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MULTIPLE POINTS OF LÉVY PROCESSES

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We prove a conjecture of Hendricks and Taylor that a Lévy process in $\mathbb{R}^d$ with 1-potential kernel $u(x)$ will have k-multiple points if

$$\int_{|x| \le 1} (u(x))^k \, dx < \infty$$

and $u(0) > 0$.

1. Introduction. In this paper we provide a simple condition which will insure that a Lévy process has k-multiple points, a condition essentially conjectured by Hendricks and Taylor (1975). Let X_t be a Lévy process in $\mathbb{R}^d$. A point x is a k-multiple for X if $x = X_{t_1} = X_{t_2} = \cdots = X_{t_k}$ for distinct $t_1, \dots, t_k$. It is assumed that X_t has a density. We state our theorem in terms of the density $p_t(x)$ of X_t.

THEOREM 1. *If, for some* $\varepsilon, T > 0$,

$$\text{(A)} \qquad \int_{|x| \le \varepsilon} \left(\int_0^T p_t(x)\, dt \right)^k dx < \infty$$

and

$$\text{(B)} \qquad \int_0^T p_t(0)\, dt > 0,$$

then X has k-multiple points, a.s.

This implies that the following conditions, in terms of the potential kernel

$$u(x) = \int_0^\infty e^{-t} p_t(x)\, dt,$$

guarantee that X has k-multiple points a.s.:

$$\text{(A}'\text{)} \qquad \int_{|x| \le \varepsilon} (u(x))^k \, dx < \infty$$

and

$$\text{(B}'\text{)} \qquad u(0) > 0.$$

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Hendricks and Taylor (1975) have conjectured that (A′) together with the obvious requirement that no projection of X be a subordinator are necessary and sufficient conditions for the existence of k-multiple points. (B′) is our substitute for this latter requirement.

We note that for a symmetric process, i.e., $p_t(x) = p_t(-x)$, condition (B) is automatically satisfied since

$$p_t(0) = \int \left(p_{t/2}(x) \right)^2 dx > 0.$$

Work on multiple points of Lévy processes has a long history, beginning with the study of Brownian motion by Dvoretzky, Erdös and Kakutani (1950, 1954) and Dvoretzky, Erdös, Kakutani and Taylor (1957). Stable processes were investigated by Taylor (1966, 1967) and stable components by Hendricks (1979). More general work includes Hendricks and Taylor (1975), Orey (1967), Hawkes (1978), Takeuchi (1964), Evans (1987a) and Dynkin (1981). In the latter two works we find proofs that (A) is sufficient in the symmetric case. Evans (1987b) improves on our result by removing the assumption that X_t has a density.

Our method of proof is closely related to the local time techniques which were used in particular by Geman, Horowitz and Rosen (1984) in order to study intersections of independent Brownian paths. The basic idea is to construct a nontrivial measure supported by the k-tuples $(t_1, \dots, t_k)$ of distinct times such that $X_{t_1} = \cdots = X_{t_k}$. Of course the existence of such a measure implies the nonemptiness of the set of k-multiple points. Some of our arguments are also inspired by Hawkes' work (1978). However, in contrast with the latter paper and Evans (1987a, b) we do not use any potential-theoretic tools.

When X has k-multiple points we would like to know how many. Several of the above references investigate the Hausdorff dimension of the set of k-multiple points. We let

$$E_k = \left\{ (t_1, \dots, t_k) | X_{t_1} = \cdots = X_{t_k} \right\},$$

be the set of k-multiple times.

Recall the definition of the lower index β'' of Blumenthal and Getoor (1961):

$$\beta'' = \sup\{ \alpha \geq 0;\ |y|^{-\alpha} \operatorname{re} \psi(y) \to \infty \text{ as } |y| \to \infty \},$$

where

$$E(e^{iy \cdot X_t}) = e^{-t\psi(y)}.$$

We prove

THEOREM 2. *If* $u(0) > 0$ *and* $k - (k-1)d/\beta'' > 0$, *then* $\dim E_k \geq k - (k-1)d/\beta''$.

REMARK. The existence of k-multiple points under the conditions of Theorem 2 was first conjectured by Orey (1967). Note that the Fourier transform of u is in $L^{k/(k-1)}$; hence u itself is in L^k, so the existence of k-multiple points follows from Theorem 1.

See also Le Gall (1987) for further results on the Hausdorff measure of multiple points for Lévy processes.

The paper is organized as follows. In Section 2 we establish some preliminary results concerning intersections of independent Lévy processes. These results play a key role in the proof of Theorem 1 which is developed in Section 3. Theorem 2 is established in Section 4.

2. Intersections of independent Lévy processes. Throughout this work we shall assume that the semigroup $(p_t, t \geq 0)$ associated with X is strong Feller. According to Hawkes (1979), this assumption is equivalent to the absolute continuity of the transition kernels with respect to Lebesgue measure. Moreover, the strong Feller assumption implies the existence of a (unique) family $\{p_t, t > 0\}$ of probability densities of $\mathbb{R}^d$ such that:

1. $(t, x) \to p_t(x)$ is jointly measurable.
2. For any $t > 0$, $x \to p_t(x)$ is lower semicontinuous.
3. $p_t * p_s = p_{t+s}$ everywhere.
4. $P_t f(x) = \int f(y) p_t(y - x)\, dy$, for any measurable $f\colon \mathbb{R}^d \to R_+$.

Consider k independent Lévy processes $X^1, \dots, X^k$, with the same semigroup as X. We assume that X^i starts from x^i. We prove that, under the assumptions (A) and (B) and if $x^1, \dots, x^k$ are close enough to each other, then the paths of $X^1, \dots, X^k$ have a common point with positive probability. For any $s \leq t$ we denote by $X^i(s, t)$ the path of X^i on the time interval $[s, t]$,

$$X^i(s,t) = \{X_u^i;\ s \leq u \leq t\}.$$

THEOREM 3. *Suppose that assumptions* (A) *and* (B) *hold for some* $\varepsilon, T > 0$. *Then there exist* $s > 0$ *and a neighborhood* U *of* 0 *in* $\mathbb{R}^d$ *such that* $p_s(0) > 0$ *and, whenever* $x^1, \dots, x^k$ *belong to* U, *for any* $r > 0$,

$$(1)\qquad P\left[X^1(0,s) \cap \cdots \cap X^k(0,s) \neq \phi;\ X_s^i \in B(0,r),\ i = 1, \dots, k\right] > 0.$$

Here $B(0, r)$ *denotes the ball of radius* r *centered at* 0.

PROOF. By assumption (B), we have

$$\int_0^\infty ds \int_0^{s\wedge T} p_t(0) p_{s-t}(0)\, dt = \int_0^T p_t(0)\, dt \int_0^\infty p_s(0)\, ds > 0,$$

where $s \wedge T$ denotes $\inf(s, T)$. Thus we may choose $s > 0$ such that

$$\int_0^{s\wedge T} p_t(0) p_{s-t}(0)\, dt > 0.$$

By the lower semicontinuity of $x \to p_t(x)$, for any t with $p_t(0)p_{s-t}(0) > 0$, we can find a neighborhood U' of 0 such that

$$p_t(x) p_{s-t}(-x) > 0, \quad \text{for } x \in U'.$$

Hence,

$$p_s(0) = \int dx\, p_t(x) p_{s-t}(-x) > 0.$$

Using Fatou's lemma and again the lower semicontinuity of $x \to p_t(x)$, we can choose a neighborhood V of 0 such that, for any $x, y \in V$,

$$\int_0^{s \wedge T} p_t(x) p_{s-t}(y)\, dt > 0.$$

We may assume that V is contained in the ball $B(0, \varepsilon/2d)$. We take U so small that, for any $x, y \in U$, $x - y \in V$.

For any integer $n \geq 1$, and $p = (p_1, \dots, p_d) \in \mathbb{Z}^d$ such that $-\varepsilon 2^n/2d \leq p_i \leq \varepsilon 2^n/2d - 1$, let $C_p^{(n)}$ denote the cube

$$C_p^{(n)} = \{x = (x_1, \dots, x_d) \in \mathbb{R}^d,\ p_i 2^{-n} \leq x_i \leq (p_i + 1)2^{-n}\}.$$

Set

$$I = [0, s \wedge T]^k.$$

We fix $r > 0$ and we consider the random measure α_n on I defined by

$$\alpha_n(B) = \left(\prod_{i=1}^k 1_{B(0,r)}(X_s^i) \right) c_n \int_B ds_1 \cdots ds_k \left(\sum_p \left(\prod_{i=1}^k 1_{C_p^{(n)}}(X_{s_i}^i) \right) \right). \tag{2}$$

Here B is any Borel subset of I and the constant c_n is defined by

$$c_n = \left(\operatorname{vol} C_p^{(n)}\right)^{-(k-1)} = 2^{dn(k-1)}.$$

When it is necessary to emphasize the dependence of measures such as $\alpha_n(B)$ on ω we write $\alpha_n(\omega, B)$.

Lemma 4. *There exist a positive constant C and a monotonically decreasing continuous function ϕ: $(0, \infty) \to (1, \infty)$ with $\phi(0+) = \infty$, such that for any n large enough,*

$$E[\alpha_n(I)] \geq C^{-1}, \tag{3}$$

$$E\left[\iint_{I \times I} \alpha_n(ds_1 \cdots ds_k) \alpha_n(dt_1 \cdots dt_k) \prod_{i=1}^k \phi(|t_i - s_i|) \right] \leq C. \tag{4}$$

We first assume the lemma and complete the proof of Theorem 3. Note that, since $\phi \geq 1$, condition (4) implies that, for n large,

$$E\left[\alpha_n(I)^2\right] \leq C. \tag{5}$$

We use the following elementary inequality: For any $0 < \lambda < 1$,

$$P[\alpha_n(I) \geq \lambda E[\alpha_n(I)]] \geq (1-\lambda)^2 (E[\alpha_n(I)])^2 / E\left[\alpha_n(I)^2\right]. \tag{6}$$

It follows from (3) and (5) that we may find $\gamma, \delta > 0$ such that for n large

$$P\left[\gamma^{-1} \leq \alpha_n(I) \leq \gamma\right] \geq \delta.$$

For any n, set

$$A_n = \left\{ \gamma^{-1} \le \alpha_n(I) \le \gamma, \iint_{I\times I} \alpha_n(d\mathbf{s})\alpha_n(d\mathbf{t}) \prod_{i=1}^{k} \phi(|t_i - s_i|) \le \gamma \right\},$$

where we use the notation $\mathbf{s} = (s_1, \dots, s_n)$ and $\mathbf{t} = (t_1, \dots, t_n)$. Changing γ and δ if necessary and using (4) it follows that for n large,

$$P[A_n] \ge \delta > 0.$$

Set $A_\infty = \limsup A_n$, so that $P[A_\infty] \ge \delta$. Since I is a compact set, for any $\omega \in A_\infty$, we may find a subsequence $(\alpha_n(\omega, \cdot))$ which converges weakly to a measure $\alpha(\omega, \cdot)$ such that

$$\gamma^{-1} \le \alpha(I) \le \gamma$$

and

$$\iint_{I\times I} \alpha(d\mathbf{s})\alpha(d\mathbf{t}) \prod_{i=1}^{k} \phi(|t_i - s_i|) \le \gamma. \tag{7}$$

In order to deduce (7) we simply apply the definition of weak convergence to the bounded continuous function

$$\prod_{i=1}^{k} (\phi(|t_i - s_i|) \wedge N),$$

where $a \wedge b = \inf(a, b)$ and let N go to infinity. Observe that $A_\infty \subset \{X_s^i \in B(0, r),\ i = 1, \dots, k\}$.

The next step of the proof is to show that for $\omega \in A_\infty$, $\alpha(\omega, \cdot)$ is supported by

$$J^* = \left\{ (s_1, \dots, s_k) \in I;\ X^i(s_i \pm) = X^{i-1}(s_{i-1} \pm), i = 2, \dots, k \right\}.$$

The notation $X^i(s_i \pm) = X^{i-1}(s_{i-1} \pm)$ means that either $X^i(s_i) = X^{i-1}(s_{i-1})$, $X^i(s_i -) = X^{i-1}(s_{i-1})$, $X^i(s_i) = X^{i-1}(s_{i-1} -)$ or $X^i(s_i -) = X^{i-1}(s_{i-1} -)$. For any $\varepsilon > 0$, define

$$\begin{aligned} B_\varepsilon = \Big\{ (s_1, \dots, s_k) \in I; &\text{ for some } i \in \{2, \dots, k\}, \\ &\inf\big(|X^i(s_i) - X^{i-1}(s_{i-1})|, |X^i(s_i -) - X^{i-1}(s_{i-1})|, \\ &\quad |X^i(s_i) - X^{i-1}(s_{i-1} -)|, |X^i(s_i -) - X^{i-1}(s_{i-1} -)|\big) > \varepsilon \Big\}. \end{aligned}$$

Note that B_ε is open and that

$$J^* = I - \bigcup_{\varepsilon > 0} B_\varepsilon.$$

The definition of α_n implies that, for n large, $\alpha_n(B_\varepsilon) = 0$, so that, for $\omega \in A_\infty$,

$$\alpha(B_\varepsilon) \le \liminf \alpha_n(B_\varepsilon) = 0.$$

We conclude that $\alpha(I - J^*) = 0$.

We will now prove that α is actually supported on

$$J = \left\{ (s_1, \dots, s_k);\ X^i(s_i) = X^{i-1}(s_{i-1}), i = 2, \dots, k \right\}.$$

It is clear enough to prove that, for $\omega \in A_\infty$,

$$\alpha\Big(\big\{(s_1,\dots,s_k);\ X^1(s_1) \neq X^1(s_1-)\big\}\Big) = 0.$$

Since X^1 has only a countable number of discontinuities, it suffices to verify that, almost surely for any $a \in [0;T]$,

$$\alpha\big(\{(s_1,\dots,s_k);\ s_1 = a\}\big) = 0.$$

But the latter statement follows at once from condition (7) and the fact that $\phi(0+) = \infty$.

In order to complete the proof we simply note that, for $\omega \in A_\infty$, $\alpha(\omega,\cdot)$ is a nontrivial measure supported by J and thus

$$P\big(J \neq \varnothing,\ X_s^i \in B(0,r),\ i = 1,\dots,k\big) \geq P[A_\infty] > 0.$$

PROOF OF LEMMA 4. We first prove (3):

$$\begin{aligned}
E[\alpha_n(I)] &= c_n \int_I ds_1 \cdots ds_k \sum_p \prod_{i=1}^k \left(\int_{C_p^{(n)} \times B(0,r)} dy\, dz\, p_{s_i}(y - x_i) p_{s-s_i}(z - y) \right) \\
&= c_n \sum_p \left(\prod_{i=1}^k \int_0^{s\wedge T} dt \int_{C_p^{(n)} \times B(0,r)} dy\, dz\, p_t(y - x_i) p_{s-t}(z - y) \right) \\
&\geq \int_{UC_p^{(n)}} dy \prod_{i=1}^k \inf\left\{ \int_{B(0,r)} dz \int_0^{s\wedge T} dt\, p_t(y' - x_i) p_{s-t}(z - y');\right. \\
&\qquad\qquad \left. |y - y'| \leq d2^{-n} \right\}.
\end{aligned}$$

Observe that $U \subset \bigcup_p C_p^{(n)}$ for n large. By Fatou's lemma and the lower semicontinuity of $x \to p_s(x)$ we obtain

$$\liminf_{n\to\infty} E[\alpha_n(I)] \geq \int_U dy \prod_{i=1}^k \left(\int_{B(0,r)} dz \int_0^{s\wedge T} dt\, p_t(y - x_i) p_{s-t}(z - y) \right) > 0,$$

by our choice of U.

We now consider (4). Our first task is to choose a suitable function ϕ. We first note that for any positive decreasing function ϕ, writing $\overline{T} = s \wedge T$, we have

$$\begin{aligned}
&\int_{|x|\leq\varepsilon} \left(\int_0^{\overline{T}} \phi(t) p_t(x)\, dt \right)^k dx \\
&\quad = \int_I dt \left(\prod_{i=1}^k \phi(t_i) \right) \int_{|x|\leq\varepsilon} dx \prod_{i=1}^k p_{t_i}(x) \\
&\quad \leq \int_I dt\, \phi(t^*)^k \int_{|x|\leq\varepsilon} dx \prod_{i=1}^k p_{t_i}(x),
\end{aligned}$$

where $t^* = \inf\{t_i;\ i = 1, \dots, k\}$. Assumption (A) implies that

$$\sum_{m=1}^{\infty} \int_I d\mathbf{t}\, 1_{(2^{-m} < t^* \le 2^{-m+1})} \int_{|x| \le \varepsilon} dx \prod_{i=1}^{k} p_{t_i}(x) < \infty.$$

Thus we may find an increasing sequence of positive numbers $(a_m;\ m = 1, 2, \dots)$ such that $\lim a_m = +\infty$ and

$$\sum_{m=1}^{\infty} a_m \int_I d\mathbf{t}\, 1_{(2^{-m} < t^* \le 2^{-m+1})} \int_{|x| \le \varepsilon} dx \prod_{i=1}^{k} p_{t_i}(x) < \infty.$$

We take ϕ such that $\phi(2^{-m}) = a_m^{1/k}$ for m large. We then extend ϕ to be a decreasing continuous function ϕ: $(0, \infty) \to [1, \infty]$, with $\phi(0+) = \infty$. By the above

$$\int_{|x| \le \varepsilon} \left(\int_0^T p_t(x) \phi(t)\, dt \right)^k dx < \infty.$$

Then

$$E\left[\iint_{I \times I} \alpha_n(d\mathbf{s}) \alpha_n(d\mathbf{t}) \prod_{i=1}^{k} \phi(|t_i - s_i|) \right]$$
$$\le c_n^2 \sum_{p,q} \iint_{I \times I} d\mathbf{s}\, d\mathbf{t} \prod_{i=1}^{k} \phi(|t_i - s_i|) \int_{(C_p^{(n)})^k \times (C_q^{(n)})^k} d\mathbf{y}\, d\mathbf{z} \prod_{i=1}^{k} P^i_{s_i, t_i}(y_i, z_i),$$

where

$$P^i_{s,t}(y, z) = \begin{cases} p_s(y - x^i) p_{t-s}(z - y), & \text{if } s < t, \\ p_t(z - x^i) p_{s-t}(y - z), & \text{if } t < s. \end{cases}$$

Next we use Fubini's theorem to change the order of integration and for fixed p, q we apply the generalized Hölder inequality with respect to the measure $d\mathbf{y}\, d\mathbf{z}$ on $(C_p^{(n)})^k \times (C_q^{(n)})^k$. It follows that

$$E\left[\iint_{I \times I} \alpha_n(d\mathbf{s}) \alpha_n(d\mathbf{t}) \prod_{i=1}^{k} \phi(|t_i - s_i|) \right]$$
$$\le c_n^2 \sum_{p,q} \prod_{i=1}^{k} \left[\int_{(C_p^{(n)})^k \times (C_q^{(n)})^k} d\mathbf{y}\, d\mathbf{z} \right.$$
$$\left. \times \left(\int_0^T \int_0^T ds_i\, dt_i\, \phi(|t_i - s_i|) P^i_{s_i, t_i}(y_i, z_i) \right)^k \right]^{1/k}$$
$$= \sum_{p,q} \prod_{i=1}^{k} \left\{ \int_{C_p^{(n)} \times C_q^{(n)}} dy\, dz \left(\int_0^T \int_0^T ds\, dt\, \phi(|t - s|) \left[P^i_{s,t}(y, z) \right]^k \right) \right\}^{1/k}.$$

Once again we apply the generalized Hölder inequality in the form

$$\left|\sum_i a_{1,i}a_{2,i}\cdots a_{k,i}\right| \le \prod_{j=1}^{k}\left(\sum_i |a_{j,i}|^k\right)^{1/k}$$

to bound the above by

$$\prod_{i=1}^{k}\left[\sum_{p,q}\int_{C_p^{(n)}\times C_q^{(n)}} dy\,dz\left(\int_0^{\overline{T}}\int_0^{\overline{T}} ds\,dt\,\phi(|t-s|)P^i_{s,t}(y,z)\right)^k\right]^{1/k}.$$

Then, using the fact that $UC_p^{(n)} \subset B(0,\varepsilon/2)$, we have for any $i = 1,\dots,k$,

$$\begin{aligned}
&\sum_{p,q}\int_{C_p^{(n)}\times C_q^{(n)}} dy\,dz\left(\int_0^{\overline{T}}\int_0^{\overline{T}} ds\,dt\,\phi(|t-s|)P^i_{s,t}(y,z)\right)^k\\
&\quad\le 2^{k+1}\int_{B(0,\varepsilon)\times B(0,\varepsilon)} dy\,dz\left(\int_0^{\overline{T}} ds\,p_s(y-x^i)\right)^k\left(\int_0^{\overline{T}} dt\,\phi(t)p_t(z)\right)^k\\
&\quad= c < \infty
\end{aligned}$$

by the choice of ϕ. This completes the proof of the lemma. □

REMARK. It is not hard to extend the result of Theorem 3 to the case of independent Lévy processes with different semigroups. Suppose that $X^1,\dots,X^k$ are independent Lévy processes in $\mathbb{R}^d$ such that, for any $i = 1,\dots,k$, the semigroup $(P_t^i, t > 0)$ associated with X^i is strong Feller. Assume that for $i = 1,\dots,k$ the canonical transition densities (p_t^i) satisfy conditions (A) and (B) of Theorem 1. Then the conclusion of Theorem 3 still holds. In fact the same proof goes through without change. It would even be possible to weaken significantly the above assumptions on the p_t^i's. We shall leave this extension to the reader since we are mainly interested in multiple points for the single process.

3. Proof of Theorem 1. It has been noted for a long time [see, e.g., Hawkes (1978)] that the existence of k-multiple points for a single Lévy process X can often be reduced to the study of the intersection of the ranges of k independent copies of X. However, a simple example will show that this reduction needs some rigorous justification in our general context. Suppose that X is a stable subordinator of index $\alpha > \frac{1}{2}$. Then of course X has no double points, but it is not hard to see that two independent copies of X, starting from arbitrary points, will intersect with probability 1. Note that, under the assumptions of Theorem 1, no projection of X on a line can be a subordinator and thus the above situation cannot occur.

To complete the proof of Theorem 1, we will show that, with the choice of s given by Theorem 3, we have

$$P\left[\left(\bigcap_{i=0}^{k-1} X(2is,(2i+1)s)\right) \ne \varnothing\right] > 0. \tag{8}$$

Note that the stationarity of increments implies, for any $m \geq 1$,

$$P\left[\left(\bigcap_{i=0}^{k-1} X((2mk+2i)s,(2mk+2i+1)s)\right) \neq \varnothing\right]$$
$$= P\left[\left(\bigcap_{i=0}^{k-1} X(2is,(2i+1)s)\right) \neq \varnothing\right].$$

Thus, once we have established (8), Theorem 1 will follow through an application of Borel–Cantelli lemma, using the independence of the increments of X. It reamins to prove (8). Without loss of generality we assume $X_0 = 0$.

Our process $(X_t, 0 \leq t \leq (2k-1)s)$ is defined by a measure P on $D[0,(2k-1)s]$, the set of paths which are right continuous and have left limits. We may assume that P is complete.

Via the map

$$X \to (X^1,\dots,X^k), \quad \text{with } X^i = X^i(X)$$

defined by

$$X_t^i = X_{(2i-2)s+t} - X_{(2i-2)s},$$

P induces a measure P^* on $D^k[0,s]$. Since under P, X has stationary and independent increments, P^* equals the k-fold product measure of $P_{|D[0,s]}$ when restricted to the product σ-algebra. Actually, it is easy to see that P^* is complete since P is.

The set $A' \subset [0,s]^k \times D^k[0,s]$ given by

$$A' = \{(t_1,\dots,t_k,w_1,\dots,w_k)|(w_1(t_1) = \cdots = w_k(t_k))\}$$

is measurable with respect to the product σ-algebra. If π denotes the projection on $D^k[0,s]$,

$$A = \pi(A') = \{(w_1,\dots,w_k)|\exists(t_1,\dots,t_k) \in [0,s]^k \text{ with } w_1(t_1) = \cdots = w_k(t_k)\}$$

will be measurable with respect to P^* because of the above mentioned properties of P^* [Dellacherie and Meyer (1978), pages 43 and 58]. We let E, E^* denote expectations with respect to P, P^*. With the notation

$$h(x_1,\dots,x_k;w_1,\dots,w_k) = 1_A(x_1+w_1,\dots,x_k+w_k),$$

Theorem 3 can be restated as

$$E^*\left(\prod_{i=1}^k 1_{B(0,r)}(x_i+w_i(s))h(x_1,\dots,x_k;w_1,\dots,w_k)\right) > 0$$

for all $x_1,\dots,x_k \in U$ and $r > 0$. Equivalently

$$\int \prod_{i=1}^k 1_{B(0,r)}(x_i+y_i)E^*(h(x_1,\dots,x_k;w_1,\dots,w_k)|w_i(s) = y_i, \forall\, i) \times \prod_{i=1}^k p_s(y_i)\,dy_i > 0. \tag{9}$$

Returning to X we see that

$$h\big(0, X_{2s}, \dots, X_{2(k-1)s}; X^1, \dots, X^k\big)$$

is the characteristic function of the set B which appears in (8),

$$B = \{X \mid \exists t_i \in [2(i-1)s, (2i-1)s] \text{ with } X(t_1) \cdots = X(t_k)\}.$$

If $Z_s^i = X_{2is} - X_{(2i-1)s}$, by independence,

$$\begin{aligned}
P(B) &= E\big(h\big(0, X_{2s}, \dots, X_{2(k-1)s}; X^1, \dots, X^k\big)\big) \\
&= E\left(h\left(0, X_s^1 + Z_s^1, \dots, \sum_{i=1}^{k-1} \big(X_s^i + Z_s^i\big); X^1, \dots, X^k\right)\right) \\
&= \int E\left(h\left(0, X_s^1 + z_1, \dots, \sum_{i=1}^{k-1} \big(X_s^i + z_i\big); X^1, \dots, X^k\right)\right) \prod_{i=1}^{k-1} p_s(z_i)\, dz_i \\
&= \int E^*\left(h\left(0, w_1(s) + z_1, \dots, \sum_{i=1}^{k-1} (w_i(s) + z_i); w_1, \dots, w_k\right)\right) \prod_{i=1}^{k-1} p_s(z_i)\, dz_i \\
&= \iint E^*\left(h\left(0, y_1 + z_1, \dots, \sum_{i=1}^{k-1} (y_i + z_i); w_1, \dots, w_k\right) \Big| w_i(s) = y_i, \forall\, i\right) \\
&\quad \times \prod_{i=1}^{k} p_s(y_i)\, dy_i \prod_{j=1}^{k-1} p_s(z_j)\, dz \\
&= \iint E^*(h(0, x_1, \dots, x_{k-1}; w_1, \dots, w_k) | w_i(s) = y_i, \forall\, i) \\
&\quad \times \prod_{i=1}^{k} p_s(y_i)\, dy_i \prod_{j=1}^{k-1} p_s(x_j - x_{j-1} - y_j)\, dx_j, \qquad x_0 = 0 \\
&\geq \iint \prod_{i=1}^{k} 1_{B(0,r)}(x_{i-1} + y_i) E^*(h(0, x_1, \dots, x_{k-1}; w_1, \dots, w_k) | w_i(s) = y_i, \forall\, i) \\
&\quad \times \prod_{i=1}^{k} p_s(y_i)\, dy_i \prod_{j=1}^{k-1} p_s(x_j - x_{j-1} - y_j)\, dx_j.
\end{aligned}$$

Since $p_s(0) > 0$, if we take the x_j's and r sufficiently small, the factor $1_{B(0,r)}(x_{i-1} + y_i)$ also forces the y_i's to be small, so that the factor

$$\prod_{j=1}^{k-1} p_s(x_j - x_{j-1} - y_j) > 0.$$

By (9), our last integral, and hence $P(B)$ is > 0. This completes the proof of Theorem 1.

4. Proof of Theorem 2. The condition $u(0) > 0$ implies that for some $a > 0$

$$\int_a^{a(1+1/2k)} p_t(0)\,dt > 0.$$

Let $q_\varepsilon(x) = e^{-|x|^2/2\varepsilon}/(2\pi\varepsilon)^{d/2}$, $x \in \mathbb{R}^d$, and let

$$L_\varepsilon(B) = \int_B q_\varepsilon(X_{t_2} - X_{t_1}) \cdots q_\varepsilon(X_{t_k} - X_{t_{k-1}})\,d\mathbf{t}. \tag{10}$$

We will show that if $\beta'' > (k-1)d/k$, then for some $\gamma > 0$,

$$|L_\varepsilon(B) - L_{\varepsilon'}(B')| \le c_\omega |(\varepsilon, B) - (\varepsilon', B')|^\gamma \tag{11}$$

for all rectangles $B, B' \subset D = \prod_{i=1}^k [ai, ai(1 + 1/2k)]$, where if $B = \prod_{i=1}^k [a_i, b_i]$, by (ε, B) we mean $(\varepsilon, a_1, b_1, a_2, b_2, \dots, a_k, b_k)$. (11) will show the existence of $L(B) = \lim_{\varepsilon \to 0} L_\varepsilon(B)$, with

$$|L(B) - L(B')| \le c_\omega |(B) - (B')|^\gamma, \tag{12}$$

so that $L(\cdot)$ defines a measure on D with no hyperplane mass. Let B_δ be defined as in the proof of Lemma 4. We note that $L_\varepsilon(B_\delta) \le \varepsilon^{-N} e^{-\delta^2/2\varepsilon} \to 0$ as $\varepsilon \to 0$, so that $L(B_\delta) = 0$. The argument in the proof of Theorem 3 concerning the support of α now shows that $L(\cdot)$ is supported on $E_k = \{(t_1, \dots, t_k) | X_{t_1} = \cdots = X_{t_k}\}$.

In addition to (11) we will show that for any $\beta < \beta''$,

$$L(B) \le |B|^{1-(k-1)d/\beta k} \tag{13}$$

for all dyadic hypercubes B in D of sufficiently small edge length, where $|B|$ denotes the Lebesgue measure of B.

We postpone for the moment the proof of (11) and (13). The proof will also show that $L_\varepsilon(B)$ converges in L^p for all p.

We have

$$\begin{aligned}
E(L(D)) &= \lim_{\varepsilon \to 0} E(L_\varepsilon(D)) \\
&= \lim_{\varepsilon \to 0} \int_{\mathbb{R}^{dk}} q_\varepsilon(x_2 - x_1) \cdots q_\varepsilon(x_k - x_{k-1}) \\
&\qquad \times \int_D p_{t_1}(x_1) p_{t_2 - t_1}(x_2 - x_1) \cdots p_{t_k - t_{k-1}}(x_k - x_{k-1})\,dt\,d\mathbf{x} \\
&= \int_D p_{t_2 - t_1}(0) p_{t_3 - t_2}(0) \cdots p_{t_k - t_{k-1}}(0)\,dt \\
&\ge \frac{a}{2k} \left(\int_a^{a(1+1/2k)} p_t(0)\,dt \right)^{k-1} > 0
\end{aligned}$$

by our choice of D. Note that $p_t(x)$ is continuous in x and t, $t \ge 0$, since $\beta'' > 0$.

A standard argument [see, e.g., Adler (1981)] then shows that

$$\dim E_k \cap D \ge k - d(k-1)/\beta''$$

on $\{L(D) > 0\}$, which by the last inequality has positive probability. We can replace D with

$$D_m = \prod_{i=0}^k \left[amk + ai, amk + ai\left(1 + \frac{1}{2k}\right) \right]$$

and a Borel–Cantelli argument then gives

$$\dim E_k \geq k - d(k-1)/\beta'' \quad \text{a.s.}$$

The proof of (11) is similar to the proof in Rosen [(1986), Section 2.4], only replace the bound

$$\int_0^1 e^{-tv^2}\, dt \leq \frac{c}{1+v^2}$$

of that paper by

$$\int_0^1 |e^{-t\psi(v)}|\, dt \leq \frac{c}{1+|v|^{\beta}}, \quad \text{for } \beta < \beta'',$$

and verify that [see Rosen (1986), (4.10)]

$$\int \frac{1}{\left(1+|v|^{\beta}\right)^{k/(k-1)}}\, dv < \infty.$$

This is true if $\beta > (k-1)d/k$.

The proof of (13) also follows from the same article, since [Rosen (1986), (4.16)], for $\beta < \beta''$,

$$E\left(L_\varepsilon(B)\right)^m \leq c(m)|B|^{m(1-(k-1)d/\beta k)} \tag{14}$$

for any m, and $B \subset D$, uniformly in $\varepsilon > 0$, hence also for $\varepsilon = 0$.

Let $B_1, B_2, \dots$ be an enumeration of the dyadic cubes in D and choose any $\bar{\beta} < \beta < \beta''$. Then from (14), for $\varepsilon = 0$,

$$\begin{aligned}
&\sum_i P\left(L(B_i) \geq |B_i|^{1-d(k-1)/\bar{\beta}k}\right) \\
&\quad \leq c(m) \sum_i |B_i|^{m(k-1)d/k(1/\bar{\beta}-1/\beta)} \\
&\quad \leq c(m) \sum_{n=1}^{\infty} 2^{nk} 2^{-mnk(k-1)d/k(1/\bar{\beta}-1/\beta)} < \infty
\end{aligned}$$

for m large. The Borel–Cantelli lemma now proves (13), which completes the proof of Theorem 2.

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