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THE RANGE OF STABLE RANDOM WALKS

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Limit theorems are proved for the range of d-dimensional random walks in the domain of attraction of a stable process of index β. The range R_n is the number of distinct sites of $\mathbb{Z}^d$ visited by the random walk before time n. Our results depend on the value of the ratio β/d. The most interesting results are obtained for $2/3 < \beta/d \le 1$. The law of large numbers then holds for R_n, that is, the sequence $R_n/E(R_n)$ converges toward some constant and we prove the convergence in distribution of the sequence $(\operatorname{var} R_n)^{-1/2}(R_n - E(R_n))$ toward a renormalized self-intersection local time of the limiting stable process. For $\beta/d \le 2/3$, a central limit theorem is also shown to hold for R_n, but, in contrast with the previous case, the limiting law is normal. When $\beta/d > 1$, which can only occur if $d = 1$, we prove the convergence in distribution of $R_n/E(R_n)$ toward some constant times the Lebesgue measure of the range of the limiting stable process. Some of our results require regularity assumptions on the characteristic function of X.

1. Introduction. Let $X = (X_n, n \ge 0)$ be a random walk on the d-dimensional integer lattice $\mathbb{Z}^d$,

$$X_n = x_0 + \sum_{i=1}^{n} Y_i,$$

where $x_0 \in \mathbb{Z}^d$ and the random variables $(Y_i, i \ge 1)$ are independent identically distributed with values in $\mathbb{Z}^d$. We assume for convenience that the law of Y_1 is not supported on a proper subgroup of $\mathbb{Z}^d$. In particular, d is the genuine dimension of the random walk. The range R_n of the random walk is the cardinality of the set $\{X_0, X_1, \dots, X_n\}$, that is, the number of distinct sites visited by the random walk up to time n. Asymptotic properties of the sequence (R_n) have been investigated by many authors after the pioneering work of Dvoretzky and Erdös [2]. Following Jain and Pruitt [7], we consider two specific issues:

1. The law of large numbers: Does $R_n/E(R_n)$ converge almost surely?
2. The central limit theorem: Does $(\operatorname{var} R_n)^{-1/2}(R_n - E(R_n))$ converge in distribution?

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Two basic results concerning question 1 have been obtained by Kesten, Spitzer and Whitman (see [21], page 40) and Jain and Pruitt [7]. Kesten, Spitzer and Whitman proved that the strong law holds for all transient random walks. More precisely,

$$\lim_{n\to\infty} \frac{1}{n} R_n = q \quad \text{a.s.,} \tag{1.a}$$

where $q = P[X_1 \neq x_0, X_2 \neq x_0, \dots] > 0$ if X is transient. Jain and Pruitt proved that the strong law holds for all recurrent two-dimensional random walks.

Question 2 was investigated by Jain and Pruitt [6, 8]. They proved the central limit theorem for R_n with a normal limit law for all random walks in $\mathbb{Z}^d$, $d \geq 3$. More recently, Le Gall [9] was able to prove the central limit theorem for R_n in the case of two-dimensional random walks with zero mean and finite second moments. It is remarkable that in this case the limit law is not normal but is a renormalized self-intersection local time of the planar Brownian motion.

The goal of the present paper is to study questions 1 and 2 in the case of stable random walks, meaning that Y_1 is in the domain of attraction of a stable law. More precisely, we will assume the existence of a constant $\beta \in (0, 2]$ and a function $b(n)$ of regular variation of index $1/\beta$ such that

$$b(n)^{-1} X_n \xrightarrow[n\to\infty]{(d)} U_1, \tag{1.b}$$

where the symbol $\to_{(d)}$ means convergence in distribution and $U = (U_t, t \geq 0)$ is a nondegenerate stable process of index β in $\mathbb{R}^d$. Here the world nondegenerate means that the law of U_1 is not supported on a hyperplane. It follows from (1.b) that U must in fact be strictly stable, in the sense that $U_{\lambda t} =_{(d)} \lambda^{1/\beta} U_t$ for any $\lambda > 0$, $t \geq 0$.

Observe that we do not allow a centering sequence in (1.b). This restriction is essential to our methods, especially when $\beta > 2d/3$. In the case when a centering sequence is allowed, some partial results may be derived from the work of Jain and Orey [5] (see also Jain and Pruitt [7]).

Under (1.b), we obtain essentially three types of results, depending on the value of the ratio β/d. Some of these results require additional assumptions on the law of X: see Sections 4, 6 and 7 for more precise statements.

RESULT 1. If $\beta \leq 2d/3$, the random walk is transient and the strong law holds for R_n by (1.a). We prove in Section 4 that the central limit theorem also holds and that the limiting distribution is normal. To be specific, if $g(n) = \sum_{k=1}^{n} k^2 b(k)^{-2d}$, we have

$$\frac{R_n - E[R_n]}{\sqrt{n g(n)}} \xrightarrow[n\to\infty]{(d)} \sigma N, \tag{1.c}$$

where σ is an explicit constant and N denotes a standard normal variable.

When $\beta < 2d/3$, $g(n)$ increases to a finite limit, so our result can be stated as

$$(1.d)\qquad \frac{R_n - E[R_n]}{\sqrt{n}} \xrightarrow[n\to\infty]{(d)} \sigma N.$$

If $\beta = 2d/3$ and we introduce the slowly varying function s such that $b(n) = n^{1/\beta}s(n)$, we see that

$$g(n) = \sum_{k=1}^{n} k^{-1}s(k)^{-2d}.$$

In this form, it can easily be proved that g is slowly varying (see Lemma 2.2).

RESULT 2. If $2d/3 < \beta \le d$, the random walk may be either transient (this is automatic if $\beta < d$) or recurrent. We prove that

$$(1.e)\qquad \lim_{n\to\infty} n^{-1}h(n)R_n = 1,$$

where $h(n)$ is the truncated Green function

$$h(n) = \sum_{k=0}^{n} P[X_k = X_0].$$

Note that if the random walk is transient $h(n)$ increases to q^{-1} so that (1.e) trivially reduces to (1.a). In any case, h is easily seen to be slowly varying. The convergence in (1.e) is proved to hold almost surely, except in certain cases when $\beta = d = 1$, where we only obtain L^p-convergence (it is actually plausible that the almost sure convergence holds in all cases when $\beta = d = 1$). The central limit theorem also holds for R_n, but the limiting distribution is now a renormalized self-intersection local time of the process U, denoted by γ_U,

$$(1.f)\qquad n^{-2}h(n)^2b(n)^d(R_n - E[R_n]) \xrightarrow[n\to\infty]{(d)} -\gamma_U.$$

Again, if X is transient, (1.f) reduces to

$$(1.g)\qquad n^{-2}b(n)^d(R_n - E[R_n]) \xrightarrow[n\to\infty]{(d)} -q^2\gamma_U.$$

A precise definition of γ_U is given in Section 6. It is interesting to note that the condition $\beta > 2d/3$ is exactly the one needed to ensure the existence of γ_U (see [14], [15]).

RESULT 3. If $\beta > d$, which implies $d = 1$, the strong law does not hold for R_n. Instead we prove

$$(1.h)\qquad b(n)^{-1}R_n \xrightarrow[n\to\infty]{(d)} m(U(0;1)),$$

where m denotes the Lebesgue measure on $\mathbb{R}$ and $U(0;1) = \{U_s; 0 \le s \le 1\}$.

The following table summarizes our main results:

	LLN	CLT
$\beta \leq \frac{2d}{3}$	$\frac{1}{n} R_n \to q$ a.s.	$\frac{R_n - E[R_n]}{\sqrt{ng(n)}} \xrightarrow{(d)} \sigma N$
$\frac{2d}{3} < \beta \leq d$	$\frac{h(n)}{n} R_n \to 1$ a.s.	$\frac{b(n)^d h(n)^2}{n^2} (R_n - E[R_n]) \xrightarrow{(d)} -\gamma_U$
$\beta > d$	$\frac{R_n}{b(n)} \xrightarrow{(d)} m(U(0;1))$	

Among these results, (1.d), (1.e) and (1.h) are most satisfactory since they hold without any other assumption than (1.b). The other results, especially (1.f), require some regularity of the characteristic function of Y_1, which is discussed in Section 5. When $\beta > 1$, this regularity assumption is automatically satisfied (Proposition 5.4). In particular, the results described in the table are valid for all two-dimensional random walks that satisfy (1.b).

In view of (1.f), we can see that our basic assumption that Y_1 be in the domain of attraction of a stable process is very natural. For (1.f) shows that the limit process U actually appears and it is well known that the only possible limit processes for random walks are the stable processes.

Let us describe the connection between (1.c), $\dots$, (1.h) and the previously mentioned results in the literature. In high dimensions ($d \geq 3$), we are in the regime $\beta \leq 2d/3$ and both (1.c) and (1.d) are special cases of the results of [8]. However, (1.d) may have some interest even in the case $d = 3$, since it gives some more information on var R_n than is available in [8]. In lower dimensions ($d = 1, 2$), (1.c) and (1.d) seem to be new. When $d = 2$, (1.e) is a special case of the general result of Jain and Pruitt [7] for two-dimensional recurrent random walks. Thus the main interest of (1.e) is when $d = 1$, since not many results are known for the range of one-dimensional random walks (see the discussion in [7]). Assertions (1.f) and (1.g) extend the main theorem of [9] concerning planar random walks with zero mean and finite second moments. The limiting result (1.h) extends a theorem proved by Jain and Pruitt [7] in the case of random walks with zero mean and finite second moments. As a final remark, one may compare our results (1.f) and (1.g) with the recent work of Dynkin [3]. In the case of planar random walks with zero mean and finite second moments, Dynkin proves limit theorems for certain functionals of the path between 0 and n, with convergence towards k-multiple self-intersection functionals of the planar Brownian motion. The limiting variable γ_U of (1.f) and (1.g) is a special case of these functionals when $k = 2$. One may expect that a more precise asymptotic expansion of R_n would also involve functionals associated with k multiple self-intersections of the process U (see [12, 16, 17] for related results).

We now sketch the main arguments of the proof of our central limit theorems for R_n. We use the following convenient notation:

$$X(a,b) = \{X_n; a \le n \le b\}$$

and for any random variable V,

$$\{V\} = V - E[V].$$

We denote by $|A|$ the cardinality of a finite set A. Take $n \ge 1$ and let $p \ge 1$ be an integer which may depend on n. Then

$$\begin{aligned}\{R_n\} &= \{|X(0,n)|\} \\ &= \sum_{i=1}^{p} \left\{ \left| X\left(\frac{i-1}{p} n, \frac{i}{p} n \right) \right| \right\} \\ &\quad - \sum_{i=2}^{p} \left\{ \left| X\left(0, \frac{i-1}{p} n\right) \cap X\left(\frac{i-1}{p} n, \frac{i}{p} n \right) \right| \right\}.\end{aligned}$$

The right-hand side is the difference of two terms that we denote by $A(n,p)$, $B(n,p)$.

If $\beta \le 2d/3$, we will be able to choose $p = p(n)$ such that $p(n) \to \infty$ as $n \to \infty$ and moreover $B(n,p)$ is small in comparison with $A(n,p)$. Note that $A(n,p)$ is a sum of independent variables. An application of Lindeberg's theorem on triangular arrays then shows that the sequence $\{R_n\}$, suitably normalized, converges in distribution towards a normal law.

If $2d/3 < \beta \le d$, we are in the opposite situation. If we take p large but fixed with respect to n, we will see that $A(n,p)$ is small in comparison with $B(n,p)$. Note that $B(n,p)$ is a sum of intersection terms, each of which can be interpreted as the number of intersection points of two independent random walks. By using the results of [18], we can prove that $B(n,p)$, suitably normalized, is close in distribution to the renormalized intersection local time γ_U.

It is worth noting that these arguments can also be used to study the asymptotics of the Wiener sausage in $\mathbb{R}^d$; see [11, 12]. In fact, it is very plausible that analogues of (1.c), ..., (1.h) hold for sausages associated with stable processes in $\mathbb{R}^d$ (see also [17]).

The paper is organized as follows. Sections 2 and 3 contain a number of preliminary estimates. In particular, we obtain bounds for the moments of the number of intersection points of two independent walks, which play a major role throughout this work. Section 4 deals with the case $\beta \le 2d/3$. Several of our arguments are here taken from Jain and Pruitt [7]. In Section 5 we derive precise estimates on Green functions and hitting times of points, to be used in the last two sections. In Section 6, we treat the case $2d/3 < \beta \le d$. The general outline is the same as in [9], but we use some new ideas which allow us to simplify certain arguments and to solve some open problems of [9]. Finally, Section 7 is devoted to the case $\beta > d$.

2. Notation and preliminaries.

2.1. We consider a random walk $X = (X_0, X_1, \dots)$ taking values in the lattice $\mathbb{Z}^d$. For $x \in \mathbb{Z}^d$, the notation $P_x[\cdot]$ will be used to denote events related to the random walk starting at x. When $x = 0$, we will simply use $P[\cdot]$. Unless otherwise indicated, we will always assume that the random walk starts at 0. The transition probabilities of the random walk are denoted by

$$P_n(x,y) = P_x[X_n = y] = P[X_n = y - x].$$

We will also use the hitting times

$$T_x = \inf\{n \geq 0; X_n = x\} \qquad (\inf \varnothing = \infty).$$

Let $\beta \in (0; 2]$. The following two assumptions play a basic role in this work.

ASSUMPTION A1. The random walk X is in the domain of attraction of a nondegenerate stable process, meaning that there exists a function $b(n)$ of regular variation of index $1/\beta$ such that

$$b(n)^{-1} X_n \xrightarrow[n \to \infty]{(d)} U_1,$$

where U_1 is a nondegenerate stable random variable of index β in $\mathbb{R}^d$.

ASSUMPTION A2. The random walk X is adapted, meaning that, with P_0 probability 1, X does not stay on a proper subgroup of $\mathbb{Z}^d$.

Assumption A2 is not really restrictive. If it is not satisfied, we may consider the smallest subgroup H of $\mathbb{Z}^d$ on which the random walk takes place and then find a linear isomorphism φ from H onto $\mathbb{Z}^m$ for some $m \leq d$. Then $\varphi(X)$ is an adapted random walk on $\mathbb{Z}^m$, which satisfies Assumption A1 if X does.

Assumption A1 is fundamental. Without loss of generality (see [4], page 577–580), we may and will assume that the function b is continuous and monotonically increasing from $\mathbb{R}_+$ onto $\mathbb{R}_+$ and that $b(0) = 0$. We denote by l the inverse of b.

2.2. We will first state a few elementary facts about regularly and slowly varying functions (we refer to [1] for more detailed information on this subject). By definition of a regularly varying function we have

$$b(x) = x^{1/\beta} s(x), \qquad x > 0,$$

where $s\colon (0,\infty) \to (0;\infty)$ is a slowly varying function, meaning that for any $c > 0$,

$$\lim_{x \to \infty} \frac{s(cx)}{s(x)} = 1. \tag{2.a}$$

A basic property of slowly varying functions ([4], page 277) states that the convergence (2.a) holds uniformly when c varies over the interval $[\varepsilon, 1]$ for any $\varepsilon > 0$.

The function l is of regular variation of index β and thus

$$l(x) = x^{\beta} t(x)$$

for some slowly varying function t. It is easily checked that

$$t(x) = s(l(x))^{-\beta}. \tag{2.b}$$

The following lemma is a simple consequence of the definitions. We state it for the function l, but it is clear that a similar result holds for b, with β replaced by $1/\beta$.

LEMMA 2.1. *For any $\varepsilon > 0$, there exist two positive constants C_{ε}, C'_{ε} such that, for any y, z with $1 \le y \le z$,*

$$C_{\varepsilon} y^{\beta-\varepsilon} \le l(y) \le C'_{\varepsilon} y^{\beta+\varepsilon}, \tag{2.c}$$

$$C_{\varepsilon}\left(\frac{z}{y}\right)^{\beta-\varepsilon} \le \frac{l(z)}{l(y)} \le C'_{\varepsilon}\left(\frac{z}{y}\right)^{\beta+\varepsilon}. \tag{2.d}$$

NOTE. Throughout this work, $C, C', C'' \dots$ will denote constants whose values may vary from line to line. When we wish to emphasize the dependence of a constant on a parameter ε, we write C_{ε}, C'_{ε}, and so on.

We shall also need the following elementary fact about slowly varying functions.

LEMMA 2.2. *Let $f\colon \mathbb{R} \to (0;\infty)$ be a slowly varying function. For $x \ge 1$, set*

$$k(x) = \sum_{i=1}^{[x]} \frac{f(i)}{i},$$

where $[x]$ denotes the integer part of x. Then k is slowly varying and $f(n) = o(k(n))$ as $n \to \infty$.

PROOF. We first prove the second assertion. Take c with $0 < c < 1$. Then

$$k(n) \ge \sum_{i=[cn]}^{n} \frac{f(i)}{i} \underset{n\to\infty}{\sim} \left(\log\frac{1}{c}\right) f(n),$$

since f is slowly varying. The desired result follows since c can be taken arbitrarily small. In order to get the first assertion, note that

$$k(x) - k(cx) = \sum_{i=[cx]+1}^{[x]} \frac{f(i)}{i} \underset{x\to\infty}{\sim} \left(\log\frac{1}{c}\right) f([x]) = o(k(x)). \qquad \square$$

REMARK. Since k is increasing it also follows from Lemma 2.2 that for any $0 < \alpha < 1$, we have $f(n)^{\alpha} = o(k(n))$.

2.3. We will now obtain an equivalent form of Assumption A1 in terms of the characteristic function of X_1:

$$\varphi(\xi) = E[\exp i\xi \cdot X_1] \quad \text{for } \xi \in T^d := (-\pi, \pi]^d.$$

We first need some information on the characteristic function Φ of U_1. It is well known [13] that for any $\xi \in \mathbb{R}^d - \{0\}$,

$$\Phi(\xi) = E[\exp i\xi \cdot U_1] = \exp - |\xi|^\beta S\left(\frac{\xi}{|\xi|}\right),$$

where S is a continuous complex-valued function defined on the sphere S^{d-1}, such that

$$\operatorname{Re} S(\omega) \geq a > 0, \qquad \omega \in S^{d-1},$$

for some positive constant a. The latter property follows from the fact that we have assumed U_1 to be nondegenerate, which implies that its law is not supported on a hyperplane.

PROPOSITION 2.3. *Under Assumption* A1 *we have*

$$\varphi(\xi) = 1 - \frac{1}{l(1/|\xi|)} S\left(\frac{\xi}{|\xi|}\right) + o\left(\frac{1}{l(1/|\xi|)}\right)$$

as $|\xi|$ *tends to* 0.

PROOF. For $|\xi|$ small enough we may write

$$\varphi(\xi) = \exp - \psi(\xi),$$

where the function ψ is defined on a neighbourhood of 0 in $\mathbb{R}^d$ and takes values in $\{z; |z| < \frac{1}{2}\}$. Take $\varepsilon > 0$. By Assumption A1 we have for $|\xi| \leq 2$ and for n large enough, say $n \geq n_0$,

$$\left|\exp\left(-n\psi\left(\frac{\xi}{b(n)}\right)\right) - \exp\left(-|\xi|^\beta S\left(\frac{\xi}{|\xi|}\right)\right)\right| < \varepsilon.$$

The properties of the exponential function then imply the existence for each pair (n, ξ) $(n \geq n_0, |\xi| \leq 2)$ of an integer $k(n, \xi)$ such that

$$\left|n\psi\left(\frac{\xi}{b(n)}\right) - |\xi|^\beta S\left(\frac{\xi}{|\xi|}\right) - 2i\pi k(n, \xi)\right| < C\varepsilon,$$

where the constant C does not depend on ε (the point is that S is bounded). We may assume that $C\varepsilon < 1$. Then obviously $k(n, 0) = 0$ [note that $\psi(0) = 0$]. On the other hand, for $|\xi|$, $|\xi'| \leq 2$,

$$2\pi|k(n, \xi) - k(n, \xi')| \leq |F_n(\xi) - F_n(\xi')| + 2C\varepsilon$$

for some continuous function F_n. It follows that if $|\xi - \xi'|$ is small enough, we must have $k(n, \xi) = k(n, \xi')$. We conclude that $k(n, \xi) = 0$ for any ξ. Hence,

$$\left|n\psi\left(\frac{\xi}{b(n)}\right) - |\xi|^\beta S\left(\frac{\xi}{|\xi|}\right)\right| < C\varepsilon$$

for $|\xi| \leq 2$, $n \geq n_0$.

Now choose $\delta > 0$ so small that for any ξ with $1 - \delta \le |\xi| \le 1 + \delta$,

$$\left| |\xi|^{\beta} S\left(\frac{\xi}{|\xi|}\right) - S\left(\frac{\xi}{|\xi|}\right) \right| < \varepsilon.$$

Thus, for $1 - \delta \le |\xi| \le 1 + \delta$, $n \ge n_0$,

$$\left| n\psi\left(\frac{\xi}{b(n)}\right) - S\left(\frac{\xi}{|\xi|}\right) \right| < (C+1)\varepsilon.$$

Since b is regularly varying, we have $b(n+1)/b(n) \to 1$, so that for $\eta \in T^d$, with $|\eta|$ small enough, we may find ξ and $n = n(\eta) \ge n_0$ such that

$$\eta = \frac{\xi}{b(n)} \quad \text{and} \quad 1 - \delta \le |\xi| \le 1 + \delta.$$

It follows that

$$\left| n(\eta)\psi(\eta) - S\left(\frac{\eta}{|\eta|}\right) \right| < (C+1)\varepsilon. \tag{2.e}$$

Now note that

$$l\left(\frac{1-\delta}{|\eta|}\right) \le n(\eta) = l\left(\frac{|\xi|}{|\eta|}\right) \le l\left(\frac{1+\delta}{|\eta|}\right).$$

Hence, for $|\eta|$ small,

$$\begin{aligned} &\left| l\left(\frac{1}{|\eta|}\right)\psi(\eta) - S\left(\frac{\eta}{|\eta|}\right) \right| \\ &\quad \le \left| n(\eta)\psi(\eta) - S\left(\frac{\eta}{|\eta|}\right) \right| + \left| l\left(\frac{1}{|\eta|}\right) - n(\eta) \right| |\psi(\eta)| \\ &\quad \le (C+1)\varepsilon + C' l\left(\frac{1}{|\eta|}\right)^{-1} \left(l\left(\frac{1+\delta}{|\eta|}\right) - l\left(\frac{1-\delta}{|\eta|}\right) \right), \end{aligned} \tag{2.f}$$

where C' is a bound for $|l(1/|\eta|)\psi(\eta)|$ [the existence of such a bound follows from (2.e)]. The desired result follows from (2.f), since ε and δ can be chosen arbitrarily small. □

Since Re S is bounded below by some strictly positive constant, the proposition implies that

$$\exp - \frac{C}{l(1/|\xi|)} \le |\varphi(\xi)| \le \exp - \frac{C'}{l(1/|\xi|)} \tag{2.g}$$

for some positive constants C, C' and for $\xi \in T^d$ small enough.

If we assume that the property of the proposition is satisfied, we have at once

$$\varphi\left(\frac{\xi}{b(n)}\right)^n \xrightarrow[n\to\infty]{} \Phi(\xi), \qquad \xi \in \mathbb{R}^d.$$

Hence,

$$b(n)^{-1}X_n \xrightarrow[n\to\infty]{(d)} U_1,$$

so that Assumption A1 is satisfied. In fact one can prove more than A1. For each $n \geq 1$, let $X^{(n)} = (X_t^{(n)}, t \geq 0)$ be the process

$$X_t^{(n)} = b(n)^{-1}X_{[nt]}, \qquad (t \geq 0).$$

Then the sequence $X^{(n)}$ converges in distribution, in the sense of Skorokhod's J1 topology, toward the process $U = (U_t, t \geq 0)$, defined as the unique in law stable process in $\mathbb{R}^d$ whose marginal distribution at time 1 is U_1. We refer to Skorokhod [20] for a proof of this result as well as for the definition of the J1 topology.

2.4. In terms of the characteristic function φ, Assumption A2 means exactly that, for $\xi \in T^d$, $\varphi(\xi) = 1$ if and only if $\xi = 0$. However there may exist nonzero values of ξ for which $|\varphi(\xi)| = 1$. The random walk is called aperiodic if and only if $|\varphi(\xi)| < 1$ for all $\xi \in T^d - \{0\}$. This is equivalent to the property that the law of X_1 is not supported on a set of the form $x + H$, with $x \in \mathbb{Z}^d$ and H a proper subgroup of $\mathbb{Z}^d$ (see Spitzer [21]; the terminology in Spitzer is slightly different from ours: What we call aperiodic is strongly aperiodic for Spitzer).

Some of the results of the present work are more easily proved for aperiodic random walks and can then be extended to the adapted case. We will now describe how this extension can be made. Suppose that X is adapted and set

$$\mathscr{S} = \{\xi \in T^d, |\varphi(\xi)| = 1\}.$$

Note that for $\xi \in \mathscr{S}$, we have

$$\exp i\xi \cdot X_1 = \omega \quad \text{a.s.}$$

for some $\omega \in \mathbb{C}$ such that $|\omega| = 1$. Let us identify T^d with the quotient group $\mathbb{R}^d/(2\pi\mathbb{Z})^d$. Then $\mathscr{S}$ is a finite cyclic subgroup of T^d. Set $\tau = |\mathscr{S}|$. The number τ is called the period of X. Obviously $\tau = 1$ if and only if X is aperiodic. Set

$$H = \{x \in \mathbb{Z}^d; x \cdot \xi \equiv 0(2\pi), \forall\, \xi \in \mathscr{S}\}.$$

Then H is a sublattice of $\mathbb{Z}^d$ and $\mathbb{Z}^d/H$ is cyclic of order τ. We may choose a generator x_0 such that the law of X_1 is supported on $x_0 + H$. Moreover for $n = m\tau + r$ with $m \geq 0$, $0 \leq r < \tau$ we have

$$X_n \in rx_0 + H \quad \text{a.s.}$$

In particular the sets

$$\{X_0, X_\tau, X_{2\tau}, \dots\}, \{X_1, X_{\tau+1}, X_{2\tau+1}, \dots\}, \dots \{X_{\tau-1}, X_{2\tau-1}, \dots\}$$

are disjoint.

Set

$$Y_n = X_{n\tau} \qquad (n = 0, 1, 2, \dots).$$

Then Y is a random walk on H and hence is not adapted. However, if φ is any linear isomorphism from H onto $\mathbb{Z}^d$, $\varphi(Y)$ is an aperiodic random walk on $\mathbb{Z}^d$. The same holds for $Y'_n = X_{n\tau+1} - X_1, \dots, Y_n^{(\tau-1)} = X_{n\tau+\tau-1} - X_{\tau-1}$. We have thus reduced the study of the range of X to that of the sum of the ranges of τ aperiodic random walks.

Let us describe another trick, taken from Spitzer [21], that is sometimes useful when dealing with the nonaperiodic case. Suppose again that X is adapted and for $\lambda \in (0;1)$ set

$$P^{(\lambda)}(x, y) = (1 - \lambda) P(x, y) + \lambda I(x = y).$$

Let $X^{(\lambda)}$ denote a random walk associated with the transition kernel $P^{(\lambda)}(\cdot, \cdot)$. Then $X^{(\lambda)}$ is aperiodic. Moreover, for any $n \geq 0$,

$$X_n^{(\lambda)} =_{(d)} X_{k(n,\lambda)}, \qquad R_n^{(\lambda)} =_{(d)} R_{k(n,\lambda)},$$

where $k(n, \lambda)$ is a random variable independent of X and such that

$$P[k(n,\lambda) = m] = \binom{n}{m}(1 - \lambda)^m \lambda^{n-m} \quad \text{for } m = 0, 1, \dots, n.$$

Note that $k(n, \lambda)/n \to_{(P)} 1$ as $\lambda \to 0$, uniformly in n. It is also clear that $X^{(\lambda)}$ satisfies (A1) if X does.

2.5. We now proceed to investigate the asymptotic behaviour of $P_n(0,0)$ as $n \to \infty$. We need to introduce the family $(p_t(x), t \geq 0, x \in \mathbb{R}^d)$ of transition densities of the process U.

Proposition 2.4. *Under Assumptions* A1 *and* A2, *if τ denotes the period of X, we have*:

(i) $P_n(0,0) = 0$ *if n is not a multiple of τ.*
(ii) $\lim_{n\to\infty} b(n\tau)^d P_{n\tau}(0,0) = \tau p_1(0)$.

Moreover, there exists a constant C such that, for any $n \geq 0$, $x \in \mathbb{Z}^d$,

$$P_n(0, x) \leq C b(n)^{-d}. \tag{2.h}$$

Proof. We only treat the case when X is aperiodic ($\tau = 1$). The general case can then be handled using the remarks of Section 2.4 (it is also possible to proceed directly). By aperiodicity, we may assume that the upper bound in (2.g),

$$|\varphi(\xi)| \leq \exp\left(-\frac{C}{l(1/|\xi|)}\right),$$

holds for every $\xi \in T^d$ and not only for $|\xi|$ small. It follows that, for $n \geq 1$ and $0 < \varepsilon < \beta$,

$$\text{(2.i)} \qquad \left|\varphi\left(\frac{\xi}{b(n)}\right)^n\right| \leq \exp\left(-\frac{Cn}{l(b(n)/|\xi|)}\right) \leq \exp -C'_\varepsilon\left(|\xi|^{\beta-\varepsilon} + |\xi|^{\beta+\varepsilon}\right)$$

by Lemma 2.1 (write $n = l(b(n))$). Then

$$\begin{aligned}
P_n(0,x) &= (2\pi)^{-d}\int_{T^d} e^{-ix\cdot\xi}\varphi(\xi)^n\,d\xi \\
&\leq (2\pi)^{-d}b(n)^{-d}\int_{b(n)T^d}\left|\varphi\left(\frac{\xi}{b(n)}\right)\right|^n d\xi \\
&\leq (2\pi)^{-d}b(n)^{-d}\int_{\mathbb{R}^d}\exp -C'_\varepsilon\left(|\xi|^{\beta-\varepsilon} + |\xi|^{\beta+\varepsilon}\right)d\xi \\
&\leq Cb(n)^{-d}.
\end{aligned}$$

If $x = 0$, we can use Assumption A1 to get

$$\begin{aligned}
\lim_{n\to\infty} b(n)^d P_n(0,0) &= \lim_{n\to\infty}(2\pi)^{-d}\int_{b(n)T^d}\varphi\left(\frac{\xi}{b(n)}\right)^n d\xi \\
&= (2\pi)^{-d}\int_{\mathbb{R}^d}\Phi(\xi)\,d\xi \\
&= p_1(0),
\end{aligned}$$

where we have used (2.i) to justify dominated convergence. □

REMARK. The method of proof of Proposition 2.4 easily yields stronger results. For instance in the aperiodic case we have

$$\lim_{n\to\infty}\sup_{x\in\mathbb{Z}^d}\left|b(n)^d P_n(0,x) - p_1\left(\frac{x}{b(n)}\right)\right| = 0.$$

We leave the proof to the reader since we shall not need this result (see, e.g., [19]).

Let $G_{(n)}(x,y)$ denote the truncated Green function of X,

$$G_{(n)}(x,y) = \sum_{k=0}^{n} P_k(x,y).$$

Suppose that X is recurrent, i.e., $G_{(n)}(0,0) \to \infty$ as $n \to \infty$. A straightforward application of Proposition 2.4 shows that

$$\text{(2.j)} \qquad \lim_{n\to\infty}\left(\sum_{k=1}^{n} b(k)^{-d}\right)^{-1} G_{(n)}(0,0) = p_1(0).$$

Note that the period τ of X does not appear in (2.j).

We shall make use of the following simple identity:

$$G_{(n)}(x,y) = \sum_{j=0}^{n} P_x[T_y = j] G_{(n-j)}(0,0). \tag{2.k}$$

When X is transient, we can consider its Green function

$$G(x,y) = \sum_{k=0}^{\infty} P_k(x,y).$$

It is well known and easy to prove that $G(0,0) = q^{-1}$, where

$$q = P[X_n \neq 0 \text{ for all } n \geq 1].$$

PROPOSITION 2.5. *Under Assumption* A1, *the random walk X is transient if and only if*

$$\int_{T^d} |1 - \varphi(\xi)|^{-1}\, d\xi < \infty.$$

PROOF. Since $\mathrm{Re}(S(\omega)) \geq a > 0$ for $\omega \in S^{d-1}$, it follows from Proposition 2.3 that, for $|\xi|$ small,

$$\frac{C}{l(1/|\xi|)} \leq |1 - \varphi(\xi)| \leq \frac{C'}{l(1/|\xi|)},$$

for some positive constants C, C'. Therefore the condition of Corollary 2.5 is equivalent to

$$\int_{T^d} l(1/|\xi|)\, d\xi < \infty.$$

Easy transformations show that this condition is in turn equivalent to

$$\sum_{k=1}^{\infty} b(k)^{-d} < \infty. \tag{2.l}$$

By Proposition 2.4, (2.l) implies that X is transient.

Conversely, suppose that X is transient. Then, either $\beta < d$, in which case (2.l) holds trivially, or $\beta = d$ and then it is well known [22] that $p_1(0) > 0$, so that (2.l) follows from Proposition 2.4. □

REMARK. In general [without (A1)], it is known (see [21]) that X is transient if and only if

$$\int_{T^d} \mathrm{Re}\big((1 - \varphi(\xi))^{-1}\big)\, d\xi < \infty.$$

3. Bounds on the number of intersection points of two independent walks. We now consider two independent random walks X, X' in $\mathbb{Z}^d$. Let I_n denote the number of intersection points of the paths of X and X' up to time n:

$$I_n = |X(0,n) \cap X'(0,n)|,$$

where $X(0,n) = \{X_0, X_1, \dots, X_n\}$. The following lemma, which holds without any assumption on the law of X, gives us bounds on the moments of I_n.

LEMMA 3.1. *Suppose that X, X' are two independent random walks in $\mathbb{Z}^d$ with the same distribution and the same starting point. Then for any integer $k \geq 1$,*

$$E\big[(I_n)^k\big] \leq (k!)^2 E[I_n]^k.$$

REMARK. Our notation is different from the one used by other authors. What we write as $E[(I_n)^k]$ is often denoted by $E(I_n)^k$. What we write as $E[I_n]^k$ would be written as $(E[I_n])^k$.

PROOF. Since X and X' are independent and identically distributed, we have

$$E\big[(I_n)^k\big] = \sum_{y_1,\dots,y_k \in \mathbb{Z}^d} E\left[\prod_{i=1}^k I(y_i \in X(0,n))\right]^2.$$

Let Σ_k denote the set of permutations of $\{1,\dots,k\}$. Then

$$E\left[\prod_{i=1}^k I(y_i \in X(0,n))\right] \leq \sum_{\sigma \in \Sigma_k} P\big[T_{y_{\sigma(1)}} \leq T_{y_{\sigma(2)}} \leq \cdots \leq T_{y_{\sigma(k)}} \leq n\big].$$

Hence, using the Cauchy–Schwarz inequality,

$$\begin{aligned}
&\sum_{y_1,\dots,y_k} E\left[\prod_{i=1}^k I(y_i \in X(0,n))\right]^2 \\
&\quad \leq k! \sum_{y_1,\dots,y_k} \sum_{\sigma \in \Sigma_k} P\big[T_{y_{\sigma(1)}} \leq \cdots \leq T_{y_{\sigma(k)}} \leq n\big]^2 \\
&\quad = (k!)^2 \sum_{y_1,\dots,y_k} P\big[T_{y_1} \leq \cdots \leq T_{y_k} \leq n\big]^2.
\end{aligned}$$

On the other hand, applying the Markov property at time $T_{y_{k-1}}$,

$$\begin{aligned}
&\sum_{y_1,\dots,y_k} P\big[T_{y_1} \leq \cdots \leq T_{y_k} \leq n\big]^2 \\
&\quad \leq \sum_{y_1,\dots,y_{k-1}} P\big[T_{y_1} \leq \cdots \leq T_{y_{k-1}} \leq n\big]^2 \sum_{y_k} P_{y_{k-1}}\big[T_{y_k} \leq n\big]^2 \\
&\quad = \left(\sum_{y_1,\dots,y_{k-1}} P\big[T_{y_1} \leq \cdots \leq T_{y_{k-1}} \leq n\big]^2\right) E[I_n] \\
&\quad \leq E[I_n]^k,
\end{aligned}$$

by induction. □

REMARKS. (i) Lemma 3.1 can be easily extended to bound the moments of the number of intersection points of p independent identically distributed random walks. It suffices to replace $(k!)^2$ by $(k!)^p$. For $p = 1$ we get

$$E\left[(R_n)^k\right] \le k!\, E[R_n]^k.$$

Note that $E[R_n]$ can be bounded as follows. Writing

$$\sum_{y\in\mathbb{Z}^d} E\left[I(y \in X(0,n)) \sum_{i=0}^{2n} I(X_i = y)\right] \le 2n + 1$$

and applying the Markov property at T_y, we get

$$G_{(n)}(0,0)E[R_n] \le 2n + 1, \tag{3.a}$$

and it follows that

$$E\left[(R_n)^k\right] \le k!\left(\frac{2n+1}{G_{(n)}(0,0)}\right)^k. \tag{3.b}$$

(ii) Suppose that, for some function $f(n)$, one can prove the convergence of all moments of $f(n)I_n$ towards the corresponding moments of some nonnegative random variable V. It then follows that for each k, $E[V^k] \le c^k(k!)^2$ ($c = E[V]$). Hence the law of V is determined by its moments and $f(n)I_n$ converges in distribution toward V. This method was used in [9].

COROLLARY 3.2. *Suppose that X, X' are two independent random walks satisfying Assumptions* A1, A2 *with the same index β and the same function $b(n)$. For each $k \ge 1$, there exists a constant C_k such that for $n \ge 1$,*

$$E\left[(I_n)^k\right] \le C_k\left(\frac{\sum_{i=1}^n ib(i)^{-d}}{\left(\sum_{i=1}^n b(i)^{-d}\right)^2}\right)^k.$$

REMARK. We allow different limit laws U_1, U_1' for $b(n)^{-1}X_n$, $b(n)^{-1}X_n'$, as well as different starting points X_0, X_0'. This will be important in our applications.

PROOF. We may assume that X and X' are identically distributed. Indeed,

$$\begin{aligned} E\left[(I_n)^k\right] &= \sum_{y_1,\dots,y_k} E\left[\prod_{i=1}^k I(y_i \in X(0,n))\right] E\left[\prod_{i=1}^k I(y_i \in X'(0,n))\right] \\ &\le \left(\sum_{y_1,\dots,y_k} E\left[\prod_{i=1}^k I(y_i \in X(0,n))\right]^2\right)^{1/2} \\ &\quad \times \left(\sum_{y_1,\dots,y_k} E\left[\prod_{i=1}^k I(y_i \in X'(0,n))\right]^2\right)^{1/2}. \end{aligned}$$

According to Lemma 3.1, it suffices to treat the case $k = 1$. We may also assume that X is aperiodic. Indeed, with the notation of the end of Section 2.4, a bound on the first moment of

$$|X^{(\lambda)}(0,2n) \cap X'^{(\lambda)}(0,2n)|$$

immediately gives a bound on $E[I_n]$.

Set

$$J_n = \sum_{i=0}^{n} \sum_{j=0}^{n} I(X_i = X'_j)$$
$$= \sum_{y \in \mathbb{Z}^d} \left(\sum_{i=0}^{n} I(X_i = y) \right) \left(\sum_{i=0}^{n} I(X'_i = y) \right).$$

The same argument as for (3.a) leads to

$$G_{(n)}(0,0)^2 E[I_n] \le E[J_{2n}].$$

If $p_1(0) > 0$, (2.j) implies

$$G_{(n)}(0,0) \ge C \left(\sum_{i=1}^{n} b(i)^{-d} \right) \tag{3.c}$$

for some $C > 0$. Now $p_1(0)$ can only vanish if $\beta < d$ (see [22]), in which case $\Sigma b(i)^{-d} < \infty$ and (3.c) also holds. We thus obtain

$$E[I_n] \le C \frac{E[J_{2n}]}{\left(\Sigma_{i=1}^{n} b(i)^{-d}\right)^2}. \tag{3.d}$$

It remains to bound $E[J_n]$. Note that

$$E[J_n] = \sum_{i=0}^{n} \sum_{j=0}^{n} (2\pi)^{-d} \int_{T^d} \varphi(\xi)^i \varphi(-\xi)^j \, d\xi.$$

The arguments of the proof of Proposition 2.4 imply that

$$\int_{T^d} \varphi(\xi)^i \varphi(-\xi)^j \, d\xi \le C b(i+j)^{-d}.$$

Hence,

$$E[J_n] \le C \sum_{i=0}^{n} \sum_{j=0}^{n} b(i+j)^{-d}$$
$$\le C \sum_{i=0}^{2n} (i+1) b(i)^{-d}$$
$$\le C' \sum_{i=1}^{2n} i b(i)^{-d}$$

and the desired result follows from (3.d), since it is clear that, for $n \geq 1$,

$$\sum_{i=1}^{2n} ib(i)^{-d} \leq C \sum_{i=1}^{n} ib(i)^{-d}. \qquad \square$$

REMARK. If X is transient, $\Sigma b(i)^{-d} < \infty$ and we can replace the bound of Corollary 3.2 by

$$E\left[(I_n)^k\right] \leq C_k \left(\sum_{i=1}^{n} ib(i)^{-d} \right)^k \leq \begin{cases} C_k & \text{if } \beta < d/2, \\ C_k \left(\displaystyle\sum_{i=1}^{n} \frac{1}{is(i)^d} \right)^k & \text{if } \beta = d/2, \\ C_k \dfrac{n^{2k}}{b(n)^{dk}} & \text{if } \beta > d/2. \end{cases}$$

In particular, if $\beta < d/2$, we see that $I_\infty < \infty$ a.s. The other bounds are also sharp. For instance, in the case of random walks with zero mean and finite second moments in $\mathbb{Z}^4$ ($\beta = 2 = d/2$, $s(n) = 1$), we obtain

$$E\left[(I_n)^k\right] \leq C_k (\log n)^k,$$

which agrees with the results of [10], where it is proved that, for some constant C independent of k,

$$E\left[(I_n)^k\right] \underset{n \to \infty}{\sim} C^k E[N^{2k}] (\log n)^k,$$

where N denotes a standard normal variable.

4. The case $\beta \leq 2d/3$.

4.1. Throughout this section we assume that the random walk (X_n) satisfies Assumptions A1 and A2, with $\beta \leq 2d/3$. As follows from the estimates of Proposition 2.4, the random walk is then transient and thus the strong law of large numbers holds for the range R_n:

$$\lim_{n \to \infty} \frac{R_n}{n} = q \quad \text{a.s.},$$

where $q = P_0[X_n \neq 0 \text{ for all } n \geq 1] > 0$. If $q = 1$, then $R_n = n + 1$ for each n. Therefore we will also assume that $q < 1$. This assumption is automatically satisfied if $\beta > 1$. Indeed, if $\beta > 1$, it is well known [22] that $p_1(0) > 0$, which implies (Proposition 2.4) that $P_k(0,0) > 0$ for infinitely many k. On the other hand if $\beta < 1$, simple examples show that we may have $q = 1$.

Our goal in this section is to prove the central limit theorem for R_n. We shall follow closely the method used by Jain and Pruitt [6] for random walks in high dimensions ($d \geq 3$). Note that if $\beta < d/2$, the random walk is strongly transient and thus the central limit theorem holds for R_n by a theorem of Jain and Orey [5].

We need to introduce some notation. For $x, y \in \mathbb{Z}^d$ we set

$$H_x = \inf\{n \geq 1; X_n = x\}, \qquad F(x,y) = P_x[H_y < \infty].$$

It is clear that $G(x,y) \geq F(x,y)$. More precisely, if $x \neq y$,

$$G(x,y) = F(x,y)G(0,0) = q^{-1}F(x,y).$$

Recall the definition of the truncated Green function:

$$G_{(n)}(x,y) = \sum_{k=0}^{n} P_k(x,y).$$

We also set

$$P_k^z(x,y) = P_x[X_k = y; H_z \geq k].$$

Changing $(Y_1, \dots, Y_k)$ into $(Y_k, \dots, Y_1)$ (where $X_k = X_0 + \Sigma_1^k Y_i$), we see that, for any x, y,

$$P_k^x(x,y) = P_k^y(x,y).$$

4.2. The following lemmas are straightforward extensions of results in Jain and Pruitt [6]. The only difference is that we need to replace the bound of Lemma 1 in [6] by the one of Proposition 2.4:

$$P_k(0,x) \leq Cb(k)^{-d}. \tag{4.a}$$

Recall that we assume $d/\beta \geq 3/2$.

LEMMA 4.1. *We have*

$$\sum_x G_{(n)}(0,x)(G(u,x) + G(x,u)) = \begin{cases} O(1) & \textit{if } \beta < d/2, \\ O\left(\sum_{k=1}^{n} \dfrac{1}{ks(k)^d}\right) & \textit{if } \beta = d/2, \\ O\left(n^2 b(n)^{-d}\right) & \textit{if } \beta > d/2, \end{cases}$$

$$\sum_x G_{(n)}(0,x)G(u,x)G(x,v) = \begin{cases} O(1) & \textit{if } \beta < 2d/3, \\ O\left(\sum_{k=1}^{n} \dfrac{1}{ks(k)^{2d}}\right) & \textit{if } \beta = 2d/3, \end{cases}$$

uniformly in $u, v \in \mathbb{Z}^d$.

PROOF. The proof is very similar to that of Lemma 3 in [6]. For $k \geq 1$, we have

$$\begin{aligned}
&\sum_x P_k(0,x)(G(u,x) + G(x,u)) \\
&\quad = \sum_{j=0}^{\infty} \sum_x P_k(0,x)\big(P_j(u,x) + P_j(x,u)\big) \\
&\quad \leq C\left(\sum_{j=0}^{k-1} \sum_x b(k)^{-d}\big(P_j(u,x) + P_j(-u,-x)\big) \right. \\
&\qquad\qquad \left. + \sum_{j=k}^{\infty} \sum_x P_k(0,x) b(j)^{-d}\right) \\
&\quad \leq C' k b(k)^{-d}.
\end{aligned}$$

The first assertion of the lemma follows by summing over $k = 0, 1, \dots, n$. Similarly, using Lemma 2.2 if $\beta = d/2$, we have, for $k \geq 1$,

$$\begin{aligned}
&\sum_x P_k(0,x) G(u,x) G(x,v) \\
&\quad = \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} \sum_x P_k(0,x) P_i(u,x) P_j(x,v) \\
&\quad \leq C\left(\sum_{i=0}^{k-1} \sum_{j=0}^{\infty} b(k)^{-d} P_{i+j}(u,v) + \sum_{i=k}^{\infty} \sum_{j=0}^{\infty} b(i)^{-d} P_{k+j}(0,v)\right) \\
&\quad \leq C'\left(\sum_{i=0}^{k-1} b(k)^{-d}(i+1) b(i+1)^{-d} + \sum_{i=k}^{\infty} b(i)^{-d} k b(k)^{-d}\right) \\
&\quad = \begin{cases} O\big(b(k)^{-d}\big) & \text{if } \beta < d/2, \\ O\left(b(k)^{-d} \displaystyle\sum_{i=1}^{k} \frac{1}{i s(i)^d}\right) & \text{if } \beta = d/2, \\ O\big(k^2 b(k)^{-2d}\big) & \text{if } \beta > d/2, \end{cases}
\end{aligned}$$

and the desired result follows by summing over k. □

LEMMA 4.2. *We have for* $m, n \geq 1$,

$$\begin{aligned}
&\sum_x G_{(n)}(0,x) P_x[m < H_x < \infty, H_0 < \infty] \\
&\qquad = \begin{cases} O\big(m b(m)^{-d}\big) & \textit{if } \beta < d/2, \\ O\left(m b(m)^{-d} \displaystyle\sum_{1}^{n} \frac{1}{k s(k)^d}\right) & \textit{if } \beta = d/2, \\ O\big(m b(m)^{-d} n^2 b(n)^{-d}\big) & \textit{if } \beta > d/2. \end{cases}
\end{aligned}$$

PROOF. For any y, z, we have

$$P_y[m < H_z < \infty] \le \sum_{k=m+1}^{\infty} P_k(y,z) \le Cmb(m)^{-d}.$$

Then, as in [6] (proof of Lemma 4), if $x \neq 0$,

$$\begin{aligned}
&P_x[m < H_x < \infty, H_0 < \infty] \\
&\quad \le P_x[m < H_x < \infty]F(x,0) + P_x\left[\frac{m}{2} < H_0 < \infty\right]F(0,x) \\
&\qquad + F(x,0)P_0\left[\frac{m}{2} < H_x < \infty\right] \\
&\quad \le Cmb(m)^{-d}(G(0,x) + G(x,0)).
\end{aligned}$$

An application of Lemma 4.1 completes the proof. □

LEMMA 4.3 (Lemma 5 of [6]). *For $x \neq 0$, set*

$$a(x) = P_x[H_0 < \infty]P_x[H_x = \infty] - P_x[H_0 < \infty, H_x = \infty]$$

Then

$$a(x) = P_x[H_0 < H_x < \infty]P_x[H_0 = \infty] = \frac{1 - F(x,0)}{1 - F(x,0)F(0,x)}qF(x,0)F(0,x).$$

4.3.

THEOREM 4.4. *Under Assumptions* A1 *and* A2, *if $q < 1$ and $\beta < 2d/3$, we have*

$$\operatorname{var} R_n \underset{n\to\infty}{\sim} \sigma^2 n,$$

where σ^2 is the positive constant defined by

$$\sigma^2 = q(1-q) + 2q\sum_{x\neq 0} G(0,x)a(x),$$

[*$a(x)$ is defined in Lemma* 4.3].

REMARK. Lemma 4.1 shows that $\sigma^2 < \infty$.

PROOF. Once again we follow closely the arguments of [6] (proof of Theorem 1). Let

$$\begin{aligned}
Z_i^n &= I[S_i \neq S_{i+1}, \dots, S_i \neq S_n] \quad \text{for } 0 \le i < n,\ Z_n^n = 1, \\
Z_i &= I[S_i \neq S_{i+1}, S_i \neq S_{i+2}, \dots] \quad \text{for } i \ge 0, \\
W_i^n &= Z_i^n - Z_i \quad \text{for } 0 \le i < n.
\end{aligned}$$

We note that $R_n = \sum_{i=0}^n Z_i^n$, so that $R_n = Y_n + W_n + 1$, where

$$Y_n = \sum_{i=0}^{n-1} Z_i, \qquad W_n = \sum_{i=0}^{n-1} W_i^n.$$

We have, for $i < j$,

$$\begin{aligned} E[W_i^n W_j^n] &= \sum_{x \neq 0} P_{j-i}^0(0,x) P_x[n-j < H_0 < \infty, n-j < H_x < \infty] \\ &\leq \sum_x P_{j-i}(0,x) P_x[n-j < H_x < \infty, H_0 < \infty] \end{aligned}$$

and the bound is also valid for $i = j$. Then, treating separately the three cases $\beta < d/2$, $\beta = d/2$, $\beta > d/2$, we deduce from Lemma 4.2 that, for $0 < j < n$,

$$\text{(4.b)} \quad \sum_{i=0}^{j} E[W_i^n W_j^n] = \begin{cases} O\big((n-j)b(n-j)^{-d}\big) & \text{if } \beta < d/2, \\ O\left(\left(\sum_1^j \dfrac{1}{ks(k)^d}\right)(n-j)b(n-j)^{-d}\right) & \text{if } \beta = d/2, \\ O\big(j^2 b(j)^{-d}(n-j)b(n-j)^{-d}\big) & \text{if } \beta > d/2. \end{cases}$$

Summing on j and using the Cauchy–Schwarz inequality if $\beta \geq d/2$, we obtain

$$\text{(4.c)} \qquad E[W_n^2] = o(n).$$

Next we estimate $\operatorname{var} Y_n$:

$$\operatorname{var} Y_n = \sum_{i=0}^{n-1} \operatorname{var} Z_i + 2 \sum_{0 \leq i < j \leq n-1} \operatorname{cov}(Z_i, Z_j).$$

Clearly $\operatorname{var} Z_i = q(1-q)$, whereas it is easily seen that, for $i < j$,

$$\operatorname{cov}(Z_i, Z_j) = \sum_{x \neq 0} P_{j-i}^0(0,x) a(x).$$

It follows that

$$\operatorname{var} Y_n = nq(1-q) + 2\sum_{j=1}^{n-1} a_j,$$

where

$$a_j = \sum_{i=1}^{j} \sum_{x \neq 0} P_i^0(0,x) a(x).$$

The sequence (a_j) increases and its limit

$$a = \sum_{i=1}^{\infty} \sum_{x \neq 0} P_i^0(0,x) a(x) = \sum_{x \neq 0} F(0,x) a(x)$$

is finite by Lemma 4.1. We conclude that

$$\text{(4.d)} \qquad \operatorname{var} Y_n \sim (q(1-q) + 2a)n.$$

Theorem 4.4 follows from (4.c), (4.d) and the identity $R_n = Y_n + W_n + 1$. □

4.4. We are now ready to state and prove the central limit theorem for R_n. Our method of proof is somewhat different from the one in [6]. It will depend on both Theorem 4.4 and Corollary 3.2.

THEOREM 4.5. *Under Assumptions* A1 *and* A2, *if* $\beta < 2d/3$,

$$n^{-1/2}(R_n - E[R_n]) \xrightarrow[n\to\infty]{(d)} \sigma N,$$

where N denotes a standard normal variable and σ is defined as in Theorem 4.4.

PROOF. For each $n \geq 1$, let $p = p(n) \geq 2$ be an integer to be fixed later. We have

$$R_n = \sum_{i=1}^{p} \left| X\left(\frac{i-1}{p}n, \frac{i}{p}n\right) \right| - \sum_{i=2}^{p} \left| X\left(0, \frac{i-1}{p}n\right) \cap X\left(\frac{i-1}{p}n, \frac{i}{p}n\right) \right|,$$

where we use the notation

$$X(a,b) = \{X_k; a \leq k \leq b\} \quad \text{for } a, b \in \mathbb{R}.$$

Let $i \in \{2, \dots, p\}$ be fixed for the moment and set $j = [(i-1)n/p]$. For $k \in \{0, 1, \dots, j\}$, set $X'_k = X_{j-k} - X_j$, $X''_k = X_{j+k} - X_j$. Then X', X'' are two independent random walks, defined on the time interval $\{0, \dots, j\}$ and

$$\left| X\left(0, \frac{i-1}{p}n\right) \cap X\left(\frac{i-1}{p}n, \frac{i}{p}n\right) \right| \leq |X'(0,j) \cap X''(0,j)| + 1.$$

We can apply Corollary 3.2 (or rather, the final remark of Section 3) to get bounds on the moments of the right-hand side. Notice that X', X'' are not identically distributed, but they both satisfy Assumptions A1 and A2 with the same function $b(n)$.

We then sum over i and use the triangle inequality to get, for any $\varepsilon > 0$,

$$E\left[\left(\sum_{i=2}^{p} \left| X\left(0, \frac{i-1}{p}n\right) \cap X\left(\frac{i-1}{p}n, \frac{i}{p}n\right) \right|\right)^2\right]^{1/2} \leq \begin{cases} C_\varepsilon p n^{2+\varepsilon-d/\beta} & \text{if } \beta \geq d/2, \\ Cp & \text{if } \beta < d/2. \end{cases}$$

Taking $p \sim n^\delta$ with $\delta > 0$ sufficiently small (such that $2 + \delta - d/\beta < \frac{1}{2}$), we obtain

$$\text{(4.e)} \qquad E\left[\left(R_n - \sum_{i=1}^{p} \left| X\left(\frac{i-1}{p}n, \frac{i}{p}n\right) \right|\right)^2\right] = o(n).$$

Set $R_{n,i} = |X(((i-1)/p)n, (i/p)n)|$, $1 \leq i \leq p$, and let

$$\{R_{n,i}\} = R_{n,i} - E[R_{n,i}].$$

Note that the random variables $\{R_{n,i}\}$ are independent and that, by Theorem 4.4,

$$E\left[\{R_{n,i}\}^2\right] \underset{n\to\infty}{\sim} \sigma^2 \frac{n}{p}.$$

Taking (4.e) into account, we see that Theorem 4.5 will be proved if we can apply Lindeberg's theorem on triangular arrays to the family $(\{R_{n,i}\}, 1 \le i \le p(n))$. It remains to verify Lindeberg's condition. This condition is clearly satisfied if we know that

$$E\left[\{R_n\}^4\right] \le Cn^2 \tag{4.f}$$

(where again $\{R_n\} = R_n - E[R_n]$). In order to prove (4.f), take $n \ge 2$ and set $n_1 = [n/2]$, where $[x]$ denotes the integral part of x and $n_2 = n - n_1$. Then

$$R_n = |X(0, n_1)| + |X(n_1, n)| - |X(0, n_1) \cap X(n_1, n)|.$$

Therefore,

$$E\left[\{R_n\}^4\right]^{1/4} \le E\left[\{|X(0,n_1)| + |X(n_1,n)|\}^4\right]^{1/4} + E\left[\{|X(0,n_1) \cap X(n_1,n)|\}^4\right]^{1/4}.$$

Corollary 3.2 implies

$$E\left[\{|X(0,n_1) \cap X(n_1,n)|\}^4\right]^{1/4} = o(n^{1/2}).$$

On the other hand, by Theorem 4.4,

$$E\left[\{|X(0,n_1)| + |X(n_1,n)|\}^4\right] \le E\left[\{R_{n_1}\}^4\right] + E\left[\{R_{n_2}\}^4\right] + Cn^2$$

(using the independence of $|X(0, n_1)|$ and $|X(n_1, n)|$). It follows that

$$E\left[\{R_n\}^4\right]^{1/4} \le \left(E\left[\{R_{n_1}\}^4\right] + E\left[\{R_{n_2}\}^4\right] + Cn^2\right)^{1/4} + o(n^{1/2}). \tag{4.g}$$

For any $k \ge 1$, set

$$\alpha_k = \sup\left\{2^{-k/2} E\left[\{R_n\}^4\right]^{1/4}; 2^k \le n \le 2^{k+1}\right\}.$$

(4.g) implies

$$\alpha_{k+1} \le \left(\tfrac{1}{2}\alpha_k^4 + C\right)^{1/4} + o(1).$$

Therefore the sequence α_k is bounded, which completes the proof of (4.f). □

4.5. We now investigate the case $\beta = 2d/3$ ($d = 1$, 2 or 3). We shall need the asymptotic behaviour of the Green function $G(0, x)$. It will be proved in the next section that, under a regularity assumption on the characteristic function of X (see Assumption A3) and if $d = 1$ or 2, one has, with the notation $l(x) = l(|x|)$,

$$G(0,x) \underset{|x|\to\infty}{\sim} \frac{l(x)}{|x|^d}\Omega\left(\frac{x}{|x|}\right), \tag{4.h}$$

where $\Omega(\xi) = \int_0^\infty p_t(\xi)\,dt$ is the potential kernel of U. Note that $\Omega(\cdot)$ is continuous on the unit sphere (indeed, it is so if $\beta > (d-1)/2$; see [13]). Hence (4.h) implies

$$G(0,x) \le C\frac{l(x)}{|x|^d}. \tag{4.h'}$$

THEOREM 4.6. *Suppose that X satisfies Assumptions* A1 *and* A2 *with $\beta = 2d/3$ and that* (4.h) *holds. If $d = 1$, suppose in addition that U is not a subordinator. Set*

$$g(n) = \sum_{k=1}^{n} \frac{1}{ks(k)^{2d}}, \qquad 1 \le n \le \infty.$$

Then, (i) *if $g(\infty) < \infty$,*

$$\operatorname{var} R_n \underset{n\to\infty}{\sim} \sigma^2 n,$$

where $\sigma^2 = q(1-q) + 2q\sum_{x\neq 0} G(0,x)a(x) < \infty$;
(ii) *if $g(\infty) = \infty$,*

$$\operatorname{var} R_n \underset{n\to\infty}{\sim} c^2 n g(n),$$

where $c^2 = (3/d)q^4\int_{S^{d-1}}\Omega(\omega)^2\Omega(-\omega)\,d\omega > 0$.

PROOF. Once again we follow closely [6] (proof of Theorem 2). We first note that, by Lemma 2.2,

$$\frac{1}{s(n)^{\alpha}} = o(g(n)) \quad \text{for } 0 < \alpha \le 2d. \tag{4.i}$$

With the notation of the proof of Theorem 4.4, we have

$$R_n = Y_n + W_n + 1$$

and

$$\begin{aligned} E\left[(W_n)^2\right] &= \sum_{i,j=0}^{n-1} E\left[W_i^n W_j^n\right] \\ &\le 2\sum_{\substack{i,j=0\\ i\le j}}^{n-1}\sum_x P_{j-i}(0,x)P_x[n-j < H_x < \infty, H_0 < \infty] \\ &= O\left(\sum_{j=1}^{n-1} j^2 b(j)^{-d}(n-j)b(n-j)^{-d}\right), \end{aligned}$$

by Lemma 4.2. Next we evaluate

$$
\begin{aligned}
&\sum_{j=1}^{n-1} j^2 b(j)^{-d}(n-j)b(n-j)^{-d} \\
&\quad \le n\sum_{j=1}^{n-1} j^{-1/2}(n-j)^{-1/2}s(j)^{-d}s(n-j)^{-d} \\
&\quad \le 2n\sum_{j=1}^{[n/2]} j^{-1/2}(n-j)^{-1/2}s(j)^{-d}s(n-j)^{-d} \\
&\quad \le 2n\left(\sum_{j=1}^{[n/2]}\frac{1}{js(j)^{2d}}\right)^{1/2}\left(\sum_{j=[n/2]}^{n}\frac{1}{js(j)^{2d}}\right)^{1/2} \\
&\quad = 2ng([n/2])^{1/2}(g(n)-g([n/2]-1))^{1/2}.
\end{aligned}
$$

Since g is slowly varying, we get

$$
E\left[(W_n)^2\right] = o(ng(n)). \tag{4.j}
$$

We now estimate $\operatorname{var} Y_n$. As above in the proof of Theorem 4.4, we have

$$
\operatorname{var} Y_n = 2\sum_{j=1}^{n-1} a_j + nq(1-q), \tag{4.k}
$$

where

$$
a_j = \sum_{i=1}^{j}\sum_{x\neq 0} P_i^0(0,x)a(x).
$$

For any $j \ge 0$, set $\rho_j = P_0[j < H_0 < \infty]$. Then it is easily verified that, for $x \neq 0$,

$$
\sum_{i=1}^{j} P_i^0(0,x) = \sum_{i=1}^{j} P_i^x(0,x) = P_0[H_x \le j] = \sum_{i=1}^{j}(q+\rho_{j-i})P_i(0,x)
$$

(consider the last time before j when X visits x). It follows that

$$
a_j = \sum_{i=1}^{j}\sum_{x\neq 0}(q+\rho_{j-i})P_i(0,x)a(x).
$$

We first bound the contribution of large values of x. Set

$$
\begin{aligned}
B_j &= \{x \in \mathbb{Z}^d; 1 \le |x| \le b(j)\}, \\
C_j &= \{x \in \mathbb{Z}^d; |x| > b(j)\}.
\end{aligned}
$$

Then, using (4.h′) and Lemma 4.3,

$$
\sum_{i=1}^{j}\sum_{x\in C_j} P_i(0,x)a(x) \le C\sum_{i=1}^{j}\sum_{x\in C_j} P_i(0,x)\frac{l(b(j))^2}{b(j)^{2d}} \le \frac{C}{s(j)^{2d}}.
$$

Hence, by (4.i),

$$a_j = \sum_{i=1}^{j} \sum_{x \in B_j} (q + \rho_{j-i}) P_i(0,x) a(x) + o(g(j)). \tag{4.l}$$

Observe that, for $j \geq 1$,

$$\rho_j \leq \sum_{k=j+1}^{\infty} P_k(0,0) \leq Cj^{-1/2} s(j)^{-d},$$

by Proposition 2.4. Using again Proposition 2.4, we bound

$$\begin{aligned}
&\sum_{i=1}^{j-1} \sum_{x \in B_j} \rho_{j-i} P_i(0,x) a(x) \\
&\quad \leq C \sum_{i=1}^{j-1} i^{-3/2} s(i)^{-d} (j-i)^{-1/2} s(j-i)^{-d} \sum_{x \in B_j} G(0,x) G(x,0) \\
&\quad \leq C' \sum_{i=1}^{j-1} i^{-3/2} s(i)^{-d} (j-i)^{-1/2} s(j-i)^{-d} j^{1/2} s(j)^{-d},
\end{aligned}$$

where we have used (4.h′) to get the bound

$$\begin{aligned}
\sum_{x \in B_j} G(0,x) G(x,0) &\leq C \int_{\{1 \leq |u| \leq b(j)\}} du \frac{l(u)^2}{|u|^{2d}} \\
&\leq C \int_1^{b(j)} r^{2\beta - d - 1} t(r)^2 \, dr \\
&\leq C b(j)^{2\beta - d} t(b(j))^2 \\
&\leq C j^{1/2} s(j)^{-d},
\end{aligned}$$

by (2.b). We then consider separately

$$\sum_{i=1}^{[j/2]} (\cdots) \leq \frac{C}{s(j)^d} \sum_{i=1}^{[j/2]} i^{-3/2} s(i)^{-d} s(j-i)^{-d} = O\left(s(j)^{-2d}\right)$$

and

$$\begin{aligned}
\sum_{i=[j/2]+1}^{j-1} (\cdots) &\leq 2 \left(\sum_{i=[j/2]+1}^{j-1} i^{-1} s(i)^{-d} (j-i)^{-1/2} s(j-i)^{-d} \right) s(j)^{-d} \\
&= O\left(s(j)^{-3d} j^{-1/2}\right).
\end{aligned}$$

Using (4.i) we conclude that

$$\sum_{i=1}^{j} \sum_{x \in B_j} \rho_{j-i} P_i(0,x) a(x) = o(g(j)). \tag{4.m}$$

It remains to study

$$\sum_{i=1}^{j} \sum_{x \in B_j} P_i(0,x)a(x) = \sum_{x \in B_j} G_{(j)}(0,x)a(x).$$

A straightforward calculation shows that

$$\text{(4.n)} \qquad \sum_{x \in B_j} \big(G(0,x) - G_{(j)}(0,x)\big)a(x) \le Cj^{-1/2}s(j)^{-d} \sum_{x \in B_j} a(x) = O\big(s(j)^{-2d}\big).$$

Putting (4.l), (4.m) and (4.n) together, we obtain

$$\text{(4.o)} \qquad a_j = q \sum_{x \in B_j} G(0,x)a(x) + o(g(j)).$$

From Lemma 4.3, it is clear that

$$\text{(4.p)} \qquad a(x) \underset{|x| \to \infty}{\sim} q^3 G(x,0)G(0,x).$$

Then two possibilities may occur. Either $g(\infty) < \infty$, in which case (4.h′) leads to

$$\sum_x G(x,0)G(0,x)^2 \le C\int_{\{|u| \ge 1\}} du|u|^{-d}t(u)^3 \le C'\int_1^\infty \frac{dr}{r}s(r)^{-2d} < \infty$$

[using (2.b) to get from the integral of $r^{-1}t(r)^3$ to the integral of $r^{-1}s(r)^{-2d}$] and

$$\lim_{j \to \infty} a_j = a := q \sum_{x \neq 0} G(0,x)a(x) < \infty,$$

which implies

$$\operatorname{var} Y_n \underset{n \to \infty}{\sim} n(q(1-q) + 2a).$$

Alternatively, $g(\infty) = \infty$ and then a simple calculation using (4.o), (4.p) and (4.h) shows that

$$\text{(4.q)} \qquad a_j \underset{j \to \infty}{\sim} q^4 \int_{\{1 \le |u| \le b(j)\}} du \frac{l(u)^3}{|u|^{3d}} \Omega\left(\frac{u}{|u|}\right)^2 \Omega\left(\frac{-u}{|u|}\right) \underset{j \to \infty}{\sim} \frac{1}{2}c^2 g(j),$$

where $c^2 = (3q^4/d)\int_{S^{d-1}} \Omega(\omega)^2\Omega(-\omega)\,d\omega$.

If $d \ge 2$, it is clear that $c > 0$ [indeed, in this case, $\beta > 1$, which implies $\Omega(\omega) > 0$ on S^{d-1}]. If $d = 1$, this follows from our assumption that U is not a subordinator. Putting (4.j), (4.k) and (4.q) together completes the proof of Theorem 4.6. □

4.6. The central limit theorem for R_n follows from our estimates for $\operatorname{var} R_n$ exactly as in the case $\beta < 2d/3$.

THEOREM 4.7. *Under the assumptions and with the notation of Theorem* 4.6, *we have*

(i) *if* $g(\infty) < \infty$,

$$n^{-1/2}(R_n - E[R_n]) \xrightarrow[n\to\infty]{(d)} \sigma N,$$

(ii) *if* $g(\infty) = \infty$,

$$(ng(n))^{-1/2}(R_n - E[R_n]) \xrightarrow[n\to\infty]{(d)} cN,$$

where N *denotes a standard normal variable.*

PROOF. The proof is essentially the same as that of Theorem 4.5. By (4.i), we may choose $p = p(n)$ such that

$$\lim_{n\to\infty} p(n) = \infty$$

and

$$\frac{p(n)^2}{s(n)^{2d}} = o(g(n)). \tag{4.r}$$

Arguing as in the proof of Theorem 4.5, we get from Corollary 3.2 that, for $i = 2, \ldots, p$,

$$E\left[\left|X\left(0, \frac{i-1}{p}n\right) \cap X\left(\frac{i-1}{p}n, \frac{i}{p}n\right)\right|^2\right]^{1/2} \le C\frac{n^2}{b(n)^d}.$$

Therefore,

$$E\left[\left(R_n - \sum_{i=1}^{p}\left|X\left(\frac{i-1}{p}n, \frac{i}{p}n\right)\right|\right)^2\right]^{1/2} \le Cp\frac{n^{1/2}}{s(n)^d} = o\big((ng(n))^{1/2}\big), \tag{4.s}$$

by (4.r). Thus, if

$$R_{n,i} = \left|X\left(\frac{i-1}{p}n, \frac{i}{p}n\right)\right|, \qquad 1 \le i \le p,$$

we have

$$\sum_{i=1}^{p} E\big[\{R_{n,i}\}^2\big] \sim E\big[\{R_n\}^2\big] \sim \begin{cases} \sigma^2 n & \text{if } g(\infty) < \infty, \\ c^2 ng(n) & \text{if } g(\infty) = \infty, \end{cases}$$

by Theorem 4.6. Theorem 4.7 follows from (4.s) and an application of Lindeberg's theorem to the family $(\{R_{n,i}\}, 1 \le i \le p(n))$. In order to verify Lindeberg's condition, we prove that

$$E\big[\{R_n\}^4\big] \le cn^2 g(n)^2. \tag{4.t}$$

The proof of (4.t) is similar to that of (4.f) and will be left to the reader. □

REMARK. In the case of random walks with zero mean and finite second moments in $\mathbb{Z}^3$ ($d = 3$, $\beta = 2$), we have $g(n) \sim \log n$ and we recover Theorem 5 of Jain and Pruitt [6].

5. Estimates for the Green function and the distribution of hitting times.

5.1. Our goal in this section is to obtain certain estimates to be used in the next sections, especially in Section 6 to deal with the case $2d/3 < \beta \le d$. We will always assume that the random walk X satisfies Assumptions A1 and A2 of Section 2. We will sometimes make the following additional assumption on the characteristic function φ of X.

ASSUMPTION A3. The function φ is continuously differentiable on $T^d - \mathscr{S}$, where $\mathscr{S} = \{\xi;\ |\varphi(\xi)| = 1\}$. Moreover, there exists a constant C such that, for $|\xi|$ small,

$$|\nabla\varphi(\xi)| \le \frac{C}{|\xi| l(1/|\xi|)}. \tag{5.a}$$

Writing $\varphi = \exp -\psi$ as in (2.3) and using Proposition 2.3, we can replace (5.a) by

$$|\nabla\psi(\xi)| \le \frac{C}{|\xi|}|\psi(\xi)|. \tag{5.b}$$

This form of Assumption A3 was used in [18]. Note however that it was assumed in [18] that (5.b) holds for every $\xi \in T^d$. Here we restrict this condition to a neighbourhood of 0 since we are working with adapted not necessarily aperiodic random walks.

If $\beta > 1$, X has first order moments ([4], page 578) and $E[X_1] = 0$. It follows that φ is continuously differentiable on T^d and that $\nabla\varphi(0) = 0$. We shall prove in Section 5.4 that Assumption A3 is always satisfied in this case.

5.2. For $\lambda \ge 0$ and $x, y \in \mathbb{Z}^d$, set

$$G_\lambda(x,y) = \sum_{n=0}^{\infty} e^{-\lambda n} P_n(x,y) \quad [\text{in particular } G_0(x,y) = G(x,y)].$$

Following [9] (Section 3), we propose to study the asymptotic law of $T_x/l(|x|)$ as $|x|$ tends to infinity. Note that, for $\lambda > 0$,

$$E[\exp - \lambda T_x/l(|x|)] = \frac{G_{\lambda/l(|x|)}(0,x)}{G_{\lambda/l(|x|)}(0,0)}, \tag{5.c}$$

so that we need to investigate the asymptotic behaviour of $G_{\lambda/l(|x|)}(0,x)$. For

$\lambda > 0$, we have

$$G_\lambda(0,x) = (2\pi)^{-d} \int_{T^d} (1 - e^{-\lambda}\varphi(\xi))^{-1} e^{-ix\xi}\, d\xi$$

and this formula also holds for $\lambda = 0$ if the random walk is transient, in which case $|1 - \varphi(\xi)|^{-1}$ is integrable (Proposition 2.5). After some easy transformations, we get

$$\begin{aligned} G_{\lambda/l(x)}(0,x) &= (2\pi)^{-d} l(x)|x|^{-d} \\ &\quad \times \int_0^\infty dt \exp(-(1 - e^{-\lambda/l(x)}) l(x)t)\Gamma(x, e^{-\lambda/l(x)}t), \end{aligned} \tag{5.d}$$

where

$$\Gamma(x,t) = \int_{|x|T^d} d\xi \exp\left(-i\frac{x}{|x|}\xi - t\left(1 - \varphi\left(\frac{\xi}{|x|}\right)\right)l(x)\right) \tag{5.e}$$

(see [9], Section 3 for similar computations). Formula (5.d) suggests that $G_{\lambda/l(x)}(0,x)$ behaves like $l(x)|x|^{-d}$ when $|x|$ tends to infinity. In order to make this precise, we need the following technical estimate.

LEMMA 5.1. *For each $\varepsilon > 0$, there exists a constant C_ε such that, for all $x \in \mathbb{Z}^d - \{0\}$, $t > 0$,*

$$|\Gamma(x,t)| \le C_\varepsilon(t^\varepsilon + t^{-\varepsilon})t^{-d/\beta}. \tag{5.f}$$

Under Assumption A3, *we also have*

$$|\Gamma(x,t)| \le C_\varepsilon(t^\varepsilon + t^{-\varepsilon})t^{-(d-1)/\beta}. \tag{5.g}$$

PROOF. Proposition 2.3 and Assumption A2 give the bound

$$\mathrm{Re}(1 - \varphi(\xi)) \ge C/l(1/|\xi|), \qquad \xi \in T^d, C > 0. \tag{5.h}$$

Then, using Lemma 2.1,

$$\begin{aligned} |\Gamma(x,t)| &\le \int_{|x|T^d} d\xi \exp(-Ctl(|x|)/l(|x|/|\xi|)) \\ &\le \int_{|\xi|\le 1} d\xi \exp\left(-C'_\varepsilon t|\xi|^{\beta+\varepsilon}\right) + \int_{|\xi|>1} d\xi \exp\left(-C''_\varepsilon t|\xi|^{\beta-\varepsilon}\right) \\ &\le C_\varepsilon(t^{-d/(\beta-\varepsilon)} + t^{-d/(\beta+\varepsilon)}). \end{aligned}$$

In order to get the other bound, we use Assumption A3. Write $x = (x_1, \dots, x_d)$ and choose j such that $|x_j| = \max|x_k|$. We integrate by parts in (5.e) in the jth direction. Observe that, by Assumption A3, φ is piecewise continuously differentiable along lines. Since the boundary terms vanish by the periodicity of φ, we get

$$
\begin{aligned}
|\Gamma(x,t)| &= \left| \frac{|x|}{x_j} \int_{|x|T^d} d\xi \, t \frac{l(x)}{|x|} \frac{\partial \varphi}{\partial \xi_j}\left(\frac{\xi}{|x|}\right) \exp\left(-i \frac{x}{|x|}\xi - t\left(1 - \varphi\left(\frac{\xi}{|x|}\right)\right) l(x) \right) \right| \\
&\le Ct \int_{|x|T^d} d\xi \frac{l(x)}{|x|} \left| \nabla\varphi\left(\frac{\xi}{|x|}\right)\right| \exp\left(-C't \frac{l(x)}{l(x/|\xi|)}\right),
\end{aligned}
\tag{5.i}
$$

by (5.h). Let $r > 0$ be so small that the bound (5.a) holds for any $\xi \in rT^d$. Set

$$
W = T^d - \bigcup_{\eta \in \mathscr{S}} (\eta + rT^d).
$$

We note that, by Assumption A3, $|\nabla\varphi|$ is bounded on W. Moreover, if $\xi \in |x|W$, we have

$$
\frac{l(x)}{|x|} \le C_\varepsilon |x|^{\beta+\varepsilon-1} \le C'_{\varepsilon, r}|\xi|^{\beta+\varepsilon-1}.
$$

It follows that

$$
\begin{aligned}
&t\int_{|x|W} d\xi \frac{l(x)}{|x|}\left|\nabla\varphi\left(\frac{\xi}{|x|}\right)\right| \exp -C't\frac{l(x)}{l(x/|\xi|)} \\
&\quad \le C_{\varepsilon,r} t \left[\int_{|\xi|\ge r} d\xi \, |\xi|^{\beta+\varepsilon-1} \exp -C'_\varepsilon t|\xi|^{\beta-\varepsilon}\right] \\
&\quad \le C_{\varepsilon,r} t^{-(d-1)/(\beta-\varepsilon)-2\varepsilon/(\beta-\varepsilon)}.
\end{aligned}
$$

It remains to bound the integral over $|x|T^d - |x|W = |x|(\bigcup_{\mathscr{S}}(\eta + rT^d))$. Note that for $\xi \in T^d$ and $\eta \in \mathscr{S}$, we have $\varphi(\xi + \eta) = \varphi(\xi)\varphi(\eta)$ and thus $|\nabla\varphi(\xi + \eta)| = |\nabla\varphi(\xi)|$. Also note that if r is small enough, for $\xi \in |x|(\eta + rT^d)$, with $\eta \ne 0$, $l(x)/l(x/|\xi|)$ is bounded below by $Cl(x)$, for some $C > 0$. It easily follows from these observations that we need only consider the case $\eta = 0$. In this case, (5.a) leads to the bound

$$
\begin{aligned}
&t\int_{|x|rT^d} d\xi \frac{l(x)}{|x|}\left|\nabla\varphi\left(\frac{\xi}{|x|}\right)\right| \exp\left(-C't\frac{l(x)}{l(x/|\xi|)}\right) \\
&\quad \le Ct\int_{|x|rT^d} \frac{d\xi}{|\xi|} \frac{l(x)}{l(x/|\xi|)} \exp\left(-C't\frac{l(x)}{l(x/|\xi|)}\right) \\
&\quad \le C_\varepsilon t\left[\int_{|\xi|\le 1} d\xi \, |\xi|^{\beta-\varepsilon-1} \exp - C'_\varepsilon t|\xi|^{\beta+\varepsilon} + \int_{|\xi|\ge 1} d\xi \, |\xi|^{\beta+\varepsilon-1} \exp -C''_\varepsilon t|\xi|^{\beta-\varepsilon}\right] \\
&\quad \le C_\varepsilon \left(t^{-(d-1)/(\beta+\varepsilon)+2\varepsilon/(\beta+\varepsilon)} + t^{-(d-1)/(\beta-\varepsilon)-2\varepsilon/(\beta-\varepsilon)}\right),
\end{aligned}
$$

which completes the proof of Lemma 5.1. □

5.3. We now apply Lemma 5.1 to study the asymptotic behavior of $G_{\lambda/l(x)}(0,x)$. First, Proposition 2.3 implies that

$$\lim_{|x|\to\infty} \exp -t\left(1-\varphi\left(\frac{\xi}{|x|}\right)\right)l(x) = \exp -t|\xi|^{\beta} S\left(\frac{\xi}{|\xi|}\right).$$

Hence, using (5.h) and Lemma 2.1 to justify dominated convergence, for any $\zeta \in S^{d-1}$,

$$\text{(5.j)} \quad \lim_{\substack{|x|\to\infty \\ x/|x|\to\zeta}} \Gamma(x,t) = \int_{\mathbb{R}^d} d\xi \exp\left(-i\zeta\cdot\xi - t|\xi|^{\beta} S\left(\frac{\xi}{|\xi|}\right)\right) = (2\pi)^d p_t(\zeta).$$

PROPOSITION 5.2. *Suppose that $\beta > d-1$ and, if $\beta \le d$, that Assumption A3 holds. Then, for any $\lambda > 0$, $\zeta \in S^{d-1}$,*

$$\text{(5.k)} \qquad \lim_{\substack{|x|\to\infty \\ x/|x|\to\zeta}} \frac{|x|^d}{l(x)} G_{\lambda/l(x)}(0,x) = \int_0^\infty dt\, e^{-\lambda t} p_t(\zeta).$$

If $\beta < d$, the above result also holds for $\lambda = 0$:

$$\text{(5.l)} \qquad \lim_{\substack{|x|\to\infty \\ x/|x|\to\zeta}} \frac{|x|^d}{l(x)} G(0,x) = \int_0^\infty dt\, p_t(\zeta),$$

and we have

$$\text{(5.m)} \qquad \frac{|x|^d}{l(x)} G(0,x) \le C$$

for some constant C. If $\beta = d$, for any $\varepsilon > 0$, there exists a constant C_ε such that for $\lambda > 0$ and for $x \in \mathbb{Z}^d - \{0\}$,

$$\text{(5.n)} \qquad \frac{|x|^d}{l(x)} G_{\lambda/l(x)}(0,x) \le C_\varepsilon(\lambda^{-\varepsilon} \vee 1).$$

PROOF. We pass to the limit in (5.d) using (5.j). The use of dominated convergence is justified by Lemma 5.1 and the trivial bound

$$\exp -(1-e^{-\lambda/l(x)})l(x)t \le \exp -\lambda t/2$$

for $|x|$ large. Note that if $\beta > d$, (5.f) suffices to apply dominated convergence, so that we do not need Assumption A3 in this case. Also, if $d-1 < \beta < d$, (5.f) and (5.g) allow us to pass to the limit in (5.d) for $\lambda = 0$. In this way, we get (5.k) and (5.l). To prove (5.m) use Lemma 5.1 and notice that if $\beta < d$,

$$\frac{|x|^d}{l(x)} G(0,x) = (2\pi)^{-d} \int_0^\infty \Gamma(x,t)\, dt.$$

It remains to prove (5.n). Clearly, we may assume $\lambda \le 1$. Then, taking into account (5.d), it suffices to bound

$$\int_0^\infty dt \exp(-C\lambda t)|\Gamma(x,t)| \le C_\varepsilon\left(1 + \int_1^\infty dt \exp(-C\lambda t)t^{-1+\varepsilon}\right) \le C'_\varepsilon \lambda^{-\varepsilon}. \quad \square$$

Proposition 5.2 will immediately give some useful information on the asymptotic distribution of the hitting times

$$T_x = \inf\{n \ge 0,\, X_n = x\}.$$

Recall that if the random walk is transient, in particular if $\beta < d$, we have $G(0,0) < \infty$ and we set $q = G(0,0)^{-1}$. In case $\beta = d$ and the random walk is recurrent, we will use the following notation: For any $r > 0$,

$$h(r) = \sum_{0 \le k \le r} P_k(0,0) \qquad \bigl(h(n) = G_{(n)}(0,0)\bigr).$$

By (2.j),

$$h(r) \underset{r\to\infty}{\sim} p_1(0) \sum_{0\le k\le r} \frac{1}{ks(k)^d},$$

so that h is slowly varying by Lemma 2.2. Notice that when $\beta \ge d$, we always have $p_1(0) > 0$.

COROLLARY 5.3. *Suppose that X satisfies Assumption* A3 *if $\beta \le d$. Then*:

(i) *Assume that $d - 1 < \beta \le d$ and that X is transient. Then, for any $t > 0$, $\zeta \in S^{d-1}$*,

$$\lim_{\substack{|x|\to\infty \\ x/|x|\to\zeta}} \frac{|x|^d}{l(x)} P[T_x \le tl(x)] = q\int_0^t ds\, p_s(\zeta).$$

In addition, if $\beta < d$, there exists a constant C such that for all $x \ne 0$,

$$\frac{|x|^d}{l(x)} P[T_x < \infty] \le C. \tag{5.o}$$

If $\beta = d$, for any $\varepsilon > 0$, there exists a constant C_ε such that for all $t \ge 0$ and $x \ne 0$,

$$\frac{|x|^d}{l(x)} P[T_x \le tl(x)] \le C_\varepsilon(t^\varepsilon \vee 1). \tag{5.p}$$

(ii) *If $\beta = d$ and X is recurrent, for $t > 0$ and $\zeta \in S^{d-1}$*,

$$\lim_{\substack{|x|\to\infty \\ x/|x|\to\zeta}} \frac{|x|^d h(l(x))}{l(x)} P[T_x \le tl(x)] = \int_0^t ds\, p_s(\zeta).$$

Moreover, for any $\varepsilon > 0$, *there exists a constant* C_ε *such that for* $t \geq 1$, $x \neq 0$,

$$\frac{|x|^d h(l(x))}{l(x)} P[T_x \leq tl(x)] \leq C_\varepsilon t^\varepsilon. \tag{5.q}$$

(iii) *If* $\beta > d$,

$$\lim_{\substack{|x|\to\infty \\ x/|x|\to\zeta}} P[T_x \leq tl(x)] = P[\sigma_\zeta \leq t],$$

where $\sigma_\zeta = \inf\{t\colon U_t = \zeta\}$.

REMARK. The first assertion of (ii) is also valid in (i).

PROOF. (i) Combining (5.c) with Proposition 5.2 gives

$$\lim_{\substack{|x|\to\infty \\ x/|x|\to\zeta}} \left(|x|^d/l(x)\right) E[\exp -\lambda T_x/l(x)] = q\int_0^\infty dt\, e^{-\lambda t} p_t(\zeta).$$

Let ν_x denote the law of $T_x/l(x)$ and set $\mu_x = |x|^d l(x)^{-1}\nu_x$. The continuity theorem of Feller ([4], page 433) implies that, as $|x| \to \infty$, $x/|x| \to \zeta$, μ_x converges weakly towards the measure $\mu(dt) = qp_t(\zeta)\,dt$. This gives the first assertion of (i). The bound (5.o) follows from (5.m). Similarly, (5.p) follows from (5.n): For $t \geq 1$,

$$\left(|x|^d/l(x)\right)P[T_x \leq tl(x)] \leq C\left(|x|^d/l(x)\right)E[\exp -T_x/tl(x)] \leq C_\varepsilon t^\varepsilon.$$

(ii) The main arguments are the same as in (i). The only difference is that when applying (5.c), we now use

$$G_{\lambda/l(x)}(0,0) \underset{|x|\to\infty}{\sim} h(l(x)). \tag{5.r}$$

(5.r) follows easily from Proposition 2.4:

$$G_\lambda(0,0) \underset{\lambda\to 0}{\sim} \left(\sum_{k=1}^{\infty} b(k)^{-d} e^{-\lambda k}\right) p_1(0) \underset{\lambda\to 0}{\sim} \left(\sum_{1\leq k\leq 1/\lambda} b(k)^{-d}\right) p_1(0).$$

(iii) When $\beta > d$, which forces $d = 1$, it is well known that the process U is recurrent and hits points with probability 1. Furthermore, the strong Markov property at time σ_ζ shows that

$$E[e^{-\lambda\sigma_\zeta}] = \frac{\int_0^\infty ds\, e^{-\lambda s} p_s(\zeta)}{\int_0^\infty ds\, e^{-\lambda s} p_s(0)}.$$

We have

$$\begin{aligned} G_{\lambda/l(x)}(0,0) &= (2\pi)^{-1}\int_{-\pi}^{\pi} \left(1 - e^{-\lambda/l(x)}\varphi(\xi)\right)^{-1} d\xi \\ &= (2\pi)^{-1}\frac{l(x)}{|x|}\int_{-\pi|x|}^{\pi|x|} \left(l(x)\left(1 - e^{-\lambda/l(x)}\varphi\left(\frac{\xi}{|x|}\right)\right)\right)^{-1} d\xi. \end{aligned}$$

By Proposition 2.3,

$$\lim_{|x|\to\infty} l(x)\left(1 - e^{-\lambda/l(x)}\varphi\left(\frac{\xi}{|x|}\right)\right) = \lambda + |\xi|^\beta S\left(\frac{\xi}{|\xi|}\right).$$

It follows that

$$\lim_{|x|\to\infty} \frac{|x|}{l(x)} G_{\lambda/l(x)}(0,0) = (2\pi)^{-1}\int_{-\infty}^{\infty}\left(\lambda + |\xi|^\beta S\left(\frac{\xi}{|\xi|}\right)\right)^{-1} d\xi$$
$$= \int_0^\infty ds\, e^{-\lambda s} p_s(0),$$

by Fourier inversion. The use of dominated convergence is here justified by Proposition 2.3 and the fact that $|1 - \varphi(\xi)|$ is then bounded by a positive constant on the set $[-\pi, -\delta] \cup [\delta, \pi]$ for any $\delta > 0$. Using (5.c) and Proposition 5.2, we obtain

$$\lim_{\substack{|x|\to\infty \\ x/|x|\to\zeta}} E[\exp - \lambda T_x/l(x)] = \int_0^\infty ds\, e^{-\lambda s} p_s(\zeta) \Big/ \int_0^\infty ds\, e^{-\lambda s} p_s(0) = E[e^{-\lambda \sigma_\zeta}].$$

We conclude that $T_x/l(x)$ converges in distribution towards σ_ζ. The desired result follows since the law of σ_ζ has no atoms. □

5.4.

PROPOSITION 5.4. *Under Assumption* A1, *Assumption* A3 *holds if* $\beta > 1$.

PROOF. We claim that if $\beta > 1$, for every $r \in (1, \beta)$, there exists a constant C such that for every $n \geq 1$,

$$E\left[|b(n)^{-1} X_n|^r\right] \leq C. \tag{5.s}$$

Let us first show that Assumption A3 follows from this bound. The sequence $(b(n)^{-1}X_n)$ is then uniformly integrable, so that

$$\lim_{n\to\infty} b(n)^{-1} E\left[X_n \exp\left(i\xi \cdot b(n)^{-1}X_n\right)\right] = E[U_1 \exp(i\xi U_1)] = -i\,\nabla\Phi(\xi)$$

uniformly when ξ varies over a compact subset of $\mathbb{R}^d$. On the other hand,

$$b(n)^{-1}E\left[X_n \exp\left(i\xi \cdot b(n)^{-1}X_n\right)\right]$$
$$= nb(n)^{-1}E\left[Y_1 \exp\left(i\xi \cdot b(n)^{-1}Y_1\right)\right]\varphi\left(\frac{\xi}{b(n)}\right)^{n-1}$$
$$= -inb(n)^{-1}\nabla\varphi\left(\frac{\xi}{b(n)}\right)\varphi\left(\frac{\xi}{b(n)}\right)^{n-1}.$$

We already know that $\varphi(b(n)^{-1}\xi)^{n-1}$ converges toward $\Phi(\xi)$ uniformly on the compact subsets of $\mathbb{R}^d$. It follows that for any compact $K \subseteq \mathbb{R}^d$ for n large enough and for every $\xi \in K$,

$$nb(n)^{-1}\left|\nabla\varphi\left(\frac{\xi}{b(n)}\right)\right| \leq C.$$

Assumption A3 follows, taking for instance $K = \{1 \leq |\xi| \leq 2\}$.

It remains to prove (5.s). Clearly, we can deal separately with the different coordinates of X_n and therefore assume that $d = 1$. We will use the following estimates from Feller [4] (pages 577–579). There exists a constant c such that for every $a \geq 1$,

$$E\left[(Y_1)^2 1_{(|Y_1| \leq a)}\right] \leq c\frac{a^2}{l(a)}. \tag{5.t}$$

Furthermore, for every $\rho \in [0, \beta)$, there exists a constant $C_{(\rho)} \in [0, \infty)$ such that

$$\lim_{a\to\infty} a^{-\rho} l(a) E\left[|Y_1|^\rho 1_{(|Y_1| > a)}\right] = C_{(\rho)} \tag{5.u}$$

(see formulas XVII (5.21) and (5.23) in [4]; when $\beta = 2$, we have $C_{(\rho)} = 0$ for every ρ, but we shall not need that).

Since $E[Y_1] = 0$, it follows from (5.u) that for n large,

$$\left|E\left[Y_1 1_{(|Y_1| \leq b(n))}\right]\right| = \left|E\left[Y_1 1_{(|Y_1| > b(n))}\right]\right| \leq 2C_{(1)}\frac{b(n)}{n}. \tag{5.v}$$

Hence,

$$\begin{aligned} &E\left[\left(b(n)^{-1}\sum_{i=1}^{n} Y_i 1_{(|Y_i| \leq b(n))}\right)^2\right] \\ &\quad = nb(n)^{-2}E\left[(Y_1)^2 1_{(|Y_1| \leq b(n))}\right] + n(n-1)b(n)^{-2}E\left[Y_1 1_{(|Y_1| \leq b(n))}\right]^2 \\ &\quad \leq C\left(nb(n)^{-2}\frac{b(n)^2}{n} + n^2 b(n)^{-2}\left(\frac{b(n)}{n}\right)^2\right) \leq C', \end{aligned}$$

by (5.t) and (5.v). Thus it is enough to bound

$$\begin{aligned} &E\left[\left|b(n)^{-1}\sum_{i=1}^{n} Y_i 1_{(|Y_i| > b(n))}\right|^r\right] \\ &\qquad \leq b(n)^{-r}E\left[\left(\sum_{i=1}^{n} 1_{(|Y_i| > b(n))}\right)^{r-1}\sum_{i=1}^{n}|Y_i|^r 1_{(|Y_i| > b(n))}\right], \end{aligned}$$

by the Hölder inequality. Next we condition on

$$A_n = \sum_{i=1}^{n} 1_{(|Y_i| > b(n))}.$$

Observe that the law of A_n is dominated by a geometric law of parameter

$$\text{(5.w)}\qquad \begin{aligned} \alpha_n &= P[A_n \geq 1] = 1 - \left(1 - P[|Y_1| > b(n)]\right)^n \\ &\underset{n\to\infty}{\sim} 1 - \exp -nP[|Y_1| > b(n)] \end{aligned}$$

(notice that $nP[|Y_1| > b(n)] \leq C$ by (5.u) with $\rho = 0$). Then

$$\begin{aligned} E&\left[\left|b(n)^{-1}\sum_{i=1}^{n} Y_i 1_{(|Y_i| > b(n))}\right|^r\right] \\ &\leq E\left[A_n^r E\left[b(n)^{-r}|Y_1|^r \,\middle|\, |Y_1| > b(n)\right]\right] \\ &\leq C\sum_{k=1}^{\infty} k^r (\alpha_n)^k \frac{1}{nP[|Y_1| > b(n)]} \leq C', \end{aligned}$$

by (5.w). We have used (5.u) to bound

$$E\left[|Y_1|^r \,\middle|\, |Y_1| > b(n)\right] \leq C\frac{b(n)^r}{nP[|Y_1| > b(n)]}.$$

This completes the proof of (5.s). □

6. The case $2d/3 < \beta \leq d$.

6.1. We now suppose that $2d/3 < \beta \leq d$ and we shall first assume that the random walk X is transient (this assumption is automatically satisfied unless $\beta = d$). We propose to show the central limit theorem for R_n by following the approach of [9]. In contrast with the case $\beta \leq 2d/3$, the limiting variable will not be normal, but will be a self-intersection local time of the limiting process U.

Let X' denote a random walk independent of X. Assume that both X and X' are transient and satisfy Assumptions A1 and A2 with the same index $\beta \in (2d/3; d]$ and the same function $b(n)$ (but not necessarily the same limiting process U). Set

$$\begin{aligned} I_n &= |X(0,n) \cap X'(0,n)|, \\ J_n &= \sum_{j=0}^{n}\sum_{k=0}^{n} I(X_j = X'_k). \end{aligned}$$

According to [18] (Theorem 5) we have

$$\text{(6.a)}\qquad \frac{b(n)^d}{n^2} J_n \xrightarrow[n\to\infty]{(d)} \alpha_{U,U'}\left([0,1]^2\right),$$

where $\alpha_{U,U'}$ denotes the intersection local time of the two (independent) limiting processes U, U', formally defined by

$$\alpha_{U,U'}\left([0,1]^2\right) = \iint_{[0,1]^2} ds\,dt\,\delta_{(0)}(U_s - U'_t).$$

In fact, (6.a) is only proved in [18] in the special case $d = 2$, $U = U'$. However, it is not hard to see that the arguments of [18] can be extended to our more general situation. Also it is assumed in [18] that the random walks are aperiodic. This assumption can be removed using the ideas of the end of Section 2.4.

Let us briefly recall a rigorous definition of the intersection local time. Let $[a, b] \times [c, d]$ be a rectangle in $(\mathbb{R}_+)^2$. With probability 1, the random measure on $\mathbb{R}^d$,

$$\varphi \to \int_a^b ds \int_c^d dt \, \varphi(U_s - U_t'),$$

has a continuous density with respect to the d-dimensional Lebesgue measure. We define $\alpha_{U,U'}([a, b] \times [c, d])$ as the value at 0 of this density. One may choose a version of the collection $(\alpha_{U,U'}([a, b] \times [c, d]))$ that is jointly continuous in a, b, c, d (see [14]). Then, the mapping $[a, b] \times [c, d] \to \alpha_{U,U'}([a, b] \times [c, d])$ defines a Radon measure on $(\mathbb{R}_+)^2$, which is supported on $\{(s, t), U_s = U_t'\}$.

Let us now discuss self-intersection local times. The basic idea is to interpret the self-intersections of U as intersections of independent processes constructed from U. This can be done as follows. For every $s \in [0, \frac{1}{2}]$, set

$$U_s' = U_{1/2-s}, \qquad U_s'' = U_{1/2+s}.$$

The above discussion allows us to consider

$$\alpha_{U',U''}\left((0, \tfrac{1}{2}]^2\right) = \iint_{(0,1/2]^2} ds\, dt \, \delta_{(0)}(U_s' - U_t'')$$

(the fact that U', U'' are only defined on $[0, \frac{1}{2}]$ is unimportant). We then define

$$\alpha\left([0, \tfrac{1}{2}) \times (\tfrac{1}{2}, 1]\right) := \alpha_{U',U''}\left((0, \tfrac{1}{2}]^2\right)$$

so that, formally,

$$\alpha\left([0, \tfrac{1}{2}) \times (\tfrac{1}{2}, 1]\right) = \iint_{[0, \frac{1}{2}) \times (\frac{1}{2}, 1]} ds\, dt \, \delta_{(0)}(U_s - U_t).$$

Now we would like to define the integral of the right-hand side on the set $\{(s, t), 0 \le s < t \le 1\}$, in order to take account of all self-intersections on the time interval $[0, 1]$. This requires a renormalization procedure. For every $j \ge 1$, for $1 \le i \le 2^{j-1}$, set

$$A_i^j = \left[\frac{2i-2}{2^j}, \frac{2i-1}{2^j}\right) \times \left(\frac{2i-1}{2^j}, \frac{2i}{2^j}\right].$$

Notice that $\{(s, t), 0 \le s < t \le 1\}$ is the disjoint union of the sets A_j^i. Arguing as previously, we can define

$$\alpha\left(A_j^i\right) = \iint_{A_j^i} ds\, dt \, \delta_{(0)}(U_s - U_t).$$

Then $\sum_{i,j} \alpha(A_j^i) = +\infty$ a.s. However, simple scaling arguments (see [14]) show that the renormalized series $\sum_{i,j}\{\alpha(A_j^i)\}$ converges in the L^2-norm. We set

$$\gamma_U = \sum_{j=1}^{\infty} \sum_{i=1}^{2^{j-1}} \left\{\alpha\left(A_j^i\right)\right\},$$

so that, formally,

$$\gamma_U = \iint_{\{(s,t),\,0\le s<t\le 1\}} ds\,dt\left(\delta_{(0)}(U_s - U_t) - E\left[\delta_{(0)}(U_s - U_t)\right]\right).$$

Note that this formal equality can also be made rigorous by replacing the Dirac measure $\delta_{(0)}$ with a suitable sequence of approximating functions.

We shall need the following strengthening of (6.a). For $n \ge 1$, $t \ge 0$, set

$$X_t^{(n)} = b(n)^{-1} X_{[nt]}, \qquad X_t^{\prime(n)} = b(n)^{-1} X'_{[nt]}.$$

Skorokhod's extension of Donsker's theorem ([20]) states that

$$(6.\text{b}) \qquad (X^{(n)}, X^{\prime(n)}) \xrightarrow[n\to\infty]{(d)} (U, U'),$$

where convergence holds in distribution in the sense of Skorokhod's topology. The proof of (6.a) given in [18] shows that the limiting results (6.a) and (6.b) hold jointly:

$$(6.\text{c}) \qquad \left(X^{(n)}, X^{\prime(n)}, \frac{b(n)^d}{n^2} J_n\right) \xrightarrow[n\to\infty]{(d)} \left(U, U', \alpha_{U,U'}\left([0,1]^2\right)\right).$$

In what follows, we shall be mainly concerned by the asymptotic behaviour of I_n. Our first task is to prove that asymptotically I_n behaves like some constant times J_n.

6.2. Our first lemma is a simple consequence of Corollary 5.3(i). For $u \in \mathbb{R}^d$, $u = (u_1, \dots, u_d)$ we set $[u] = ([u_1], \dots, [u_d])$.

LEMMA 6.1. *Suppose that X satisfies Assumptions* A1, A2 *and* A3 *with $d - 1 < \beta \le d$ and that X is transient. Then for each $u \in \mathbb{R}^d - \{0\}$,*

$$(6.\text{d}) \qquad \lim_{n\to\infty} \frac{b(n)^d}{n} P\left[T_{[b(n)u]} \le n\right] = q\int_0^1 ds\, p_s(u).$$

Moreover, for any $\varepsilon > 0$, there exists a constant C_ε such that for any $n \ge 1$, $u \in \mathbb{R}^d$,

$$(6.\text{e}) \qquad \frac{b(n)^d}{n} P\left[T_{[b(n)u]} \le n\right] \le C_\varepsilon\left(|u|^{\beta-d+\varepsilon} + |u|^{\beta-d-\varepsilon}\right).$$

PROOF. Set $x = x(n) = [b(n)u]$. Then,

$$\left(T_{[b(n)u]} \le n\right) = \left(T_x \le n\right) = \left(T_x \le l(x)\frac{l(b(n))}{l([b(n)u])}\right).$$

Since l is regularly varying of index β we have

$$\frac{l(b(n))}{l([b(n)u])} \xrightarrow[n\to\infty]{} |u|^{-\beta}. \tag{6.f}$$

On the other hand,

$$\frac{|x|^d}{l(x)} = \frac{|[b(n)u]|^d}{l([b(n)u])} \underset{n\to\infty}{\sim} |u|^{d-\beta}\frac{b(n)^d}{n}. \tag{6.g}$$

(6.d) now follows from (6.f), (6.g) and the first assertion of Corollary 5.3, noting that

$$|v|^{-d}p_{s/|v|^\beta}\left(\frac{v}{|v|}\right) = p_s(v).$$

Let us prove (6.e). We first assume that $|[b(n)u]| > 2$ and we apply the bounds of Corollary 5.3(i) to $x = [b(n)u]$. It follows that

$$P\left[T_{[b(n)u]} \le n\right] \le C_\varepsilon|[b(n)u]|^{-d}l([b(n)u])(|u|^{-\varepsilon} \vee 1).$$

Hence,

$$\begin{aligned}\frac{b(n)^d}{n}P\left[T_{[b(n)u]} \le n\right] &\le C_\varepsilon\frac{|[b(n)u]|^{-d}}{b(n)^{-d}}\frac{l([b(n)u])}{l(b(n))}(|u|^{-\varepsilon} \vee 1)\\ &\le C'_\varepsilon(|u|^{\beta-d-2\varepsilon} + |u|^{\beta-d+\varepsilon}),\end{aligned}$$

by the properties of functions of regular variation. If $|[b(n)u]| \le 2$, we simply write

$$\frac{b(n)^d}{n}P\left[T_{[b(n)u]} \le n\right] \le \frac{b(n)^d}{n} \le b(n)^{d-\beta}\frac{b(n)^\beta}{n} \le C_\varepsilon|u|^{\beta-d-\varepsilon}. \qquad \square$$

REMARK. The bound (6.e) is far from optimal when $|u|$ is large. As a matter of fact, we will only use it for $|u| \le K$ in our applications. A similar remark applies to the bounds of Lemma 6.2.

LEMMA 6.2. *Under the assumptions of Lemma* 6.1, *we have, for any* $u, v \in \mathbb{R}^d - \{0\}$, $u \ne v$,

$$\text{(i)} \qquad \lim_{n\to\infty}\left(\sup_{k\le n}\left|\frac{b(n)^d}{n}P_{[b(n)u]}\left[T_{[b(n)v]} \le k\right] - q\int_0^{k/n}p_s(v-u)\,ds\right|\right) = 0,$$

$$\text{(ii)} \qquad \begin{aligned}\lim_{n\to\infty}\left(\sup_{k\le n}\left|\frac{b(n)^d}{n}E_{[b(n)u]}\left[\sum_{i=0}^{k} I(X_i = [b(n)v])\right]\right.\right.\\ \left.\left. -\int_0^{k/n}p_s(v-u)\,ds\right|\right) = 0.\end{aligned}$$

Moreover, for any $\varepsilon > 0$, *there exists a constant* C_ε *such that for all* $u, v \in \mathbb{R}^d$,

$$\text{(iii)}\quad \frac{b(n)^d}{n} P_{[b(n)u]}\big[T_{[b(n)v]} \le n\big] \le C_\varepsilon\big(|v-u|^{\beta-d+\varepsilon} + |v-u|^{\beta-d-\varepsilon}\big),$$

$$\text{(iv)}\quad \frac{b(n)^d}{n} E_{[b(n)u]}\left[\sum_{i=0}^{k} I(X_i = [b(n)v])\right] \le C_\varepsilon\big(|v-u|^{\beta-d+\varepsilon} + |v-u|^{\beta-d-\varepsilon}\big).$$

PROOF. All assertions are easy consequences of Lemma 6.1. First, a slight extension of (6.d) shows that for $0 \le \alpha \le 1$,

$$\lim_{n\to\infty} \frac{b(n)^d}{n} P_{[b(n)u]}\big[T_{[b(n)v]} \le \alpha n\big] = q\int_0^\alpha p_s(v-u)\,ds.$$

(i) follows: The uniformity in k is a consequence of the monotonicity of the mappings

$$k \to P_{[b(n)u]}\big[T_{[b(n)v]} \le k\big] \quad\text{and}\quad k \to \int_0^{k/n} p_s(v-u)\,ds.$$

Assertion (ii) follows from (i) by using (2.k) and the following simple fact. For any $\delta > 0$, we may choose N large enough so that

$$q^{-1} - \delta \le E_0\left[\sum_{i=0}^{N} I(X_i = 0)\right] = G_{(N)}(0,0) \le q^{-1}.$$

Finally, (iii) follows from (6.e), whereas (iv) is a trivial consequence of (iii). □

6.3. We will now use Lemma 6.2 to prove the main estimate needed in the proof of the central limit theorem for R_n. We use the notation and assumptions of Section 6.1. In particular, $2d/3 < \beta \le d$.

PROPOSITION 6.3. *Suppose that X and X' are independent and both satisfy the assumptions of Lemma* 6.1. *Then*

$$\frac{b(n)^d}{n^2}(I_n - qq'J_n) \xrightarrow[n\to\infty]{(P)} 0.$$

PROOF. Let $K > 0$. We shall first prove the statement of Proposition 6.3 with I_n and J_n replaced by

$$I_n^K = |X(0,n) \cap X'(0,n) \cap \{x, |x| \le Kb(n)\}|,$$

$$J_n^K = \sum_{j=0}^{n}\sum_{k=0}^{n} I\big(X_j = X'_k\big) I\big(|X_j| \le Kb(n)\big).$$

Then

$$E\Big[\big(I_n^K - qq'J_n^K\big)^2\Big] = E\Big[\big(I_n^K\big)^2\Big] - 2qq'E\big[I_n^K J_n^K\big] + (qq')^2 E\Big[\big(J_n^K\big)^2\Big].$$

We investigate separately the asymptotic behaviour of the three terms of the right-hand side. We have

$$\begin{aligned}
&\frac{b(n)^{2d}}{n^4} E\Big[\big(I_n^K\big)^2\Big] \\
&\quad = \frac{b(n)^{2d}}{n^4} \sum_{\substack{y,z\in\mathbb{Z}^d \\ |y|,|z|\le Kb(n)}} P[y\in X(0,n), z\in X(0,n)] \\
&\qquad\qquad \times P[y\in X'(0,n), z\in X'(0,n)] \qquad (6.\mathrm{h})\\
&\quad = \frac{b(n)^{4d}}{n^4} \int_{\{|[b(n)u]|\le Kb(n),\, |[b(n)v]|\le Kb(n)\}} \vartheta_n(u,v)\vartheta_n'(u,v)\,du\,dv,
\end{aligned}$$

where we have set: $\vartheta_n(u,v) = P[[b(n)u] \in X(0,n),\ [b(n)v] \in X(0,n)]$. Now note that, if $u \neq v$ and n is large enough,

$$\begin{aligned}
\vartheta_n(u,v) &= P\Big[T_{[b(n)u]} \le n;\, P_{[b(n)u]}\big[T_{[b(n)v]} \le k\big]_{k=n-T_{[b(n)u]}}\Big] \\
&\quad + P\Big[T_{[b(n)v]} \le n;\, P_{[b(n)v]}\big[T_{[b(n)u]} \le k\big]_{k=n-T_{[b(n)v]}}\Big] - E_n(u,v), \qquad (6.\mathrm{i})
\end{aligned}$$

where the error term $E_n(u,v)$ involves visiting u both before and after visiting v, or the same with u replaced by v. Using Lemma 6.2(iii) we obtain, provided $u, v \neq 0$,

$$E_n(u,v) \le C_{u,v}\left(\frac{n}{b(n)^d}\right)^3.$$

Hence

$$\lim_{n\to\infty} \frac{b(n)^{2d}}{n^2} E_n(u,v) = 0$$

[note that since X is transient we must have: $\lim_{n\to\infty} n/b(n)^d = 0$]. Coming back to (6.i) and using (i) of Lemma 6.2 we obtain

$$\begin{aligned}
\lim_{n\to\infty} \frac{b(n)^{2d}}{n^2}\vartheta_n(u,v) &= q^2\bigg(\int_0^1 ds\, p_s(u)\int_0^{1-s} dt\, p_t(v-u) \\
&\qquad + \int_0^1 ds\, p_s(v)\int_0^{1-s} dt\, p_t(u-v)\bigg), \qquad (6.\mathrm{j})
\end{aligned}$$

whenever $u \neq v$, $u, v \neq 0$.

Set

$$f_\varepsilon(u) = |u|^{\beta-d-\varepsilon} + |u|^{\beta-d+\varepsilon}.$$

Then Lemma 6.2(iii) easily implies

$$\frac{b(n)^{2d}}{n^2}\vartheta_n(u,v) \le C_\varepsilon\big(f_\varepsilon(u)f_\varepsilon(v-u)+f_\varepsilon(v)f_\varepsilon(u-v)\big) \tag{6.k}$$

for all $u, v \in \mathbb{R}^d$, $n \ge 1$. Similar statements hold when $\vartheta_n(u,v)$ is replaced by $\vartheta'_n(u,v)$. (6.j) and (6.k) can then be used to pass to the limit in (6.h). Note that (6.k) allows us to apply dominated convergence, since f_ε is locally square integrable, provided ε is small enough. We finally obtain

$$\begin{aligned}
&\lim_{n\to\infty}\frac{b(n)^{2d}}{n^4}E\Big[\big(I_n^K\big)^2\Big]\\
&\quad= (qq')^2\int_{B_K^2}du\,dv\bigg(\int_0^1 ds\,p_s(u)\int_0^{1-s}dt\,p_t(v-u)\\
&\qquad\qquad+\int_0^1 ds\,p_s(v)\int_0^{1-s}dt\,p_t(u-v)\bigg)\\
&\qquad\times\bigg(\int_0^1 ds\,p'_s(u)\int_0^{1-s}dt\,p'_t(v-u)+\int_0^1 ds\,p'_s(v)\int_0^{1-s}dt\,p'_t(u-v)\bigg),
\end{aligned}\tag{6.l}$$

where $B_K = \{u; |u| \le K\}$. We will now prove that the same result holds when $(I_n^K)^2$ is replaced by $qq'I_n^K J_n^K$ or by $(qq')^2(J_n^K)^2$. It will then follow that

$$\lim_{n\to\infty}\frac{b(n)^{2d}}{n^4}E\Big[\big(I_n^K-qq'J_n^K\big)^2\Big]=0. \tag{6.m}$$

We first consider

$$\begin{aligned}
&\frac{b(n)^{2d}}{n^4}E\big[I_n^K J_n^K\big]\\
&\quad=\frac{b(n)^{2d}}{n^4}\sum_{\substack{y,z\in\mathbb{Z}^d\\ |y|,|z|\le Kb(n)}}E\bigg[y\in X(0,n),\ \sum_{i=0}^n I(X_i=z)\bigg]\\
&\qquad\qquad\times E\bigg[y\in X'(0,n),\ \sum_{i=0}^n I\big(X'_i=z\big)\bigg]\\
&\quad=\frac{b(n)^{4d}}{n^4}\int_{\{\|[b(n)u]\|\le Kb(n),\ \|[b(n)v]\|\le Kb(n)\}}\varphi_n(u,v)\varphi'_n(u,v)\,du\,dv,
\end{aligned}\tag{6.n}$$

where $\varphi_n(u,v) = E[[b(n)u] \in X(0,n); \Sigma_{i=0}^n I(X_i = [b(n)v])]$. Note that

$$\varphi_n(u,v) = \varphi_n^1(u,v) + \varphi_n^2(u,v),$$

where

$$\varphi_n^1(u,v) = E\left[T_{[b(n)u]} \le n;\ \sum_{i=T_{[b(n)u]}}^{n} I(X_i = [b(n)v])\right],$$

$$\varphi_n^2(u,v) = \sum_{i=0}^{n} E\big[I(X_i = [b(n)v])I(i < T_{[b(n)u]} \le n)\big].$$

The asymptotic behaviour of $\varphi_n^1(u,v)$ is easily obtained by applying the Markov property at time $T_{[b(n)u]}$ and using Lemma 6.2: If $u \neq v$ and $u, v \neq 0$,

$$\lim_{n\to\infty} \frac{b(n)^{2d}}{n^2}\varphi_n^1(u,v) = q\int_0^1 ds\, p_s(u)\int_0^{1-s} dt\, p_t(v-u) \tag{6.o}$$

and

$$\frac{b(n)^{2d}}{n^2}\varphi_n^1(u,v) \le C_\varepsilon f_\varepsilon(u) f_\varepsilon(v-u).$$

In order to study the asymptotic behaviour of $\varphi_n^2(u,v)$, we first replace $\varphi_n^2(u,v)$ by

$$\begin{aligned}\tilde{\varphi}_n^2(u,v) &= \sum_{i=0}^{n} E\big[I(X_i = [b(n)v])I([b(n)u] \in X(i+1,n))\big]\\ &= \sum_{i=0}^{n} E\big[I(X_i = [b(n)v])P_{[b(n)v]}\big[T_{[b(n)u]} \le n-i\big]\big].\end{aligned}$$

Indeed we have

$$0 \le \tilde{\varphi}_n^2(u,v) - \varphi_n^2(u,v) \le \overline{E}_n(u,v),$$

where

$$\overline{E}_n(u,v) = \sum_{i=0}^{n} E\big[I(T_{[b(n)u]} \le i)I(X_i = [b(n)v])I([b(n)u] \in X(i+1,n))\big]$$

and, as above for $E_n(u,v)$, we have

$$\lim_{n\to\infty}\frac{b(n)^{2d}}{n^2}\overline{E}_n(u,v) = 0, \qquad u \neq v,\ u,v \neq 0.$$

On the other hand, Lemma 6.2 implies

$$\lim_{n\to\infty} \frac{b(n)^{2d}}{n^2}\tilde{\varphi}_n^2(u,v) = q\int_0^1 ds\, p_s(v)\int_0^{1-s} dt\, p_t(u-v) \tag{6.p}$$

and

$$\frac{b(n)^{2d}}{n^2}\tilde{\varphi}_n^2(u,v) \le C_\varepsilon f_\varepsilon(v) f_\varepsilon(u-v).$$

Putting (6.o) and (6.p) together and applying dominated convergence in (6.n), we obtain the analogue of (6.l) with $E[(I_n^K)^2]$ replaced by $qq'E[I_n^K J_n^K]$. It would remain to study $E[(J_n^K)^2]$. This case will be left to the reader since it is easier than the previous ones. The main tools are again (ii) and (iv) of Lemma 6.2. This completes the proof of (6.m).

It remains to pass from (6.m) to the statement of Proposition 6.3. Skorokhod's extension of Donsker's theorem, recalled in Section 6.1, implies that

$$\lim_{K\to\infty}\left(\limsup_{n\to\infty} P\left[\sup_{k\le n}|X_k| > Kb(n)\right]\right) = 0. \tag{6.q}$$

Putting (6.m) and (6.q) together gives the result of Proposition 6.3. □

Corollary 6.4 follows by combining (6.c) and Proposition 6.3.

COROLLARY 6.4. *Suppose that X and X' are transient and both satisfy Assumptions* A1, A2 *and* A3 *with $2d/3 < \beta \le d$. Then, with the notation of Section* 6.1,

$$\left(X^{(n)}, X'^{(n)}, \frac{b(n)^d}{n^2} I_n\right) \xrightarrow[n\to\infty]{(d)} \left(U, U', qq'\alpha_{U,U'}([0,1]^2)\right),$$

where the convergence of processes is in the sense of Skorokhod's topology.

6.4. Before proving the central limit theorem for R_n we need to get a bound on $\operatorname{var} R_n$.

LEMMA 6.5. *Under Assumptions* A1 *and* A2, *if $2d/3 < \beta \le d$, there exists a constant C such that for any $n \ge 1$,*

$$\operatorname{var} R_n \le C\frac{n^4}{b(n)^{2d}}.$$

REMARK. Lemma 6.5 does not require Assumption A3.

PROOF. We use essentially the same method as in the proof of (4.f). We take $n \ge 2$ and $n_1 = [n/2]$, $n_2 = n - n_1$. Then

$$R_n = |X(0,n_1)| + |X(n_1,n_2)| - |X(0,n_1)\cap X(n_1,n)|.$$

Hence, with the usual notation $\{V\} = V - E[V]$,

$$\begin{aligned} E\left[\{R_n\}^2\right]^{1/2} &\le \left(E\left[\{R_{n_1}\}^2\right] + E\left[\{R_{n_2}\}^2\right]\right)^{1/2} + E\left[|X(0,n_1)\cap X(n_1,n)|^2\right]^{1/2} \\ &\le \left(E\left[\{R_{n_1}\}^2\right] + E\left[\{R_{n_2}\}^2\right]\right)^{1/2} + C\frac{n^2}{b(n)^d}, \end{aligned}$$

by Corollary 3.2 (or rather the remark following that corollary). The point is that, due to the independence of increments, we may interpret $|X(0,n_1)\cap X(n_1,n)|$ as the number of intersection points of two independent random walks both satisfying Assumptions A1 and A2. For any $k \geq 1$, set

$$a_k = \sup\left\{ \frac{b(2^k)^d}{2^{2k}} E\left[\{R_n\}^2\right]^{1/2}; 2^k \leq n \leq 2^{k+1} \right\}.$$

We obtain

$$a_{k+1} \leq \frac{1}{4} \frac{b(2^{k+1})^d}{b(2^k)^d} 2^{1/2} a_k + C' \leq 2^{d/\beta - 3/2 + \varepsilon} a_k + C'$$

for any $\varepsilon > 0$ and for k large enough (depending on ε). Since $\beta > 2d/3$ we may take ε so small that $d/\beta - \frac{3}{2} + \varepsilon < 0$. It follows that the sequence (a_k) is bounded. □

THEOREM 6.6. *Under Assumptions* A1, A2 *and* A3, *if* $2d/3 < \beta \leq d$ *and if the random walk* X *is transient, we have*

$$\frac{b(n)^d}{n^2}(R_n - E[R_n]) \xrightarrow[n\to\infty]{(d)} -q^2 \gamma_U.$$

PROOF. For every $p \geq 1$, we have

$$\begin{aligned}
\{R_n\} &= \{|X(0,n)|\} \\
&= \sum_{i=1}^{2^p} \left\{ \left| X\left(\frac{i-1}{2^p} n, \frac{i}{2^p} n \right) \right| \right\} - \sum_{j=1}^{p} \sum_{i=1}^{2^{j-1}} \left\{ \left| X\left(\frac{2i-2}{2^j} n, \frac{2i-1}{2^j} n \right) \right. \right. \\
&\qquad\qquad \left. \left. \cap X\left(\frac{2i-1}{2^j} n, \frac{2i}{2^j} n \right) \right| \right\} \\
&= A(n,p) - B(n,p).
\end{aligned}$$

Lemma 6.5 and the independence of the increments of X give the bound

$$E\left[A(n,p)^2\right] \leq 2^p \sup_i E\left[\left\{ \left| X\left(\frac{i-1}{2^p} n, \frac{i}{2^p} n \right) \right| \right\}^2 \right] \leq C 2^{-3p} n^4 b\left(\frac{n}{2^p} \right)^{-2d},$$

where the constant C does not depend on n, p. Notice that for a fixed p,

$$b(n)^{2d} b\left(\frac{n}{2^p} \right)^{-2d} \leq (1+\varepsilon) 2^{2dp/\beta}$$

for any $\varepsilon > 0$ and n large enough. Hence,

$$n^{-4} b(n)^{2d} E\left[A(n,p)^2\right] \leq C(1+\varepsilon) 2^{p(2d/\beta - 3)},$$

provided n is large enough. Since $2d/\beta - 3 < 0$, it follows that if p is large, $n^{-2} b(n)^d A(n,p)$ is small (in L^2-norm) uniformly in n. Thus we need only study $B(n,p)$ for p large.

For $1 \le j \le p$ and $1 \le i \le 2^{j-1}$, let A_j^i and $\alpha(A_j^i)$ be as in Section 6.1. We claim that

$$\left(X^{(n)}; \frac{b(n)^d}{n^2}\left\{\left|X\left(\frac{2i-2}{2^j}n; \frac{2i-1}{2^j}n\right) \cap X\left(\frac{2i-1}{2^j}n; \frac{2i}{2^j}n\right)\right|\right\}, \right.$$
$$\left. 1 \le j \le p, 1 \le i \le 2^{j-1}\right)$$
$$\xrightarrow[n\to\infty]{(d)} \left(U; q^2\left\{\alpha\left(A_i^j\right)\right\}, 1 \le j \le p, 1 \le i \le 2^{j-1}\right).$$

Indeed, let us first fix i and j. The same arguments as in the proof of Theorem 4.5 allow us to interpret the quantity

$$\left|X\left(\frac{2i-2}{2^j}n; \frac{2i-1}{2^j}n\right) \cap X\left(\frac{2i-1}{2^j}n; \frac{2i}{2^j}n\right)\right|$$

as the number of intersection points of two independent random walks taken on the interval $[0, 2^{-j}n]$. The convergence of this (suitably normalized) quantity then follows from Corollary 6.4 and our construction of $\alpha(\cdot)$ from the intersection local time of independent processes. Moreover, Corollary 6.4 also shows that this convergence holds jointly with that of $X^{(n)}$. The latter fact and an easy tightness argument imply that the convergence holds jointly for all pairs (i, j).

In particular, we get

$$\frac{b(n)^d}{n^2}B(n,p) \xrightarrow[n\to\infty]{(d)} q^2 \sum_{j=1}^{p} \sum_{i=1}^{2^{j-1}} \left\{\alpha\left(A_i^j\right)\right\}.$$

The proof of Theorem 6.6 is now completed by noting that, by the definition of γ_U, the right-hand side in the previous equation is close to $q^2\gamma_U$ when p is large. □

6.5. We now turn to the case when $\beta = d$ and X is recurrent. We keep the notation introduced in Section 5:

$$h(n) = G_{(n)}(0,0) = \sum_{k=0}^{n} P_k(0,0).$$

We have already noticed that h is slowly varying and that

$$h(n) \underset{n\to\infty}{\sim} p_1(0) \sum_{k=1}^{n} b(k)^{-d}, \qquad p_1(0) > 0.$$

We first state the analogue of Lemma 6.5.

LEMMA 6.7. *Under Assumptions* A1 *and* A2, *if* $\beta = d$ *and if* X *is recurrent, there exists a constant* C *such that for* $n \ge 1$,

$$\operatorname{var} R_n \le C\frac{n^4}{b(n)^{2d}h(n)^4}.$$

The proof is exactly similar to that of Lemma 6.5 and will thus be left to the reader. The main tool is the bound

$$E\left[|X(0,n/2)\cap X(n/2,n)|^2\right]\le C\frac{n^4}{b(n)^{2d}h(n)^4},$$

which follows from Corollary 3.2.

THEOREM 6.8. *Under Assumptions* A1, A2 *and* A3, *if* $\beta = d$ *and if the random walk* X *is recurrent, we have*

$$\frac{b(n)^d h(n)^2}{n^2}(R_n - E[R_n]) \xrightarrow[n\to\infty]{(d)} -\gamma_U.$$

The proof of Theorem 6.8 follows the outline of that of Theorem 6.6. The analogues of Lemmas 6.1 and 6.2 are easily established. For instance the first two assertions of Lemma 6.2 are replaced by

$$\lim_{n\to\infty}\sup_{k\le n}\left|\frac{b(n)^d h(n)}{n}P_{[b(n)u]}\left[T_{[b(n)v]}\le k\right]-\int_0^{k/n}p_s(v-u)\,ds\right|=0$$

and

$$\lim_{n\to\infty}\sup_{k\le n}\left|\frac{b(n)^d}{n}E_{[b(n)u]}\left[\sum_{i=0}^{k}I(X_i=[b(n)v])\right]-\int_0^{k/n}p_s(v-u)\,ds\right|=0.$$

The proof uses (ii) of Corollary 5.3 instead of (i) of this result in the transient case. In proving the analogue of the second assertion of Lemma 6.2, we also note that for any $\varepsilon,\delta>0$, we have for n large:

$$\sum_{1\le i\le \varepsilon n} P_i(0,0)\ge(1-\delta)G_{(n)}(0,0).$$

This follows from the fact that h is slowly varying.

The analogue of Proposition 6.3 is then established as in the transient case. It states that

$$\frac{b(n)^d}{n^2}\left(h(n)^2 I_n - J_n\right)\xrightarrow[n\to\infty]{(P)} 0$$

and then (6.c) implies

$$\left(X^{(n)}, X'^{(n)}, \frac{b(n)^d h(n)^2}{n^2}I_n\right)\xrightarrow[n\to\infty]{(d)}\left(U,U',\alpha_{U,U'}([0,1]^2)\right).$$

The remaining part of the argument is entirely similar to the transient case.

6.6. Jain and Pruitt [7] have proved the strong law of large numbers for R_n for all recurrent random walks in the plane. We shall recover this result in our special situation and extend it to certain one-dimensional walks.

THEOREM 6.9. *Under Assumptions* A1 *and* A2, *if* $d = \beta$ *and if the random walk* X *is recurrent, we have*

$$E[R_n] = \frac{n}{h(n)} + p_1(0)\frac{n}{h(n)^2 s(n)^d} + o\left(\frac{n}{h(n)^2 s(n)^d}\right) \tag{6.r}$$

and

$$\lim_{n\to\infty} \frac{h(n)}{n} R_n = 1,$$

where convergence holds in the L^2*-norm, and almost surely if* $s(n) \geq 1$ *for each* n. *If we also satisfy Assumption* A3 *we have*

$$R_n = \frac{n}{h(n)} + \frac{n}{h(n)^2 s(n)^d}(p_1(0) - \gamma_n), \tag{6.s}$$

where the sequence γ_n *is bounded in* L^p *for any* $p < \infty$ *and converges in distribution towards* γ_U.

PROOF. In order to investigate the asymptotic behaviour of $E[R_n]$ we need some more notation. Set

$$u_n = P_n(0,0),$$
$$r_n = P_0[H_0 > n], \quad \text{where } H_0 = \inf\{n \geq 1, X_n = 0\}.$$

Then

$$E[R_n] = \sum_{i=0}^{n} r_i,$$

and by considering the last zero of X before time n,

$$\sum_{i=0}^{n} u_i r_{n-i} = 1. \tag{6.t}$$

Since (r_n) is nonincreasing, (6.t) implies

$$r_n \leq \left(\sum_{i=0}^{n} u_i\right)^{-1} = \frac{1}{h(n)}.$$

We now propose to get a lower bound for r_n. Take $\varepsilon > 0$. Then, using (6.t) with n replaced by $[(1+\varepsilon)n]$,

$$\left(\sum_{i=0}^{[\varepsilon n]} u_i\right) r_n \geq 1 - \sum_{i=[\varepsilon n]+1}^{[(1+\varepsilon)n]} u_i r_{[(1+\varepsilon)n]-i} = 1 - \sum_{i=0}^{n-1} u_{[(1+\varepsilon)n]-i} r_i,$$

hence

$$r_n \geq \frac{1}{h([\varepsilon n])} - \frac{1}{h([\varepsilon n])}\sum_{i=0}^{n-1} \frac{1}{h(i)} u_{[(1+\varepsilon)n]-i}. \tag{6.u}$$

Notice that

$$\begin{aligned}h(n) - h([\varepsilon n]) &= \sum_{i=[\varepsilon n]+1}^{n} u_i \sim p_1(0) \sum_{i=[\varepsilon n]+1}^{n} \frac{1}{is(n)^d}\\ &\sim p_1(0) \log\frac{1}{\varepsilon} s(n)^{-d}.\end{aligned} \tag{6.v}$$

It is easily checked that

$$\sum_{i=0}^{n-1} \frac{1}{h(i)} u_{[(1+\varepsilon)n]-i} \underset{n\to\infty}{\sim} \frac{1}{h(n)} \sum_{i=0}^{n-1} u_{[(1+\varepsilon)n]-i}. \tag{6.w}$$

Use the fact that $1/h$ is slowly varying and observe that for $\delta > 0$ small,

$$\sum_{i=0}^{\delta n} \frac{1}{h(i)} u_{[(1+\varepsilon)n]-i} \le Cb(n)^{-d} \sum_{i=0}^{\delta n} \frac{1}{h(i)} \le C'b(n)^{-d} \frac{\delta n}{h(\delta n)},$$

where the constant C' does not depend on δ. It follows from (6.u), (6.v) and (6.w) that

$$\begin{aligned}r_n &\ge \frac{1}{h([\varepsilon n])} - \frac{1}{h([\varepsilon n])h(n)}(h([(1+\varepsilon)n]) - h([\varepsilon n]))\\ &\quad + o\left(\frac{1}{h(n)^2 s(n)^d}\right).\end{aligned} \tag{6.x}$$

Notice that (6.v) implies

$$\frac{1}{h([\varepsilon n])} - \frac{1}{h(n)} = p_1(0) \log\frac{1}{\varepsilon} \frac{1}{h(n)^2 s(n)^d} + o\left(\frac{1}{h(n)^2 s(n)^d}\right).$$

It then follows from (6.x) and (6.v) that

$$\begin{aligned}r_n &\ge \frac{1}{h(n)} + p_1(0)\frac{1}{h(n)^2 s(n)^d}\left(\log\frac{1}{\varepsilon} - \log\frac{1+\varepsilon}{\varepsilon}\right) + o\left(\frac{1}{h(n)^2 s(n)^d}\right)\\ &= \frac{1}{h(n)} - p_1(0)\frac{1}{h(n)^2 s(n)^d}\log(1+\varepsilon) + o\left(\frac{1}{h(n)^2 s(n)^d}\right).\end{aligned}$$

Since ε was arbitrary we conclude that

$$r_n = \frac{1}{h(n)} + o\left(\frac{1}{h(n)^2 s(n)^d}\right).$$

Notice that, by Lemma 2.2, $h(n)s(n)^d \to \infty$. Then

$$E[R_n] = \sum_{i=0}^{n} r_i = \sum_{i=0}^{n} \frac{1}{h(i)} + o\left(\frac{n}{h(n)^2 s(n)^d}\right)$$

[note that, if f is slowly varying, $\sum_{i=0}^{n} f(i) = O(nf(n))$]. On the other hand,

$$\sum_{i=0}^{n} \frac{1}{h(i)} = \frac{n+1}{h(n)} + \sum_{i=0}^{n} \frac{h(n)-h(i)}{h(n)h(i)}$$

and

$$\sum_{i=0}^{n} \frac{h(n)-h(i)}{h(i)} = \sum_{j=1}^{n} u_j \sum_{i=0}^{j-1} \frac{1}{h(i)} \sim \sum_{j=1}^{n} \frac{ju_j}{h(j)} \sim p_1(0) \frac{n}{h(n)s(n)^d}.$$

We finally obtain

$$E[R_n] = \frac{n}{h(n)} + p_1(0)\frac{n}{h(n)^2 s(n)^d} + o\left(\frac{n}{h(n)^2 s(n)^d}\right).$$

We also have, by Lemma 6.7,

$$\operatorname{var} R_n \le C \frac{n^2}{h(n)^4 s(n)^{2d}}.$$

It follows that

$$E\left[\left(\frac{h(n)}{n}R_n - 1\right)^2\right] \le C\frac{1}{h(n)^2 s(n)^{2d}}. \tag{6.y}$$

This gives the convergence in L^2-norm of $(h(n)/n)R_n$.

We now turn to the proof of the almost sure convergence, when $s(n) \ge 1$. It suffices to prove that for any $r > 1$, if $n_j = [r^j]$,

$$\sum_j E\left[\left(\frac{h(n_j)}{n_j}R_{n_j} - 1\right)^2\right] < \infty. \tag{6.z}$$

Indeed, if (6.z) holds, we have

$$\lim_{n\to\infty} \frac{h(n_j)}{n_j} R_{n_j} = 1 \quad \text{a.s.},$$

hence, by the monotonicity of (R_n),

$$r^{-1} \le \liminf \frac{h(n)}{n} R_n \le \limsup \frac{h(n)}{n} R_n \le r \quad \text{a.s.}$$

Let us prove (6.z). Set $m_0 = 0$ and for every $k \ge 1$,

$$m_k = \inf\{p; h(p) \ge k\}.$$

Notice that $k \le h(m_k) \le k+1$. Then

$$\sum_j \frac{1}{h(n_j)^2 s(n_j)^{2d}} = \sum_{k=1}^{\infty} \sum_{\{j;\, m_k \le n_j < m_{k+1}\}} \frac{1}{h(n_j)^2 s(n_j)^{2d}}$$
$$\le \sum_{k=1}^{\infty} \frac{1}{k^2} \sum_{\{j;\, m_k \le n_j < m_{k+1}\}} \frac{1}{s(n_j)^{2d}}.$$

By (6.v), we have, for j large,

$$s(n_j)^{-2d} \le s(n_j)^{-d} \le C\big(h(n_{j+1}) - h(n_j)\big).$$

It follows that

$$\sum_{\{j;\, m_k \le n_j < m_{k+1}\}} \frac{1}{s(n_j)^{2d}} \le C(h(m_{k+1}) - h(m_k)) + 1 \le C'.$$

We conclude that

$$\sum_j \frac{1}{h(n_j)^2 s(n_j)^{2d}} < \infty,$$

and together with (6.y), this implies (6.z).

The last assertion of Theorem 6.9 is obtained by combining our expansion for $E[R_n]$ with Theorem 6.8, except for the fact that (γ_n) is bounded in L^p. This has already been established for $p = 2$ in Lemma 6.7. The general case can be handled using arguments similar to those of the proof of (4.f). □

REMARK. If $d = 2$, we can always assume that $s(n) \ge 1$ (see Feller [4], Theorem 3, page 580) and we recover a special case of Jain and Pruitt's result.

In the transient case, we know from (1.a) that $n^{-1}R_n$ converges almost surely to q. We have the following analogue of Theorem 6.9. The proof uses similar arguments and will be left to the reader.

THEOREM 6.10. (i) *Under Assumptions* A1 *and* A2 *and if* $2d/3 < \beta < d$,

$$E[R_n] = qn + q^2 p_1(0)\left(\frac{1}{2 - d/\beta} + \frac{1}{d/\beta - 1}\right)\frac{n^2}{b(n)^d} + o\left(\frac{n^2}{b(n)^d}\right).$$

If we also satisfy Assumption A3,

$$R_n = qn + q^2\left(p_1(0)\left(\frac{1}{2 - d/\beta} + \frac{1}{d/\beta - 1}\right) - \gamma_n\right)\frac{n^2}{b(n)^d},$$

where the sequence γ_n *is bounded in* L^p *for any* $p < \infty$ *and converges in distribution toward* γ_U.

(ii) *Under Assumptions* A1 *and* A2, *if* $\beta = d$ *and if* X *is transient*,

$$E[R_n] = qn + q^2 p_1(0) n \sum_{k=n}^{\infty} \frac{1}{ks(k)^d} + o\left(n \sum_{k=n}^{\infty} \frac{1}{ks(k)^d}\right).$$

The same expansion is valid for R_n, *the convergence of the error term being in the* L^2*-norm.*

6.7. Our asymptotic study of the range R_n required information about the number of intersection points of two independent random walks. Similarly, we could have considered the number of intersection points of k independent random walks. For simplicity, we restrict our attention to the case of planar random walks with zero mean and finite second moments ($\beta = d = 2, s(n) \equiv 1$). Let $k \geq 2$ and let $X^1, \dots, X^k$ be k independent such random walks. Then Assumption A1 is satisfied for each i and the limiting process U^i is, up to some linear transformation, a Brownian motion in $\mathbb{R}^2$. We also assume that Assumption A2 is satisfied for each i. According to Proposition 2.4, we have

$$h^i(n) \sim_{n \to \infty} \left(2\pi\sigma_i^2\right)^{-1} \log n,$$

where σ_i^2 is the square root of the determinant of the covariance matrix of X_1^i. Set

$$I_n = |X^1(0,n) \cap \cdots \cap X^k(0,n)|,$$

$$J_n = \sum_{i_1=0}^{n} \cdots \sum_{i_k=0}^{n} I\left(X_{i_1}^1 = X_{i_2}^2 = \cdots = X_{i_k}^k\right).$$

PROPOSITION 6.11. *We have*

$$n^{-1}\left(J_n - (2\pi)^{-k}\sigma_1^{-2} \cdots \sigma_k^{-2}(\log n)^k I_n\right) \xrightarrow[n \to \infty]{(P)} 0.$$

The proof of Proposition 6.11 is very similar to that of Proposition 6.3 and will be left to the reader (see also the remarks after Lemma 3.1). Combining Proposition 6.11 with the Theorem 1 of [18] gives Corollary 6.12.

COROLLARY 6.12. *We have*

$$n^{-1}(\log n)^k I_n \xrightarrow[n \to \infty]{(d)} (2\pi)^k \sigma_1^2 \cdots \sigma_k^2 \alpha_{U^1,\dots,U^k}\left([0;1]^k\right),$$

where

$$\alpha_{U^1,\dots,U^k}\left([0;1]^k\right) = \int_{[0,1]^k} dt_1 \cdots dt_k \, \delta_{(0)}\left(U_{t_1}^1 - U_{t_2}^2\right) \cdots \delta_{(0)}\left(U_{t_{k-1}}^{k-1} - U_{t_k}^k\right)$$

is the intersection local time on $[0;1]^k$ *of the* k *independent processes* $U^1, \dots, U^k$.

REMARK. The result of Corollary 6.12 was obtained in [9] but only for $k = 2$ or 3. In these two cases one can show that the law of the limiting variable is characterized by its moments (for $k = 2$ this follows from the remarks after Lemma 3.1). Thus, in order to prove the convergence in distribution of the corollary, it is enough to prove the convergence of all moments of $n^{-1}(\log n)^k I_n$ toward the corresponding moments of the limiting variable (see [9]).

7. The case $\beta > d$. We now consider the case $\beta > d$, which can only occur if $d = 1$. We suppose that X is a random walk in $\mathbb{Z}$ which satisfies Assumptions A1 and A2 for some $\beta > 1$. We propose to show that the strong law of large numbers does not hold for R_n in this case. Instead we will prove the convergence in distribution of $b(n)^{-1}R_n$ toward a nondegenerate law. This result has already been obtained by Jain and Pruitt [7] in the case of random walks with zero mean and finite second moments. The main ingredient of our proof is part (iii) of Corollary 5.3.

Let $U(0;1) = \{U_s, 0 \le s \le 1\}$ be the set of points that are visited by U before time 1 and let m denote the Lebesgue measure on $\mathbb{R}$. Then $m(U(0;1))$ is a nontrivial random variable such that

$$E[m(U(0;1))] < \infty.$$

This bound can for instance be derived from the fact that $(U_s; 0 \le s \le 1)$ is a martingale bounded in L^p for any $p < \beta$.

THEOREM 7.1. *Under Assumptions* A1 *and* A2 *and if* $\beta > d = 1$,

$$b(n)^{-1}R_n \xrightarrow[n\to\infty]{(d)} m(U(0;1)).$$

PROOF. The first step is to study the asymptotic behaviour of $E[R_n]$. Let $K > 0$ and set

$$R_n^K = |X(0;n) \cap [-Kb(n); Kb(n)]|.$$

Then

$$E[R_n^K] = \sum_{x \in [-Kb(n);\, Kb(n)]} P[T_x \le n]$$

$$= b(n) \int_{\{u \in \mathbb{R};\, |[b(n)u]| \le Kb(n)\}} du\, P\big[T_{[b(n)u]} \le n\big].$$

Corollary 5.3(iii) implies that for $u \ne 0$,

$$\lim_{n\to\infty} P\big[T_{[b(n)u]} \le n\big] = P\Big[\sigma_{u/|u|} \le |u|^{-\beta}\Big] = P[\sigma_u \le 1],$$

where $\sigma_u = \inf\{t; U_t = u\}$. It follows that

$$\lim_{n\to\infty} b(n)^{-1}E[R_n^K] = E[m(U(0;1) \cap [-K;K])].$$

Letting K tend to infinity we obtain

$$\lim_{n\to\infty} b(n)^{-1}E[R_n] = E[m(U(0;1))]. \tag{7.a}$$

Here we use the following two arguments: First, the probability $P[X(0,n) \subset [-Kb(n); Kb(n)]]$ is close to 1 when K is large, uniformly in n; second, it follows from (3.b) and (2.j) that the sequence $b(n)^{-2}E[(R_n)^2]$ is bounded.

Set

$$X_t^{(n)} = b(n)^{-1}X_{[nt]},$$

so that $X^{(n)}$ converges in distribution towards U, in the sense of Skorokhod's J1-topology. By Skorokhod's representation theorem, we may find a sequence

$\tilde{X}^{(n)}$ such that for each n, $\tilde{X}^{(n)}$ and $X^{(n)}$ are identically distributed and moreover

$$\tilde{X}^{(n)} \xrightarrow[n\to\infty]{} U \quad \text{a.s.} \tag{7.b}$$

in the sense of Skorokhod's topology.

Let $\overline{U(0,1)}$ denote the closure of $U(0,1)$. Since $\overline{U(0,1)} - U(0,1)$ is countable, we have

$$m\left(\overline{U(0,1)}\right) = m(U(0,1)) \quad \text{a.s.}$$

For any $\varepsilon > 0$, let W_ε denote the ε neighbourhood of $\overline{U(0,1)}$. Then

$$m(W_\varepsilon) \underset{\varepsilon\to 0}{\downarrow} m(U(0,1)). \tag{7.c}$$

Moreover, (7.b) implies that for n large (depending on ω), for every $t \le 1$,

$$\tilde{X}_t^{(n)} \in W_\varepsilon.$$

Set $\tilde{X}^{(n)}(0,n) = \{\tilde{X}_i^{(n)}; 0 \le i \le n\}$ and $\tilde{R}_n = |\tilde{X}^{(n)}(0,n)|$, so that R_n and $\tilde{R}_n$ are identically distributed. Taking n larger if necessary, so that $b(n)^{-1} < \varepsilon$, we obtain

$$\begin{aligned} b(n)^{-1}\tilde{R}_n &= \int du\, I\left(\frac{[b(n)u]}{b(n)} \in \tilde{X}^{(n)}(0,n)\right) \\ &\le \int du\, I(u \in W_{2\varepsilon}) = m(W_{2\varepsilon}). \end{aligned}$$

Using (7.c) it follows that, a.s.,

$$\limsup_{n\to\infty} b(n)^{-1}\tilde{R}_n \le m(U(0,1)). \tag{7.d}$$

In particular, we have

$$\lim_{n\to\infty} \left(b(n)^{-1}\tilde{R}_n - m(U(0;1))\right)_+ = 0,$$

hence, since the sequence $b(n)^{-1}\tilde{R}_n$ is uniformly integrable (we have already noticed that it is bounded in L^2),

$$\lim_{n\to\infty} E\left[\left(b(n)^{-1}\tilde{R}_n - m(U(0;1))\right)_+\right] = 0.$$

By (7.a) we also have

$$\lim_{n\to\infty} E\left[b(n)^{-1}\tilde{R}_n - m(U(0;1))\right] = 0.$$

We conclude that

$$\lim_{n\to\infty} E\left[\left|b(n)^{-1}\tilde{R}_n - m(U(0;1))\right|\right] = 0. \qquad \square$$

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