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A JOINTLY CONTINUOUS LOCAL TIME FOR TRIPLE INTERSECTIONS OF A STABLE PROCESS IN THE PLANE

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We study the asymptotics of the triple intersection local time $\alpha(x, y, t)$ for the symmetric stable process X_t of index $\beta > \frac{8}{5}$ in $\mathbb{R}^2$, as $x, y \to 0$. This describes the set of "near-intersections", $\{r \le s \le t \le T \mid X_s - X_r = x, \ X_t - X_s = y\}$. With $U(x) = c(\beta)/|x|^{2-\beta}$ we show that $\alpha(x, y, T) - TU(x)U(y) - U(x)\hat{\alpha}(y, T) - U(y)\hat{\alpha}(x, T)$ has a continuous extension, where $\hat{\alpha}(x, T)$ is a continuous function, the renormalized local time for double intersections.

KEY WORDS: Stable process, triple intersections, local time.

1 INTRODUCTION

Let X_t be a symmetric stable process of index β in the plane, with density

$$p_t(x) = \frac{1}{(2\pi)^2} \int e^{iq\cdot x} e^{-|q|^\beta} \, d^2q. \tag{1.1}$$

When $\beta = 2$, X_t is a Brownian motion.

We will use the notation $X(r, s) \doteq X_s - X_r$. If $\beta > \frac{4}{3}$, then the random field

$$Y(r, s, t) = (X(r, s), X(s, t))$$

has a local time over any bounded Borel set $B \subseteq R^3_{\le} \doteq \{(r, s, t) \mid 0 \le r \le s \le t\}$, i.e. there exists a function $\alpha(x, y, B)$ such that

$$\int_B f(Y(r, s, t)) \, dr \, ds \, dt = \int f(x, y)\alpha(x, y, B) \, d^2x \, d^2y, \tag{1.2}$$

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for all bounded Borel functions $f: \mathbb{R}^4 \to \mathbb{R}$. In addition, when B lies away from both diagonals $r = s$, $s = t$, we can take $\alpha(x, y, B)$ to be a measure in the set variable B, and weakly continuous in x, y. The measure $\alpha(0, 0, \cdot)$ is supported on

$$\{(r, s, t) | X_r = X_s = X_t\}.$$

Because of this, we call $\alpha(x, y, B)$ a triple intersection local time. The results above are essentially due to Shieh [1986], and in Section 7 we shall review them.

In this paper we remove the restriction that B lies away from the diagonals, and study the behavior of $\alpha(x, y, B)$ for arbitrary bounded Borel sets $B \subseteq R^3_{\leq}$. It will turn out that for $\beta > \frac{8}{5}$, we can choose a version of $\alpha(x, y, B)$ which is a measure in B, weakly continuous in x, $y \neq (0, 0)$, see Section 6. In general, $\alpha(x, y, B)$ "blows up" as $x, y \to 0$. Let

$$B_T = \{(r, s, t) | 0 \leq r \leq s \leq t \leq T\}$$

and with $c(\beta) = (1/2^\beta \pi)\Gamma(2 - \beta/2)/\Gamma(\beta/2)$, let

$$U(x) = \int_0^\infty p_t(x)\, dt = \frac{c(\beta)}{|x|^{2-\beta}}, \qquad \beta < 2,$$

the potential of our process. The main result of this paper, established in Section 5, is that with $\alpha(x, y, T) \doteq \alpha(x, y, B_T)$, $\beta > \frac{8}{5}$,

$$\alpha(x, y, T) - TU(x)U(y) - U(x)\hat{\alpha}(y, T) - U(y)\hat{\alpha}(x, T) \tag{1.3}$$

has a continuous extension to all x, y, T. Here $\hat{\alpha}(x, T)$ is the continuous version of the renormalized intersection local time for double intersections, see Rosen (1988). It is defined by the a.s. limit

$$\hat{\alpha}(x, T) = \lim_{\varepsilon \to 0} \int_0^T \int_0^t p_{\varepsilon, x}(X_t - X_s)\, ds\, dt - TU_\varepsilon(x),$$

where $p_{\varepsilon, x}(a) = p_\varepsilon(x - a)$ and $U_\varepsilon(x) = \int_\varepsilon^\infty p_t(x)\, dt$. Thus (1.3) isolates the singularity of $\alpha(x, y, T)$ as $x, y \to 0$. We note that (1.3) has recently been established for the case $\beta = 2$, Brownian motion in the plane, using stochastic integrals, Rosen-Yor [1991].

To establish (1.3) for all $\beta > \frac{8}{5}$ we study the functional

$$I(\varepsilon, x, y, B) = \int_B \{p_{\varepsilon, x}(X(r, s))\}_0 \{p_{\varepsilon, y}(X(s, t))\}_0\, dr\, ds\, dt, \tag{1.4}$$

where for any random variable Z we use the notation $\{Z\}_0 = Z - E(Z)$. In Sections 2 and 3 we will show that with probability 1, $I(\varepsilon, x, y, B)$ has a limit $I(x, y, B)$ as $\varepsilon \to 0$, which we call the renormalized triple intersection local time. We will show that

$I(x, y, T) \doteq I(x, y, B_T)$ is jointly continuous in (x, y, T) and, modulo a regular term, is equal to (1.3).

Along the way we will find a nice approximation to $I(x, y, B_T)$. Let

$$D(2) = \bigcup_{n=1}^{\infty} \bigcup_{k=1}^{2^{n-1}} A(k, n, 2), \tag{1.4}$$

$$A(k, n, 2) = \left[\frac{2k-2}{2^n}, \frac{2k-1}{2^n}\right] \times \left[\frac{2k-1}{2^n}, \frac{2k}{2^n}\right]^2_{\leq},$$

where

$$[a, b]^2_{\leq} \doteq \{(s, t) | a \leq s \leq t \leq b\},$$

and

$$D(1) = \bigcup_{n=1}^{\infty} \bigcup_{k=1}^{2^{n-1}} A(k, n, 1),$$

$$A(k, n, 1) = \left[\frac{2k-2}{2^n}, \frac{2k-1}{2^n}\right]^2_{\leq} \times \left[\frac{2k-1}{2^n}, \frac{2k}{2^n}\right],$$

We note that $B_1 = D(1) \cup D(2)$. To see this, if $(r, s, t) \in B_1$, take n to be the smallest place in which the dyadic expansions of r, s, and t differ.

We now describe the approximation we will use for $I(x, y, D(2))$. There will be an analogous approximation for $I(x, y, D(1))$. Let

$$D(2, N, M) = \bigcup_{n=1}^{N} \bigcup_{k=1}^{2^{n-1}} A(k, n, M, 2) \tag{1.5}$$

$$A(k, n, M, 2) = \left[\frac{2k-2}{2^n}, \frac{2k-1}{2^n}\right] \times \left\{\left(\frac{2k-1}{2^n}, \frac{2k-1}{2^n}\right) + \bigcup_{m=1}^{M} \bigcup_{l=1}^{2^{m-1}} \frac{1}{2^n}\left[\frac{2l-2}{2^m}, \frac{2l-1}{2^m}\right] \times \left[\frac{2l-1}{2^m}, \frac{2l}{2^m}\right]\right\}$$

In Section 4 we will show that

$$I(x, y, D(2, N, M)) \to I(x, y, D(2))$$

in all L^p spaces, and a.s., as $N, M \to \infty$.

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