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TANAKA'S FORMULA FOR MULTIPLE INTERSECTIONS OF PLANAR BROWNIAN MOTION

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We establish a Tanaka-like formula relating the local times of r and $r+1$ fold self-intersections of a Brownian path in the plane.

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Brownian motion * multiple intersections * local time * Tanaka's formula.

1. Introduction

It is well known that, for any r, planar Brownian motion W_t has r-multiple points, i.e. points $x\in\mathbb{R}^2$ with $x=W_{t_1}=W_{t_2}=\cdots=W_{t_r}$ for distinct $t_1,\ldots,t_r$. (Dvoretzky, Erdös and Kakutani, 1954). As a purely formal measure of such r-fold intersections we study

$$\int\cdots_B\int \delta(W_{t_2}-W_{t_1})\cdots\delta(W_{t_r}-W_{t_{r-1}})\,\mathrm{d}t_1\cdots\mathrm{d}t_r \tag{1.1}$$

In a previous paper (Rosen, 1984) we showed how to interpret (1.1) as the local time of the random field

$$X_r(t_1,\ldots,t_r)=(W_{t_2}-W_{t_1},\ldots,W_{t_r}-W_{t_{r-1}}). \tag{1.2}$$

Recall that $X:R_+^r\to R^{2(r-1)}$ induces a measure $\mu_B(\cdot)$ on $\mathbb{R}^{2(r-1)}$, the occupation measure of X on $B\subseteq R^r$ defined by

$$\mu_B(A)=\lambda_r(X^{-1}(A)\cap B) \tag{1.3}$$

where λ_r denote Lesbesgue measure on $\mathbb{R}^r$. If $\mu_B\ll\lambda_{2(r-1)}$ we say that X has a local time α_r relative to B, defined by

$$\alpha_r(x,B)=\frac{\mathrm{d}\mu_B}{\mathrm{d}\lambda_{2(r-1)}}(x). \tag{1.4}$$

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By definition we have

$$\int_{\mathbb{R}^{2(r-1)}} f(x)\alpha_r(x, B)\,\mathrm{d}\lambda(x)_{2(4-1)}$$
$$= \int_B \cdots \int f(W_{t_2} - W_{t_1}, \cdots, W_{t_r} - W_{t_{r-1}})\,\mathrm{d}t_{t_1} \cdots \mathrm{d}_{t_r} \tag{1.5}$$

for all bounded Borel functions on $\mathbb{R}^{2(r-1)}$. If we formally take f to be the 'δ-function', we find that $\alpha_r(0, B)$ should give (1.1).

In (Rosen, 1984) we showed that if B is a product of disjoint intervals, $B = \times_{i=1}^{r} [a_i, b_i]$, X has a local time relative to B, and if we write $\alpha_r(x, a_1, b_1, \ldots, a_r, b_r) \doteq (x, B)$ then α_r can be taken to be a jointly continuous function of its arguments $(x, a_1, b_1, \ldots, a_r, b_r)$. We will sometimes write $I_j = [a_j, b_j]$ and will always assume that

$$a_1 < b_1 < a_2 < b_2 < \cdots < a_{r+1} < b_{r+1}. \tag{1.6}$$

Properties of a measure analogous to (1.1) for r-fold intersections of independent planar Brownian motions have been established in (German, Horowitz and Rosen, 1984) and applied by Le Gall to study intersections of Wiener Sausages (1985) and derive important information on the Hausdorff measure of r-multiple points.

The goal of this paper is to present an explicit formula for α_{r+1} in terms of α_r analogous to Tanaka's formula for the local time of one dimensional Brownian motion, and to our own formula for double points (Rosen, 1985). Our formula for double points has been analyzed and extended in (Yor, 1985), and applied in (Rosen, 1986) and (Yor, 1985) to study the asymptotics of α_2.

To describe our formula, let

$$q_t(x) = \mathrm{e}^{-t}\,\mathrm{e}^{-|x|^2/2t}/2\pi t \tag{1.7}$$

be the transition density for killed planar Brownian motion, and set

$$K(x) = \int_0^\infty q_t(x)\,\mathrm{d}t. \tag{1.8}$$

It will be seen that $\alpha_r(x, a_1, b_1, \ldots, a_r, s)$ is a continuous increasing function of s, and since K is positive and measurable the integral

$$K_r(x) = \int_{I_r} K(x - W_s)\alpha(0, a_1, b_1, \ldots, a_r, \mathrm{d}s) \tag{1.9}$$

is well defined, although a priori it may be infinite. We will show below that K_r is, in fact, continuously differentiable, and we can state Tanaka's formula for α_{r+1}:

Theorem 1. *With probability one,*

$$-\alpha_{r+1}(0, a_1, \ldots, a_{r+1}, b_{r+1}) = K_r(W_{b_{r+1}}) - K_r(W_{a_{r+1}})$$
$$- \int_{I_{r+1}} \nabla K_r(W_t)\,\mathrm{d}W_t - \int_{I_{r+1}} K_r(W_t)\,\mathrm{d}t. \tag{1.10}$$