanged, in the expressions described above all contribute a term V_0 to (5.12). This is obvious.

Second, factors of the form $\Delta G_\alpha^{(t)}$ and $\Delta^2 G_\alpha^{(t)}$ each contribute terms corresponding to $\Delta u_{k(d-\alpha)/2,1}$ and $\Delta^2 u_{k(d-\alpha)/2,1}$, respectively. But the effect of differencing is, by the mean value theorem, effectively the same as differentiating, so that these terms correspond (up to unimportant constants) to the V_1 and V_2 terms of (5.12).

Third, the fact that $1/\varepsilon$ appears as the lower limit of the integral in (5.12) comes from the fact that throughout this section we have agreed to work with test functions of support in the unit ball. Finally, the factors of ε that appear before the integral in (5.12) that do not appear, for example, in (4.30) arise since a scaling argument has already been carried out to obtain (5.12).

In view of the closing arguments of the previous section, it will therefore suffice for us to show that

$$I_n(\varepsilon) = o(\varepsilon^{n(d-k(d-\alpha))/2}). \tag{5.14}$$

Our first step towards this is the following easy lemma.

5.1 LEMMA. *Let $\beta, \gamma > 0$. Then*

$$\int_{\Re^d} u_{d+\gamma,1}(z-x) u_{d-\alpha,1}(x)\, dx \le C u_{d-\alpha,1}(z).$$

PROOF. Since $u_{d-\alpha,1}$ is bounded and $u_{d+\gamma,1} \in \mathscr{L}^1$, it is clear that the integral is bounded and so we need only consider $\|z\| \ge 1$.

We split the domain of integration into two parts. If $\|x\| \ge \|z\|/4 \ge 1/4$, note simply that there exists a C such that $u_{d-\alpha,1}(x) \le C u_{d-\alpha,1}(z)$ and then use the fact that $u_{d+\gamma,1} \in \mathscr{L}^1$ to show that the integral over this region is no greater than $C u_{d-\alpha,1}(z)$. If $\|x\| < \|z\|/4$, note that

$$\begin{aligned}
&\int_{\|x\|<\|z\|/4} u_{d+\gamma,1}(z-x) u_{d-\alpha,1}(x)\, dx \\
&\quad \le C \int_{\|x\|<\|z\|/4} u_{\alpha+\gamma,1}(z-x) u_{d-\alpha,1}(z-x) u_{d-\alpha,1}(x)\, dx \\
&\quad \le C u_{d-\alpha,1}(z) \int_{\Re^d} u_{\alpha+\gamma,1}(x-z) u_{d-\alpha,1}(x)\, dx \\
&\quad \le C u_{d-\alpha,1}(z).
\end{aligned}$$

This completes the proof. □

We now turn to the proof of (5.14). Unfortunately, the proof occasionally uses quite delicate relationships between the various parameters, and so has to be tailored to each dimension d separately. We shall give the proof only when $d = 2$, leaving the other cases ($d = 1, 3$) up to the reader. The necessary changes are not hard.

Consider, therefore, (5.12). Since $\alpha > d(k-1)/k$ and $d = 2$, we have $k(2-\alpha)/2 < k(2-\alpha) < 2$ and so it is clear from (5.13) that $V_0 \notin \mathscr{L}^1$. On the other hand, V_2 is always in $\mathscr{L}^1$. To find out what happens with V_1, we must consider two cases.

CASE 1. $V_1 \in \mathscr{L}^1$: We are in this situation when $k > 4/(3-\alpha)$.

Use Lemma 5.1 to eliminate all factors of V_1 and V_2 from (5.12), so that all that remains is the product over a chain of length $m := \#\{i: j_i = 0\}$. If $m \le 1$, then it is easy to see from (5.12), the definition of V_0 and the fact that $\alpha > d(k-1)/k$ that (5.14) is satisfied. Thus we can assume that $m \ge 2$ for the rest of this case. Write the integral in (5.12) as

$$\begin{aligned}
\int_{\|z_i\|\le\varepsilon^{-1}} \prod_{i=1}^{m} V_0(z_i - z_{i-1})\,dz_i &= \int_{\|z_i\|\le\varepsilon^{-1}} \prod_{l=1}^{m}\left(\prod_{\substack{i=1\\ i\ne l}}^{m} V_0^{1/(m-1)}(z_i - z_{i-1})\right) dz_i \\
&\le \prod_{l=1}^{m}\left(\int_{\|z_i\|\le\varepsilon^{-1}} \prod_{\substack{i=1\\ i\ne l}}^{m} V_0^{m/(m-1)}(z_i - z_{i-1})\,dz_i\right)^{1/m} \qquad (5.15)\\
&= \prod_{i=1}^{m}\left(\int_{\|z\|\le\varepsilon^{-1}} V_0^{m/(m-1)}(z)\,dz\right)^{(m-1)/m} \\
&\le C\varepsilon^{mk(2-\alpha)/2-d(m-1)},
\end{aligned}$$

where the last inequality relies on the specific form of V_0 and uses the fact that

$$\frac{m}{m-1}\,\frac{k(d-\alpha)}{2} \le k(d-\alpha) < d.$$

It is now an easy task to check coefficients in the above and (5.12) to see that (5.14) is satisfied.

Case 2. $V_1 \notin \mathscr{L}^1$: We are in this situation when $k \le 4/(3-\alpha)$.

We argue as in Case 1, but this time we can, a priori, only eliminate the V_2 factors from (5.12). The number of terms remaining is now $m = \#\{i: j_i = 0$ or $j_i = 1\}$. Corresponding to the second last line of (5.15) we obtain

$$\prod_{i=1}^{m}\left(\int_{\|z\|\le\varepsilon^{-1}} V_{j_i}^{m/(m-1)}(z)\,dz\right)^{(m-1)/m}.$$

The terms with $j_i = 0$ are treated exactly as before [i.e., as in the last line of (5.15)]. For the other terms we use

$$\left(\int_{\|z\|\le\varepsilon^{-1}} V_{j_1}^{m/(m-1)}(z)\,dz\right)^{(m-1)/m} = \begin{cases} O(1), & \text{if } \eta > 2,\\ O(\ln(\varepsilon^{-1})), & \text{if } \eta = 2,\\ O(\varepsilon^{k(3-\alpha)/2-2(m-1)/m}), & \text{if } \eta < 2,\end{cases}$$

where $\eta = mk(3-\alpha)/(2(m-1))$. The proof can then be completed as in Case 1, by comparing coefficients. Note that each of the three subcases has to be treated separately and that the comparison of coefficients will take you a few lines in each case. Nevertheless, since nothing more than elementary algebra and a little patience is now required, the details are left as an exercise for the reader.

This completes the proof of the fact that the only graphs that make asymptotically nonnegligible contributions to the moments of $\gamma_{k,\varepsilon}(f)\varepsilon^{-(1/2)(d-(d-\alpha)k)}$ are those made up only of cycles of length two, which is what we had to prove. □

(*c*) *The "recurrent case"*. We now turn to the cases $\alpha = d = 2$ and $\alpha = d = 1$, in which the underlying processes—planar Brownian motion and the real-valued Cauchy process—are neighbourhood recurrent. These two cases are special within density processes for which $\alpha > d(k-1)/k$ in that they are the only ones for which G_α^0 is not defined [cf. Formula 2.1] and so the analysis of the preceding subsections requires an essential change. Fortunately, the required change is not too drastic and, as we shall show below, the basic approach is as before, but with G_α^0 replaced by G_α^1, which does exist, throughout.

This change, however, carries with it some technical difficulties, since G_α^1 is not as simple a function as G_α^0. Nevertheless, we have already seen, on numerous occasions, that what is important in deriving normalisation factors for $\gamma_{k,\varepsilon}$ is the behaviour of the Green's function at the origin, with its behaviour elsewhere being required only to establish finiteness of some integrals. In both of the cases to be treated in this subsection, the Green's function has a logarithmic singularity at the origin, and, in fact,

$$\lim_{\varepsilon\to 0}\frac{G_2^1(\varepsilon)}{\ln(1/\varepsilon)} = 1, \qquad d = 2, \tag{5.16}$$

$$\lim_{\varepsilon\to 0}\frac{G_1^1(\varepsilon)}{\ln(1/\varepsilon)} = \frac{\pi}{\sqrt{2}}, \qquad d = 1. \tag{5.17}$$

To see how we move from reliance on G_α^0 to G_α^1, recall that the key function in the moment calculations is really neither of these, but rather $G_\alpha^{(t)}$, which, previously, we bounded via G_α^0. Note, however, that

$$\begin{aligned} G_\alpha^{(t)}(x) &:= \int_0^t dr \int_0^r p_s^\alpha(x)\,ds \\ &= \int_0^t dr \left\{ \int_0^r e^{-s} p_s^\alpha(x)\,ds + \int_0^r (1-e^{-s}) p_s^\alpha(x)\,ds \right\} \qquad (5.18) \\ &= tG_\alpha^1(x) - \int_0^t dr \int_r^\infty e^{-s} p_s^\alpha(x)\,ds + \int_0^t dr \int_0^r (1-e^{-s}) p_s^\alpha(x)\,ds. \end{aligned}$$

Since for fixed t the integrals here are bounded in x, it is clear that the behaviour of $G_\alpha^{(t)}$ at the origin is identical to that of G_α^1. Thus, in all that follows, we shall replace $G_\alpha^{(t)}$ by G_α^1 wherever the former appears. It is left to the reader to justify this replacement, following the approach taken at (4.9) and (5.2).

We have to prove (1.23)–(1.25) of Theorem 1.6. We shall proceed as follows. As a first step, we shall calculate the asymptotic second moment of $\gamma_{k,\varepsilon}$ when $\alpha = d = k = 2$. This will give the normalisation required for (1.23). We shall then indicate how to do the same thing for $\alpha = d = 2$, but $k \geq 3$. This does the same for (1.24). We shall then treat the (by now) easy case of $\alpha = d = 1$, and general k, to obtain the normalisation for (1.25). Finally, we shall show how to treat higher moments. As in the previous subsections, full weak convergence, including the convergence of the finite-dimensional distributions and tightness, is left to the reader.

STEP 1. $\alpha = d = k = 2$: By (5.1), and our above agreement to replace $G_\alpha^{(t)}$ by G_α^1, we have

$$\begin{aligned}
E\gamma_{2,\varepsilon}^2 &\approx 4t^2 \int_{\|z\|\le 1} \Phi(z) \ln(1/\|z\|) \Delta_{x,y}^2 \ln(1/\|z\|) f_\varepsilon(x) f_\varepsilon(y)\, dx\, dy\, dz \\
&\quad + 4t^2 \int_{\|z\|\le 1} \Phi(z) \Delta_x \ln(1/\|z\|) \Delta_y \ln(1/\|z\|) f_\varepsilon(x) f_\varepsilon(y)\, dx\, dy\, dz \\
&= 4\varepsilon^2 t^2 \int_{\|z\|\le 1/\varepsilon} \Phi(\varepsilon z)[\ln(1/\varepsilon) + \ln(1/\|z\|)] \\
&\qquad \times \Delta_{x,y}^2 \ln(1/\|z\|) f(x) f(y)\, dx\, dy\, dz \\
&\quad + 4\varepsilon^2 t^2 \int_{\|z\|\le 1/\varepsilon} \Phi(\varepsilon z) \Delta_x \ln(1/\|z\|) \Delta_y \ln(1/\|z\|) f(x) f(y)\, dx\, dy\, dz. \qquad (5.19)
\end{aligned}$$

Consider the first term of the last line. Use the fact that $\Phi(z) = \Phi(0) + O(\|z\|)$ for z small, say $\|z\| \le 1$, to rewrite it as

$$\begin{aligned}
&4\varepsilon^2 \Phi(0) t^2 \int_{\|z\|\le 1/\varepsilon} [\ln(1/\varepsilon) + \ln(1/\|z\|)] \Delta_{x,y}^2 \ln(1/\|z\|) \\
&\qquad \times f(x) f(y)\, dx\, dy\, dz
\end{aligned} \qquad (5.20)$$

plus lower order expressions. Break up the z integral here into the regions $0 \le \|z\| \le 1$ and $1 < \|z\| \le 1/\varepsilon$. In the first of these, write out $\Delta_{x,y}^2$ as a difference, remember that because of the finite support of f, the x and y integrals are over the unit disk, and use the fact that $\ln(1/x)$ is integrable near the origin to see that the total integral is $O(\ln(1/\varepsilon))$. For the second, use the mean value theorem to first replace $\Delta_{x,y}^2 \ln(1/\|z\|)$ by $x \cdot y \|z\|^{-2}$, with a bounded error that after integration is $O(\ln(1/\varepsilon))$. (As usual, $x \cdot y$ denotes the inner product $\Sigma_{i=1}^2 x_i y_i$.) Hence the first term of the last line of (5.19) is equal to

$$\begin{aligned}
&4\varepsilon^2 \Phi(0) t^2 \int_{1<\|z\|\le 1/\varepsilon} [\ln(1/\varepsilon) + \ln(1/\|z\|)] \big((x\cdot y)/\|z\|^2\big) \\
&\qquad \times f(x) f(y)\, dx\, dy\, dz + O(\varepsilon^2 \ln(\varepsilon)) \\
&\quad = 4\pi\varepsilon^2 \Phi(0) t^2 \ln^2(1/\varepsilon) \int x \cdot y f(x) f(y)\, dx \\
&\qquad + O(\varepsilon^2 \ln(1/\varepsilon)),
\end{aligned} \qquad (5.21)$$

the final factor of $\ln(1/\varepsilon)$ and the π coming from the z integral after a transformation to polar coordinates.

If we could now show that the second term of the last line of (5.19) was $O(\ln(1/\varepsilon))$, then this would be precisely what is required to establish the normalisation for the case $\alpha = d = k = 2$. This, however, follows as above, the lower order of $\ln(1/\varepsilon)$ coming from the lack of this factor in the integrand. □

STEP 2. $\alpha = d = 2$, $k \geq 3$: To handle this case, we must treat the second moment as given by (5.1). There are two summations here. Consider first a typical term in the first sum.

These terms can be treated precisely as in Step 1, until (5.21), which will be replaced by

$$(5.22)\qquad \begin{aligned} &2^k \varepsilon^2 \Phi(0) t^k \int_{1 < \|z\| \leq 1/\varepsilon} [\ln(1/\varepsilon) + \ln(1/\|z\|)] \\ &\quad \times \left(\prod_{i=2}^{k} \Delta^2_{x_i, y_{\pi_i}} \ln(1/\|z\|) f(x_i) f(y_i)\, dx_i\, dy_i \right) dz, \end{aligned}$$

plus a term $o(\ln(1/\varepsilon))$. But now the z integral no longer involves a singularity (which is why we cannot simplify the above via the mean value theorem) and the only divergence comes from the $\ln(1/\varepsilon)$ in the integrand. Remember that there are $(k - 1)!$ terms of the above form in the first sum in (5.1), to see that the total asymptotic contribution of this sum is $\varepsilon^2 \ln(1/\varepsilon)\Phi(0)D_f^2$, where

$$(5.23)\qquad D_f^2 := 2^k (k-1)! \pi \int \left(\int \Delta^2_{x,y} \ln(1/\|z\|) f(x) f(y)\, dx\, dy \right)^{k-1} dz$$

is the normalisation for (1.24). Thus our proof for this case will be finished once we know that the contribution of the second sum in (5.1) is of lower order than the above. But this is easy, since we once again follow the same argument, but now the integrals we end up with (because of high negative powers of $\|z\|$) are convergent and are therefore negligible compared to the $O(\ln(1/\varepsilon))$ divergence of the first sum. □

STEP 3. $\alpha = d = 1$, $k \geq 2$: This case is identical to the previous one, with the single exception that (5.22) is preceded by a factor of $(\pi/\sqrt{2})^k$ [cf. (5.17)]. In this case there is no divergence in the z integral for any k and so the only terms that are asymptotically important are those that arise from the $\ln(1/\varepsilon)$ in (5.22). Doing the z integral (this time without the necessity to introduce polar coordinates and factors of π) it follows as in the above cases that $E\gamma^2_{k,\varepsilon} \approx \varepsilon^2 \ln(1/\varepsilon)\Phi(0)E_f^2$, where

$$(5.24)\qquad E_f^2 := (\sqrt{2}\pi)^k (k-1)! \int \left(\int \Delta^2_{x,y} \ln(1/\|z\|) f(x) f(y)\, dx\, dy \right)^{k-1} dz,$$

which gives the required normalisation for (1.25).

STEP 4. HIGHER MOMENTS. We shall treat only the case $\alpha = d = k = 2$. The others are handled similarly (but nevertheless *do* involve some work) and are left to the reader.

Our starting point is the expression (4.19), to which we apply differences as in (5.1). Continuing in the spirit of subsection (b) above, it will suffice to show that, for $n \geq 3$,

$$I_n(\varepsilon) = o(\varepsilon \ln(1/\varepsilon))^n, \tag{5.25}$$

where

$$I_n(\varepsilon) = \varepsilon^{2(n-1)} \int_{\|z_i\| \leq 1/\varepsilon} \prod_{i=1}^{n} V_{j_i}(z_i - z_{i-1})\, dz_i, \tag{5.26}$$

and

$$V_0(z) := \begin{cases} \ln(1/(\varepsilon x)), & \text{if } |x| \geq \frac{1}{2}, \\ \ln(1/\varepsilon), & \text{if } |x| < \frac{1}{2}, \end{cases}$$

with $V_j = u_{j,1}$ for $j = 1, 2$. Note that, as opposed to (5.13), V_0 is still a function of ε. This is not the case for V_1 and V_2. The reasoning behind (5.26) is basically the same as for the justification of (5.12), noting only that there are some other terms in the expression of the higher moments that arise out of the scaling argument, but that are asymptotically of smaller order than $I_n(\varepsilon)$. Note that

$$\int_{\|x\| \leq 2/\varepsilon} V_j(x)\, dx = \begin{cases} O(\varepsilon^{-2}), & \text{if } j = 0, \\ O(\varepsilon^{-1}), & \text{if } j = 1, \\ O(\ln(1/\varepsilon)), & \text{if } j = 2. \end{cases}$$

It is only the first of these three cases that requires justification. In this case

$$\begin{aligned} \int_{\|x\| \leq 2/\varepsilon} V_0(x)\, dx &= \int_{\|x\| \leq 1/2} \ln(1/\varepsilon)\, dx + \int_{1/2 \leq \|x\| \leq 2/\varepsilon} \ln(1/(\varepsilon x))\, dx \\ &= O(\ln(1/(\varepsilon))) + \int_{1/2}^{2/\varepsilon} r \ln(1/(\varepsilon r))\, dr \\ &= O(\ln(1/(\varepsilon))) + O(\varepsilon^{-2}) \\ &= O(\varepsilon^{-2}), \end{aligned}$$

which is precisely what we require.

Now consider the integral (5.26). Recall the convention that $z_0 = z_n = 0$ and since the cyclic nature of the product allows us to start anywhere, assume that $j_n = 0$. It then follows, from the definition of V_0, that

$$V_{j_n}(z_n - z_{n-1}) = V_0(z_{n-1}) \leq \ln(1/\varepsilon).$$

Now integrate, in turn, over $z_{n-1}, z_{n-2}, \dots, x_1$ in (5.26), using the above trichotomy to bound the integrals. Letting a, b and c, respectively, denote the number of V_0, V_1 and V_2 terms in (5.26) (so that $a + b + c = n$), it then follows that

$$I_n(\varepsilon) \leq C\varepsilon^{2(n-1)} (\varepsilon^{-2})^{a-1} (\varepsilon^{-1})^{b} (\ln(1/\varepsilon))^{c} (\ln(1/\varepsilon)), \tag{5.27}$$

the last term coming from V_{j_n}.

Since we are in the case $k = 2$, a little thought shows that $a = c$, so that $2a + b = n$. It thus follows from (5.27) that

$$I_n(\varepsilon) \le C\varepsilon^n(\ln(1/\varepsilon))^{c+1}.$$

But since $a = c$ implies that $c \le n/2$, we have that $c + 1 < n$, so that (5.27) implies (5.25), and we are done. □

(*d*) *On going from bounded support to* $\mathscr{S}_d$. We now have to show that all the proofs of this section, for which it was assumed that the test function ϕ had bounded support, also hold for general $\phi \in \mathscr{S}_d$.

Unfortunately, this is not trivial. Furthermore, we do not have a simple recipe for lifting all proofs at once, but, rather, each proof requires its own little tricks. Thus what we shall do is show how to carry out this extension in one particular case, in the hope that this will both convince the reader that this is possible in general and show him, basically, how it is done. For the reader who remains unconvinced, there is always the option of adding the condition of compact support to the assumptions of Theorem 1.6.

The case we shall treat is the calculation of $E\gamma_{k,\varepsilon}^2$ for the case $d \ne \alpha$. This is, of course, the most important of the moment calculations and covers all but two cases. Returning to the proof in subsection (a) above, one sees that the only place that compact support was actually used was in establishing the crucial inequality (5.3). Once we have shown that this holds in general, we shall be done.

We start, as there, with some preliminary inequalities, in particular for the function

$$g(s,x) := \int_0^s p_t^\alpha(x)\,dt. \tag{5.28}$$

As before, assume that $\min(|x_1|,\dots,|x_d|) > 0$ and, without any loss of generality, that $|x_1| \ge \max(|x_2|,\dots,|x_d|)$. Integrate by parts in (1.2) to obtain

$$p_t^\alpha(x) = \frac{C}{x_1}\int tp_1\|p\|^{\alpha-2}e^{ip\cdot x - t2^{-\alpha/2}\|p\|^\alpha}\,dp, \tag{5.29}$$

so that

$$\begin{aligned} g(s,x) &= \frac{C}{x_1}\int e^{-ip\cdot x}p_1\|p\|^{\alpha-2}\left(\int_0^s te^{-t2^{-\alpha/2}\|p\|^\alpha}\,dr\right)dp \\ &= \frac{C}{x_1}\int e^{-ip\cdot x}p_1\big(se^{-s2^{-\alpha/2}\|p\|^\alpha}\|p\|^{-2} \\ &\qquad\qquad + (e^{-s2^{-\alpha/2}\|p\|^\alpha}-1)\|p\|^{-(2+\alpha)}\big)\,dp \\ &= g_1(s,x) + g_2(s,x), \end{aligned} \tag{5.30}$$

say. Consider the first of these terms. Differentiating with respect to x_1 and then integrating by parts gives

$$\frac{\partial g_1(s,x)}{\partial x_1} = \frac{C}{x_1^2}\int e^{-ip\cdot x}\Big(p_1 s e^{-s2^{-\alpha/2}\|p\|^\alpha}\|p\|^{-2} + p_1^2 s e^{-s2^{-\alpha/2}\|p\|^\alpha}\|p\|^{-4} + s^2 p_1^2 e^{-s2^{-\alpha/2}\|p\|^\alpha}\|p\|^{\alpha-3}\Big)\,dp.$$

Repeating this procedure, we obtain

$$\frac{\partial^2 g_1(s,x)}{\partial x_1^2} = \frac{C}{x_1^3}\int e^{-ip\cdot x}\Big(p_1 s e^{-s2^{-\alpha/2}\|p\|^\alpha}\|p\|^{-2}\Big)\,dp, \tag{5.31}$$

plus other similar terms, with higher powers of s and larger, negative, powers of $\|p\|$. Bounding the complex exponential in (5.31) by 1, a simple change of variables gives that the integral is bounded above by $Cs^{1-(d-1)/\alpha}$. In fact, the same is true of the other integrals and in general,

$$\left|\frac{\partial^2 g_1(s,x)}{\partial x_i\,\partial x_j}\right| \le \frac{C}{\|x\|^3}s^{(1-(d-1)/\alpha)}.$$

The same is easily seen to be true for $\partial^2 g_2/\partial x_i\,\partial x_j$, so that it follows from (5.30) and the mean value theorem that

$$|\Delta^2_{x,y}g(s,z)| \le \frac{\|x\|\,\|y\|}{\|z\|^3}s^{(1-(d-1)/\alpha)} \tag{5.32}$$

plus terms of higher order in $\|x\|$ and $\|y\|$. Since in (5.3) we are only interested in $\|x\|, \|y\| \le \varepsilon$, we can replace the numerator in (5.3) by ε^2 and note that the higher order terms are now of no asymptotic interest. Furthermore, we shall now restrict ourselves to the case $d = 2$. (The one-dimensional case is easier and the three-dimensional case is a little more difficult. Neither, however, are all that different.)

Under these restrictions, (5.32) implies

$$|\Delta^2_{x,y}g(s,z)| \le \frac{\varepsilon^2}{\|z\|^2}G^0_\alpha(z)s^{(1-1/\alpha)}. \tag{5.33}$$

We now return to the definitions (4.1) of $G^{(t)}_\alpha$ and (5.28) of g to note that

$$\begin{aligned}\Delta^2_{x,y}G^{(t)}_\alpha(z) &= \int_0^t\left(\Delta^2_{x,y}\int_0^s p^\alpha_t(z)\,dr\right)ds\\ &= \int_\delta^t\left(\Delta^2_{x,y}\int_0^s p^\alpha_t(z)\,dr\right)ds + \int_0^\delta \Delta^2_{x,y}g(s,z)\,ds\\ &= (t-\delta)\Delta^2_{x,y}G^0_\alpha(z) + \int_0^\delta \Delta^2_{x,y}g(s,z)\,ds\\ &\quad - \int_\delta^t\big[\Delta^2_{x,y}\big(G^0_\alpha(z) - g(s,z)\big)\big]\,ds.\end{aligned} \tag{5.34}$$

The first term here matches the first term of the required bound in (5.3). By (5.33), the integral over $[0, \delta]$ is bounded by $C\varepsilon^2\|z\|^{-2}G_\alpha^0(z)\delta^{2-1/\alpha}$ and this matches the second term there. As far as the last term is concerned, note that by (5.33) and (4.7),

$$\left|\int_\delta^t \left[\Delta_{x,y}^2\left(G_\alpha^0(z) - g(s,z)\right)\right] ds\right| \le C\delta^{(1-\gamma)(1-2/\alpha)}\left(\frac{\varepsilon^2 G_\alpha^0(z)}{\|z\|^2}\right)^\gamma$$

for any $\gamma \in [0, 1]$.

This is precisely the last term of (5.3) and so the proof is complete. □

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