MOMENT GENERATING FUNCTIONS FOR LOCAL TIMES OF SYMMETRIC MARKOV PROCESSES AND RANDOM WALKS

Michael B. Marcus and Jay Rosen

1. Introduction. We obtain moment generating functions for the local times of strongly symmetric Markov processes and symmetric random walks via the Dynkin Isomorphism Theorem. This allows us to reduce complex computations involving Markov processes to elementary manipulations of Gaussian random variables.

Let S be a locally compact metric space with a countable base and let $X = (\Omega, \mathcal{F}_t, X_t, P^x)$, $t \in R^+$, be a strongly symmetric standard Markov process with state space S and lifetime ζ. We assume that there is a σ–finite measure $m(\,\cdot\,)$ on S. For a precise statement of what these properties are we refer the reader to [5]. For the purposes of this note it suffices to say that X has a symmetric transition probability density function $p_t(x,y)$ with respect to m and an α–potential density

$$u^\alpha(x,y) = \int_0^\infty e^{-\alpha t} p_t(x,y)\,dt \tag{1.1}$$

which we will assume is finite for all $\alpha > 0$ and $x, y \in S$. This insures that the local time $L = \{L_t^y, (t,y) \in R^+ \times S\}$ of X exists which we normalize by taking

$$E^x \int_0^\infty e^{-\alpha t}\,dL_t^y = u^\alpha(x,y) \tag{1.2}$$

Let λ be an exponential random variable with mean $1/\alpha$, i.e. $\mathrm{Prob}(\lambda > y) = \exp(-\alpha y)$, $y > 0$, and let ϵ be a Rademacher random variable, i.e. ϵ takes on the values -1 and 1 each with probability $1/2$. Assume that X, λ and ϵ are all independent of each other. We use the Dynkin Isomorphism Theorem [2], [3] to obtain

$$E^0\left(\exp(sL_\lambda^x)\right) \qquad 0, x \in S \tag{1.3}$$

and

$$E^0\left(\exp(s(L_\lambda^x - L_\lambda^y)\right) \qquad 0, x, y \in S \tag{1.4}$$

and similar expressions with s replaced by ϵs as functions of $u^\alpha(x,y)$. (If S contains a zero we will denote it by 0. Otherwise 0 just denotes some element of S. Also whenever we write the expectation symbol, without further explanation, we mean that the expectation is taken with respect to all the random variables present).

Denote the Laplace transform of a function $f : R^+ \to R$ by

$$\mathcal{L}(f) = \int_0^\infty e^{-\alpha t} f(t)\,dt$$

The research of both authors was supported by grants from the National Science Foundation.

Considering the distribution of the random variable λ it is clear that

$$\mathcal{L}\left(E^0\left(\exp(sL^x_t)\right)\right) = \frac{1}{\alpha}\left(E^0\left(\exp(sL^x_\lambda)\right)\right) \qquad 0, x \in S \tag{1.5}$$

and similarly with s replaced by ϵs. Thus the same computation gives us both the moment generating function of L^x_λ and the Laplace transform of the moment generating function of L^x_t and similarly in the cases considered in (1.4). (When we consider the moment generating function we consider α fixed and s as a variable. When we consider the Laplace transform we consider the right–hand–side of (1.5) as functions of α, since λ depends on α, and s as a useful parameter. This will be clearer in the statement of Lemma 2.2). In [6] we obtain estimates for the inverse Laplace transforms which enables us to estimate the moment generating functions of some functions of the local time for fixed t.

We will also be concerned with symmetric random walks and even though, in some sense, it is possible to consider them within the above framework, it will be simpler and clearer to consider them separately. Let $X = \{X_n, n \geq 0\}$ be a symmetric random walk on the d–dimensional integer lattice Z^d on which we put the discrete measure, i.e.

$$X_n = \sum_{i=1}^{n} Y_i \tag{1.6}$$

where the random variables $\{Y_i, i \geq 1\}$ are symmetric, independent and identically distributed with values in Z^d. Therefore X has symmetric transition probabilities $p_n(x, y)$. In this case we define the α–potential

$$u^\alpha(x, y) = \sum_{n=0}^{\infty} e^{-\alpha n} p_n(x, y) \tag{1.7}$$

The local time $L = \{L^y_n, (n, y) \in N \times Z^d\}$ of X is simply the family of random variables

$$L^y_n = \{\text{number of times } X_j = y,\ 0 \leq j \leq n\}$$

Note that analagous to (1.2)

$$E^x \sum_{n=0}^{\infty} e^{-\alpha n}\left(L^y_n - L^y_{n-1}\right) = u^\alpha(x, y) \tag{1.8}$$

Let λ be an geometric random variable, i.e. $\text{Prob}(\lambda = k) = (1-\exp(-\alpha))\exp(-\alpha k)$, $k \in N$, and let ϵ be a Rademacher random variable as above. Assume that X, λ and ϵ are all independent of each other. Then using the Dynkin Isomorphism Theorem we can write the terms in (1.3) and (1.4) for the local times of symmetric random walks, in terms of $u^\alpha(x, y)$ as given in (1.7), where the state space is Z^d. In this case we define a discrete Laplace transform $f : N \to R$ by

$$\mathcal{L}(f) = \sum_{n=0}^{\infty} e^{-\alpha n} f(n)$$

Considering the distribution of the random variable λ, it is clear that

$$\mathcal{L}\left(E^0\left(\exp(sL_n^x)\right)\right) = \left(1-e^{-\alpha}\right)^{-1}\left(E^0\left(\exp(sL_\lambda^x)\right)\right) \qquad 0, x \in Z^d \tag{1.9}$$

and similarly for the result analagous to (1.4) and also with s replaced by ϵs in both cases.

In the case of continuous time Markov processes, we will be particularly interested in Lévy processes. Let $\{X(t), t \in R^+\}$ be a symmetric Levy process, i.e.

$$E\exp(i\lambda X(t)) = \exp(-t\psi(\lambda)) \tag{1.10}$$

where

$$\psi(\lambda) = 2\int_0^\infty (1-\cos\lambda u)\nu(du) \tag{1.11}$$

for ν a Levy measure. X has a local time if and only if $(\gamma+\psi(\lambda))^{-1} \in L^1(R^+)$ for some $\gamma > 0$, and consequently for all $\gamma > 0$. For symmeric Lévy processes the transition probability density $p_t(x,y)$ is a function of $|x-y|$ and we will denote $p_t(0,v)$ by $p_t(v)$. Similarly, we will denote $u^\alpha(0,v)$ by $u^\alpha(v)$. For symmetric Lévy processes we have

$$u^\alpha(x) = \frac{2}{\pi}\int_0^\infty \frac{\cos\lambda x}{\alpha+\psi(\lambda)}\,d\lambda \qquad \forall\alpha > 0 \tag{1.12}$$

In general $u^0(0)$ does not exist. Nevertheless

$$u^\alpha(0) - u^\alpha(x) = \frac{2}{\pi}\int_0^\infty \frac{1-\cos\lambda(x)}{\alpha+\psi(\lambda)}\,d\lambda \qquad \forall\alpha \geq 0 \tag{1.13}$$

exists whenever $(\gamma+\psi(\lambda))^{-1} \in L^1(R^+)$ for some $\gamma > 0$. When we write $u^0(0)-u^0(x)$ we mean the right–hand–side of (1.13) with $\alpha = 0$

In Section 2 we obtain expressions for (1.3) and (1.4) for the local times of symmetric Lévy processes in terms of $u^\alpha(x)$. Our results which are given in Lemma 2.2 are not new. Sid Port pointed out to us that they can also be obtained from the proof of Lemma 3.26, Chapter V, in [1]. Also it is possible to obtain the moments of L_t^x and $L_t^x - L_t^y$ using the strong Markov properties of Lévy processes and stochastic integrals, (see e.g. [4] and [7]) and given the moments one can construct the moment generating functions. However, given the Isomorphism Theorem, the derivation given here is completely trivial. Furthermore it is not much more complicated to use this approach for general strongly symmetric Markov processes than it is for Lévy processes and it indicates how other, more complicated moment generating functions, may be obtained. (For example that of finite joint distributions of the local time at different points in the state space for t fixed.) In Section 3 we will state some results in more general cases of strongly symmetric Markov processes than Lévy processes. In Section 4 we will give the evaluations of (1.3) and (1.4) for symmetric random walks. The proofs are essentially the same as the proofs for continuous time processes.

2. Lévy Processes. It will be useful to have the following list of results for normal random variables. We use the notation $z \in N(0, a^2)$ to indicate that z is a normal random variable with mean zero and variance a^2.

Lemma 2.1. *Let $\xi \in N(0, a^2)$ and $\eta \in N(0, b^2)$ be independent. Then*

$$E \exp\left(\frac{s\xi\eta}{2}\right) = \left(1 - \frac{a^2b^2s^2}{4}\right)^{-1/2} \tag{2.1}$$

$$E\xi\eta \exp\left(\frac{s\xi\eta}{2}\right) = \frac{a^2b^2s}{2}\left(1 - \frac{a^2b^2s^2}{4}\right)^{-3/2} \tag{2.2}$$

$$E\eta^2 \exp\left(\frac{s\xi\eta}{2}\right) = b^2\left(1 - \frac{a^2b^2s^2}{4}\right)^{-3/2} \tag{2.3}$$

$$E \exp\left(\frac{s\eta^2}{2}\right) = \left(1 - b^2s\right)^{-1/2} \tag{2.4}$$

$$E\eta^2 \exp\left(\frac{s\eta^2}{2}\right) = b^2\left(1 - b^2s\right)^{-3/2} \tag{2.5}$$

Proof. Let $x, y \in N(0,1)$ be independent. It is easy to verify that

$$E \exp(vxy) = E \exp\left(\frac{v^2y^2}{2}\right) = (1 - v^2)^{-1/2} \tag{2.6}$$

Differentiating (2.6) with respect to v we get

$$Exy \exp(vxy) = v(1 - v^2)^{-3/2} \tag{2.7}$$

and

$$Ey^2 \exp\left(\frac{v^2y^2}{2}\right) = \left(1 - v^2\right)^{-3/2} \tag{2.8}$$

It follows from (2.8) that

$$Ey^2 \exp(vxy) = Ey^2 \exp\left(\frac{v^2y^2}{2}\right) = (1 - v^2)^{-3/2} \tag{2.9}$$

Setting $x = \xi/a$, $y = \eta/b$ and $v = (sab)/2$ in (2.6), (2.7) and (2.9) we get (2.1), (2.2) and (2.3). Setting $y = \eta/b$ and $v^2 = sb^2$ in the middle term of (2.6) we get (2.4). Differentiating (2.4) gives us (2.5).

We can now give our results on the moment generating functions of the local times of real valued Lévy processes.

Lemma 2.2. *Let $\{X(t), t \in R^+\}$ be a symmetric Levy process for which $(\gamma + \psi(\lambda))^{-1} \in L^1(R^+)$ for some $\gamma > 0$, let u^α be as given in (1.12) and let*

$$a^2 = 2(u^\alpha(0) - u^\alpha(x - y)) \quad \textit{and} \quad b^2 = 2(u^\alpha(0) + u^\alpha(x - y)) \tag{2.10}$$

The following three equations hold for all $\alpha > 0$

$$E^0\left(\exp(sL_\lambda^x)\right) = \left(\frac{u^\alpha(x)}{u^\alpha(0)}\frac{1}{(1-u^\alpha(0)s)} + \left(1 - \frac{u^\alpha(x)}{u^\alpha(0)}\right)\right) \tag{2.11}$$

$$\begin{aligned} &E^0\left(\exp(s(L_\lambda^x - L_\lambda^y))\right) \\ &\quad = \left(\frac{(u^\alpha(x)-u^\alpha(y))s}{1-\frac{a^2b^2s^2}{4}} + \frac{2(u^\alpha(x)+u^\alpha(y))}{b^2\left(1-\frac{a^2b^2s^2}{4}\right)} + \left(1 - \frac{2(u^\alpha(x)+u^\alpha(y))}{b^2}\right)\right) \end{aligned} \tag{2.12}$$

and

$$E^0\left(\exp(\epsilon s(L_\lambda^x - L_\lambda^y))\right) = \left(\frac{2(u^\alpha(x)+u^\alpha(y))}{b^2\left(1-\frac{a^2b^2s^2}{4}\right)} + \left(1 - \frac{2(u^\alpha(x)+u^\alpha(y))}{b^2}\right)\right) \tag{2.13}$$

We prove Lemma 2.2 by using a form of Dynkin's Isomorphism Theorem.

Lemma 2.3. *Let $\mathcal{X} = \{x_i\}_{i=1}^\infty$ be a countable subset of S. Let X be a strongly symmetric standard Markov process , as described in the Introduction, with α–potential given by (1.1). Let L_λ^x be the local time of X at x at the independent exponential time λ and let $\mathbf{L} = \{L_\lambda^{x_i}, x_i \in \mathcal{X}\}$. Let $\{G(x), x \in \mathcal{S}\}$ be a mean zero real valued Gaussian process with covariance u^α and let $\mathbf{G} = \{G(x_i), x_i \in \mathcal{X}\}$. Then for all $\mathcal{C}$ measurable non–negative functions F on R^∞ we have*

$$E^0 E_G\left(F\left(\mathbf{L} + \frac{\mathbf{G}^2}{2}\right), \zeta > \lambda\right) = \alpha \int_S E_G\left(F\left(\frac{\mathbf{G}^2}{2}\right) G(0)G(v)\right) m(dv) \tag{2.14}$$

where E^0 is the expected value on the (possibly sub–) probability space $P^0 \times \mu$ where μ the probability measure of λ and $\mathcal{C}$ denotes the σ–algebra generated by the cylinder sets of R^∞. In particular

$$\begin{aligned} &E^0\left(\exp\left(\sum_{i=1}^n \beta_i L_\lambda^{x_i}\right), \zeta > \lambda\right) E_G\left(\exp\left(\sum_{i=1}^n \beta_i \frac{G^2(x_i)}{2}\right)\right) \\ &\qquad = \alpha \int_S E_G\left(\exp\left(\sum_{i=1}^n \beta_i \frac{G^2(x_i)}{2}\right) G(0)G(v)\right) m(dv) \end{aligned} \tag{2.15}$$

proof. Proofs of different versions of the Dynkin Isomorphism Theorem are given in [5]. The version given in (2.14) follows from Example 1 of this reference. In Example 1 we restrict our attention to a compact subset A of S. Here we let that subset expand to cover the whole space. Also in Example 1 we only consider the case in which $\alpha = 1$. It is not difficult to see that this is the correct generalization for arbitrary $\alpha > 0$.

Proof of Lemma 2.2. We use (2.15). For the proof of (2.11) we must evaluate

$$I = E\left(\exp\left(\frac{s(G^2(x))}{2}\right)\right) \tag{2.16}$$

and

$$II=\int_{-\infty}^{\infty}E\left(\exp\left(\frac{s(G^2(x))}{2}\right)G(0)G(v)\right)\,dv \tag{2.17}$$

Let $\xi=G(x)$ and note that $E\xi^2=u^\alpha(0)$. We write

$$G(0)=\frac{u^\alpha(x)}{u^\alpha(0)}\xi+\rho\qquad G(v)=\frac{u^\alpha(v-x)}{u^\alpha(0)}\xi+\tau(v)$$

where ρ and $\tau(v)$ are independent of ξ. Note that

$$\int_{-\infty}^{\infty}u^\alpha(v-x)\,dv=\frac{1}{\alpha}\qquad\forall x\in R \tag{2.18}$$

Using (2.18) we see that

$$II=\frac{1}{\alpha}\left(\frac{u^\alpha(x)}{(u^\alpha(0))^2}E\left(\xi^2\exp\left(\frac{s\xi^2}{2}\right)\right)\right)+E\left(\exp\left(\frac{s\xi^2}{2}\right)\right)\int_{-\infty}^{\infty}\left(E\rho\tau(v)\right)\,dv$$

By (2.18) and Lemma 2.1 this

$$=\frac{1}{\alpha}\left(\frac{u^\alpha(x)}{u^\alpha(0)}\frac{1}{(1-u^\alpha(0)s)^{3/2}}+\left(1-\frac{u^\alpha(x)}{u^\alpha(0)}\right)\frac{1}{(1-u^\alpha(0)s)^{1/2}}\right)$$

Also by Lemma 2.1

$$I=\frac{1}{(1-u^\alpha(0)s)^{1/2}}.$$

Substituting the expressions for I and II into (2.15) we get (2.11).

Essentially the same proceedure used to obtain (2.11) is used to obtain the other moment generating functions in Lemma 2.2. For (2.12) we evaluate

$$I=E\exp\left(\frac{s(G^2(x)-G^2(y))}{2}\right)$$

and

$$II=\int_{-\infty}^{\infty}E\left(\exp\left(\frac{s(G^2(x)-G^2(y))}{2}\right)G(0)G(v)\right)\,dv$$

Let $\xi=G(x)-G(y)$ and $\eta=G(x)+G(y)$. Note that $E\xi^2=a^2$, $E\eta^2=b^2$ and ξ and η are independent. We write

$$G(0)=\frac{u^\alpha(x)-u^\alpha(y)}{a^2}\xi+\frac{u^\alpha(x)+u^\alpha(y)}{b^2}\eta+\rho$$
$$G(v)=\frac{u^\alpha(x-v)-u^\alpha(y-v)}{a^2}\xi+\frac{u^\alpha(x-v)+u^\alpha(y-v)}{b^2}\eta+\tau(v)$$

where both ρ and $\tau(v)$ are independent of ξ and η. Using (2.18) we see that

$$II=\frac{1}{\alpha}\left(\frac{2(u^\alpha(x)+u^\alpha(y))}{b^4}E\left(\eta^2\exp\left(\frac{s\xi\eta}{2}\right)\right)\right.$$
$$\left.+\frac{2(u^\alpha(x)-u^\alpha(y))}{a^2b^2}E\left(\xi\eta\exp\left(\frac{s\xi\eta}{2}\right)\right)\right)+\int_{-\infty}^{\infty}E(\rho\tau(v))\,dvE\exp\left(\frac{s\xi\eta}{2}\right)$$

where

$$\int_{-\infty}^{\infty} E(\rho\tau(v))\,dv = \frac{1}{\alpha}\left(1 - \frac{2(u^\alpha(x)+u^\alpha(y))}{b^2}\right)$$

Also, obviously

$$I = E\exp\left(\frac{s\xi\eta}{2}\right)$$

We get (2.12) from these observations and Lemma 2.1. Furthermore (2.13) follows immediately from (2.12) since ϵs takes on the values s and $-s$ each with probability 1/2. This completes the proof of Lemma 2.2.

Remark 2.4. Although it is complicated to invert the Laplace transforms of the expressions given in Lemma 2.2, (recall (1.5)), it is not difficult to find expressions for the moments of the local time at fixed t in terms of the transition probability density functions. We illustrate this with a few examples. We see from (2.10) and (1.5) that

$$\mathcal{L}\left(E^0\left(\exp(sL_t^x)\right)\right) = \frac{1}{\alpha}\left(1 + \sum_{n=1}^{\infty} u^\alpha(x)(u^\alpha(0))^{n-1}s^n\right) \tag{2.19}$$

By definition

$$u^\alpha(x) = \mathcal{L}(p_t(x))$$

We consider $p_t(x)$ as a function of t and denote

$$\left(p_\cdot(x) * p_\cdot(y)\right)(u) = \int_0^u p_{(u-s)}(x)p_s(y)\,ds$$

Therefore it follows from (2.19) that

$$E^0\left(\exp(sL_t^x)\right) = 1 + \sum_{n=1}^{\infty}\int_0^t \left(p_\cdot(x) * p_\cdot(0)^{*(n-1)}\right)(u)\,du\,s^n \tag{2.20}$$

and in particular

$$E^0(L_t^x)^n = n!\int_0^t \left(p_\cdot(x) * p_\cdot(0)^{*(n-1)}\right)(u)\,du \tag{2.21}$$

Similarly it follows from (2.12) and (1.5) that

$$\begin{aligned}
&\mathcal{L}\left(E_\epsilon E^0\left(\exp(s\epsilon(L_t^x - L_t^y))\right)\right) = \\
&\quad \frac{1}{\alpha}\left(1 + \sum_{n=1}^{\infty}\left((u^\alpha(0))^2 - (u^\alpha(x-y))^2\right)^{n-1}(u^\alpha(x)-u^\alpha(y))s^{2n-1}\right. \\
&\quad \left. + \sum_{n=1}^{\infty}\left((u^\alpha(0))^2 - (u^\alpha(x-y))^2\right)^{n-1}(u^\alpha(x)+u^\alpha(y))(u^\alpha(0)-u^\alpha(x-y))s^{2n}\right)
\end{aligned} \tag{2.22}$$

and thus for all $n \geq 1$

(2.23) $$E^0 \left(L_t^x - L_t^y\right)^{2n} =$$

$$(2n)! \int_0^t \left((p_\cdot(x) + p_\cdot(y)) * (p_\cdot(0) - p_\cdot(x-y)) * \left(p_\cdot^{*2}(0) - p_\cdot^{*2}(x-y)\right)^{*(n-1)} \right)(u)\, du$$

and

(2.24)
$$\begin{aligned} & E^0 \left(L_t^x - L_t^y\right)^{2n-1} \\ = \quad & (2n-1)! \int_0^t \left((p_\cdot(x) - p_\cdot(y)) * \left(p_\cdot^{*2}(0) - p_\cdot^{*2}(x-y)\right)^{*(n-1)} \right)(u)\, du \end{aligned}$$

As a consequence of (2.13) we get the following Lemma for symmetric Levy processes.

Lemma 2.5. *Let $\{X(t), t \in R^+\}$ be a symmetric Levy process for which $(\gamma + \psi(\lambda))^{-1} \in L^1(R^+)$ for some $\gamma > 0$ and let $u^\alpha(x)$ and $u^\alpha(0) - u^\alpha(x)$ be as given in (1.12) and (1.13). Then for all $x, y, \delta \in R$*

(2.25) $$u^\alpha(x) + u^\alpha(y) \leq u^\alpha(0) + u^\alpha(x-y) \qquad \forall \alpha \geq 0$$

and

(2.26) $$|u^\alpha(x) - u^\alpha(x-\delta)| \leq u^\alpha(0) - u^\alpha(\delta) \qquad \forall \alpha \geq 0$$

The inequality in (2.26) is very interesting. Since u^α is the Fourier transform of a measure it is non–negative definite and hence

(2.27) $$|u^\alpha(x) - u^\alpha(x-\delta)| \leq \left(u^\alpha(0)\right)^{1/2} \left(u^\alpha(0) - u^\alpha(\delta)\right)^{1/2} \qquad \forall \alpha \geq 0$$

Note that (2.26) gives much more control over the increments of u^α than (2.27) does. Furthermore, the right–hand–side of (2.27) need not exist when $\alpha = 0$ whereas the right–hand–side of (2.26) does. The inequality in (2.25) can be obtained in various ways. It is related to the probability that $\{X(t), t \leq \lambda\}$ does not hit either x or y. Our proof is a completely formal consequence of (2.13). The expression in (2.26) is an immediate consequence of (2.25).

Proof of Lemma 2.5. Consider (2.13) with α fixed. Let

$$\delta = \left(1 - \frac{2(u^\alpha(x) + u^\alpha(y))}{b^2}\right)$$

We see that (2.25) follows from the observation that $\delta \geq 0$. Suppose that $\delta < 0$. Then

(2.28) $$E^0 \left(\exp(s\epsilon(L_\lambda^x - L_\lambda^y)) - \delta\right) = \frac{2(u^\alpha(x) + u^\alpha(y))}{b^2 \left(1 - \frac{a^2 b^2 s^2}{4}\right)}$$

Note that in this case $(1-\delta)^{-1}$ times the right–hand–side of (2.28) would be the moment generating function of $\xi\eta+\xi'\eta'$ for ξ and η as given in Lemma 2.1. But this random variable does not have a point mass at the origin whereas the random variable corresponding to the normalised left–hand–side of (2.28) does. Thus $\delta \geq 0$ and we get (2.25). If we substitute δ for y in (2.25) we get

$$u^\alpha(x)-u^\alpha(x-\delta)\leq u^\alpha(0)-u^\alpha(\delta)$$

and substituting $-\delta$ for y and $x-\delta$ for x in (2.25) we get

$$u^\alpha(x-\delta)-u^\alpha(x)\leq u^\alpha(0)-u^\alpha(\delta)$$

These last two inequalities give us (2.26).

3. Strongly Symetric Markov Processes. Lemma 2.3 is not restricted to real valued Lévy processes. It is just that for these processes there are simplifications that are not available when considering more general classes of processes. Besides the fact that, in general, $u^\alpha(x,y)$ is no longer a function of $|x-y|$, we must also consider the lifetime of the processes. Let ζ denote the lifetime of X, for X as defined in the Introduction. As a generalization of (2.18) we have

$$\begin{aligned}(3.1)\qquad \int_{-\infty}^{\infty}u^\alpha(x,v)\,dv&=\int_0^\infty e^{-\alpha t}\int_{-\infty}^{\infty}p_t(x,v)\,dv\,dt\\ &=\int_0^\infty e^{-\alpha t}P^x(\zeta>t)\,dt=\frac{1}{\alpha}P^x(\zeta>\lambda)\qquad\forall x\in R\end{aligned}$$

Therefore, following the proof of (2.11), but considering the general case we see that

$$\begin{aligned}(3.2)\quad &E^0\left(\exp(sL^x_\lambda),\zeta>\lambda\right)\\ &=\left(\frac{u^\alpha(0,x)}{u^\alpha(x,x)}\frac{P^x(\zeta>\lambda)}{(1-u^\alpha(x,x)s)}+\left(P^0(\zeta>\lambda)-\frac{P^x(\zeta>\lambda)u^\alpha(0,x)}{u^\alpha(x,x)}\right)\right)\end{aligned}$$

In the extension of (2.12) another simplification is lost since now

$$\int_{-\infty}^{\infty}\left(u^\alpha(x,v)-u^\alpha(y,v)\right)\,dv=P^x(\zeta>\lambda)-P^y(\zeta>\lambda)\qquad\forall x,y\in R$$

This doesn't cause any difficulties. It just results in a longer expression. There doesn't seem to be any reason to write out the extensions of (2.12)–(2.13) at this point. It is enough to note that they are available if needed.

4. Random Walks. The same proofs used in the case of Lévy processes gives analagous results for symmetric random walks as described in the Introduction. We note that as for Lévy processes the transition probabilites $p_n(x,y)=p_n(|x-y|)$ and thus similarly for the α–potential. However, there is a significant difference in that local time exists only for one dimensional Lévy processes whereas it can be defined for any random walk. This doesn't cause us any difficulties because the version of the Dynkin Isomorphism Theorem that we use requires that the local time exists but does not depend on the nature of the state space. The next Lemma is simply a restatement of Lemma 2.2 for random walks.

Lemma 4.1. *Let X be a symmetric random walk on Z^d as given in (1.6), so that λ is an integer valued random variable. Let u^α be as defined in (1.7). Then (2.11)–(2.13) hold for the local time of X if the state space is taken to be Z^d.*

Proof. Consider (2.15). A proof of this equation is given in [5], as explained in the proof of Lemma 2.3, for continuous time but the proof is almost exactly the same for discrete time. Of course, in this case ζ is infinite, the Gaussian process is defined to have covariance u^α given in (1.7) and the integral is replaced by a sum over Z^d. Also λ is integer valued. This means that α, on the right–hand–side of (2.15) is replaced by $(1-\exp(-\alpha))$. The reason for this is simple. α appears because $\{\alpha\exp(-\alpha t), t\geq 0\}$ is the probability density function in the continuous case. Similarly, $\{(1-\exp(-\alpha))\exp(-\alpha n), n\geq 0\}$ is the probability density function in the discrete case. With this difference in (2.15) in mind proceed to the proof of Lemma 2.2. The only point that is different is that instead of (2.18) we have

$$\sum_{v\in Z^d} u^\alpha(x-v)=\sum_{n=0}^\infty e^{-\alpha n}=\frac{1}{1-\exp(-\alpha)}\qquad \forall x\in Z^d$$

Thus last term cancels the corresponding term introduced into the altered version of (2.15) and we see that all the equations (2.11)–(2.13) are valid for symmetric random walks.

Remark 4.2. The statements made about the moments of the local times of symmetric Lévy processes in Remark 2.4 carry over to the moments of symmetric random walks but with the obvious changes as discussed in Lemma 4.1 and it's proof. Lemma 2.5 also carries over if we make the appropriate substitution for (1.13). Let

$$\phi(\xi)=Ee^{i(\xi,Y_1)}\qquad \xi\in T^d$$

where Y_1 is given in (1.9) and $T=[-\pi,\pi]$. It follows that for $y\in Z^d$

$$p_n(y)=\left(\frac{1}{2\pi}\right)^d\int_{T^d}\cos(\xi,y)\phi^n(\xi)\,d\xi$$

and therefore

$$u^\alpha(x)-u^\alpha(y)=\left(\frac{1}{2\pi}\right)^d\int_{T^d}\frac{\cos(\xi,x)-\cos(\xi,y)}{1-\exp(-\alpha)\phi(\xi)}\,d\xi \tag{4.1}$$

This equation is valid for all $\alpha\geq 0$. We define $u^0(0)-u^0(x)$ by the right–hand–side of (4.1). With this definition and (1.7) we get Lemma 2.5 for the symmetric random walks described in the Introduction.

References

1 Blumenthal, R.M. and Getoor, R.K. Markov processes and potential theory, 1968, Academic Press, New York.

2 Dynkin, E.B. (1983) Local times and quantum fields, In: Seminar on stochastic processes (E. Çinlar, K.L. Chung, R.K. Getoor, Eds), Birkhäuser, Boston.

3 Dynkin, E.B. (1984) Gaussian and non–Gaussian random fields associated with Markov processes, J. Funct. Anal. 55, 344–376.

4 Fitzsimmons, P. and Getoor, R. (1991) On the distribution of the Hilbert transform of the local time of a symmetric Lévy process, preprint.

5 Marcus, M. and Rosen, J. (1990) Sample path properties of the local times of strongly symmetric Markov processes via Gaussian processes, preprint.

6 Marcus, M. and Rosen, J. (1991) Laws of the iterated logarithm for the local times of symmetric Lévy processes and recurrent random walks, preprint.

7 Rosen, J. (1990) Second order limit laws for the local times of stable processes, preprint

Michael B. Marcus
Department of Mathematics
The City College of CUNY
New York, NY 10031

Jay Rosen
Department of Mathematics
College of Staten Island, CUNY
Staten Island, NY 10301