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SELF-INTERSECTIONS OF RANDOM FIELDS

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We show how to use local times to analyze the self-intersections of random fields. In particular, we compute the Hausdorff dimension of r-multiple times for Brownian motion in the plane, Brownian sheets and Lévy's multiparameter Brownian motion.

1. Introduction. A now classical theorem of Dvoretzky, Erdös and Kakutani (1954) states that for any r, planar Brownian motion W_t has r-multiple points, that is, points $x \in R^2$ with $x = W_{t_1} = W_{t_2} = \cdots = W_{t_r}$ for distinct $t_1, \cdots, t_r$. In fact, Taylor (1966) has shown that the set of r-multiple points has Hausdorff dimension 2. This is equivalent to the statement that $\{t_1, \cdots, t_r$ distinct $| W_{t_1} = \cdots = W_{t_r}\}$, the set of "$r$-multiple times", has Hausdorff dimension 1. Taylor's approach is to use potential theory, and determine which stable processes "hit" the set of r-multiple points. (See Wolpert, 1978, for an alternate proof.) Our view is that the set of r-multiple times is simply the zero level set L^r of the random field

$$X(t_1, \cdots, t_r) = (W_{t_2} - W_{t_1}, W_{t_3} - W_{t_2}, \cdots, W_{t_r} - W_{t_{r-1}}), \tag{1.1}$$

and the best way to study level sets of a random field is through its local time.

In Section 2 we will show in detail how the theory of local times can be used to prove the following theorem. Let $R^r_{\neq} = \{(t_1, \cdots, t_r) \mid 0 < t_i \text{ distinct}\}$.

THEOREM 1. $\dim(L^r \cap R^r_{\neq}) = 1$ *with probability one.*

The same proof shows that we still have dimension 1 even if we require the t_i to be separated by some fixed constant. Our method actually yields information on the Hausdorff measure function for $L^r \cap R^r_{\neq}$, a more discriminating concept than that of Hausdorff dimension (see Proposition 3).

The real advantage of our approach is its broad applicability. In this paper we will study two additional examples: the N-parameter Wiener process in R^d, $W^{N,d}$, and the index-β process W^{β}.

$W^{N,d}_t$, also known as the Brownian sheet, is the Gaussian random field indexed by $t \in R^N_+$ with d independent components $W^{N,d,i}$, each mean zero with covariance

$$E(W^{N,d,i}_s W^{N,d,i}_t) = \textstyle\prod_{i=1}^N \min(s_i, t_i) \tag{1.2}$$

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for $s = (s_1, \cdots, s_N)$, $t = (t' \cdots, t_N)$. Define $X^{N,d}$ in analogy to X:

$$X^{N,d}(t_1, \cdots, t_r) = (W_{t_2}^{N,d} - W_{t_1}^{N,d}, \cdots, W_{t_r}^{N,d} - W_{t_{r-1}}^{N,d}). \tag{1.3}$$

In Section 3 we will prove the following theorem where $L_{N,d}^r = (X^{N,d})^{-1}(0)$ is the set of r-multiple times for $W_{N,d}$.

THEOREM 2. *If $Nr > d(r-1)/2$, then*

$$\dim(L_{N,d}^r \cap (R_{\neq}^r)^N) = Nr - d(r-1)/2$$

with probability one.

W_t^β, for $0 < \beta < 1$, is the Gaussian random field indexed by $t \in R^N$ with d independent components $W^{\beta,i}$ each mean zero with covariance

$$E(W_s^{\beta,i} W_t^{\beta,i}) = (c/2)\{|s|^{2\beta} + |t|^{2\beta} - |s - t|^{2\beta}\}. \tag{1.4}$$

(When $\beta = 1/2$ this is Lévy's multiparameter Brownian motion). Define X^β as before

$$X^\beta(t_1, \cdots, t_r) = (W_{t_2}^\beta - W_{t_1}^\beta, \cdots, W_{t_r}^\beta - W_{t_{r-1}}^\beta), \tag{1.5}$$

and let $L_\beta^r = (X^\beta)^{-1}(0)$ be the set of r-multiple times for W^β. If we use the notation

$$R_{a,b}^{rN} = \{(t_1, \cdots, t_r) \mid t_i \neq 0, a \leq |t_i - t_j| \leq b, \forall i \neq j\},$$

Kono (1978) has shown that when $2N > \beta d$, then

$$L_\beta^2 \cap R_{a,b}^{2N} \neq 0 \quad \text{for any} \quad 0 < a < b.$$

Independently of this, in Section 4 we establish the following theorem:

THEOREM 3. *If $Nr > \beta d(r-1)$, then*

$$\dim(L_\beta^r \cap R_{a,b}^{rN}) = Nr - \beta d(r-1)$$

with positive probability, for any $0 < a < b$. (Of course, this also gives $\dim(L_\beta^r \cap (R^N)_{\neq}^r)$.)

Let us now describe our approach to self-intersections. Given a random field X: $R^p \to R^q$, whose components are Holder continuous of any order $< \delta$, it is a general result (see Adler 1981, Lemma 8.2.2) that

$$\dim(X^{-1}(y)) \leq p - q\delta \tag{1.6}$$

for almost every y. However, in our specific case we are not interested in knowing about "almost every" y, but about one specific y, $y = 0$. It is here that local time comes in.

We first recall the definition of local time. (For an excellent overview see Geman and Horowitz, 1980.) For any Borel set $B \subseteq R^p$ we define the occupation measure of X by

$$\mu_B(A) = \lambda_p(X^{-1}(A) \cap B) \tag{1.7}$$

for all Borel sets $A \subseteq R^q$, where λ_p denotes p-dimensional Lebesgue mesure. If $\mu_B \ll \lambda_q$ we say that X has a local time on B, and define its local time by

$$\alpha(x, B) \doteq \frac{d_{\mu_B}}{d\lambda_q}(x). \tag{1.8}$$

Of course this only determines $\alpha(x, B)$ a.e. $d\lambda_q(x)$. Intuitively, $\alpha(x, B)$ is the amount of "time" from B spent by X at x.

X is said to have a jointly continuous local time on the hypercube $I = \prod_{i=1}^p [a_i, a_i + h]$ if we can choose

$$\alpha(x, \textstyle\prod_{i=1}^p [a_i, a_i + t_i])$$

to be a continuous function of $(x, t_1, \cdots, t_p)$, $0 \le t_i \le h$. We remark that under these conditions we can choose $\alpha(y, \cdot)$ to be a finite measure supported on $X^{-1}(y)$.

Returning to level sets, it is a theorem of Adler (1978) that if X has a jointly continuous local time, then (1.6) holds for *all* y, in particular we have

$$\dim(X^{-1}(0)) \le p - \delta q. \tag{1.9}$$

Furthermore, in the examples we consider, our proof of joint continuity will easily yield

$$\alpha(0, B) \le c\,|B|^\rho \tag{1.10}$$

for any $\rho < 1 - \delta q/p$ and any sufficiently small hypercube B. The fact that $\alpha(0, \cdot)$ is a finite measure on $X^{-1}(0)$ and satisfies an inequality of the form (1.10) will show that

$$\dim(X^{-1}(0)) \ge p - \delta q, \quad \text{whenever} \quad \alpha(0, \cdot) > 0. \tag{1.11}$$

It is easy to prove that $\alpha(0, \cdot) > 0$ with positive probability, which will result in Theorem 3. To obtain the probability 1 statements of Theorem 1 and 2 we need to use some form of zero-one law. These are available for W and $W^{N,d}$—but apparently not for W^β.

As can be seen, the key step in our approach is to establish the joint-continuity of the local time for the random fields X, $X^{N,d}$ and X^β. The only general method available for proving joint continuity is that of local non-determinism (LND), see Berman (1973), Pitt (1978) and Cuzick (1981). We shall, in fact, make use of the result that W^β is LND in Section 4, but our fields X, $X^{N,d}$ and X^β do not appear to be LND and we must establish joint-continuity directly. Finally, we would like to mention that the approach of this paper is applied in Geman, Horowitz and Rosen (1983) to study the intersections of independent random fields, and in Rosen (1983) to a more detailed analysis of the local time of

$$X(s, t) = W_t - W_s$$

when W_t is two or three dimensional Brownian motion.

We have recently received the paper of Cuzick (1982) who obtains some of our results by very different methods.

2. Planar Brownian motion. In this section we carry out in detail our approach to the r-multiple times of planar Brownian motion. Let $I = \prod_{i=1}^{r} [a_i, a_i + h]$ denote a fixed hypercube in

$$R_<^r = \{(t_1, \cdots, t_r) \mid 0 < t_1 < t_2 < \cdots < t_r\},$$

so that

$$a_i + h < a_{i+1}. \tag{2.1}$$

We first show that X (see (1.1)) has a local time on I.

PROPOSITION 1. *For any Borel set $B \subseteq I$, X has a local time $\alpha(x, B)$ on B, and $\alpha(x, B) \in L^2(R^{2(r-1)}, dx)$ a.s.*

PROOF. Let μ_B be the occupation measure of X (see (1.7)). It will suffice to prove the finiteness of

$$\begin{aligned} &E\left(\int_{R^{2(r-1)}} |\hat{\mu}_B(U)|^2\, dU\right) \\ &\quad = \int_{R^{2(r-1)}} \int_B \int_B E(\exp(i\, U \cdot (X(T) - X(S))))\, dS\, dT\, dU \\ &\quad = \int_B \int_B \int_{R^{2(r-1)}} \exp\left(-\frac{1}{2} V(U \cdot (X(T) - X(S)))\right) dU\, dS\, dT \end{aligned} \tag{2.2}$$

with notation $S = (s_1, \cdots, s_r)$, $T = (t_1, \cdots, t_r)$ and $U = (u_1, \cdots, u_{r-1})$ with each $u_i \in R^2$. We now evaluate

$$\begin{aligned} V(U \cdot (X(T) - X(S))) &= V(\textstyle\sum_{i=1}^{r-1} u_i \cdot ([W_{t_{i+1}} - W_{t_i}] - [W_{s_{i+1}} - W_{s_i}])) \\ &= V(\textstyle\sum_{i=1}^{r-1} u_i \cdot ([W_{t_{i+1}} - W_{s_{i+1}}] - [W_{t_i} - W_{s_i}])) \\ &= V(\textstyle\sum_{i=1}^{r} (u_{i-1} - u_i) \cdot (W_{t_i} - W_{s_i})) \\ &= \textstyle\sum_{i=1}^{r} |u_i - u_{i-1}|^2 |t_i - s_i|, \quad u_0 \doteq u_r \doteq 0, \end{aligned} \tag{2.3}$$

where in the last step we used independence, which follows from (2.1).

For each $p = 1, \cdots, r$, $\{u_i - u_{i-1}, 1 \le i \le r, i \ne p\}$ is a set of coordinates for

$R^{2(r-1)}$. Using (2.3) and the generalized Holder's inequality for (2.2) we see that

$$
\begin{aligned}
&E\left(\int_{R^{2(r-1)}} |\hat{\mu}_B(U)|^2\, dU\right) \\
&= \int_B \int_B \int_{R^{2(r-1)}} \exp\left(-\frac{1}{2} \sum_{i=1}^r |u_i - u_{i-1}|^2 |t_i - s_i|\right) dU\, dS\, dT \\
&= \int_B \int_B \int_{R^{2(r-1)}} \prod_{p=1}^r \\
&\quad \cdot \exp\left(-\frac{1}{2(r-1)} \sum_{i=1, i\neq p}^r |u_i - u_{i-1}|^2 |t_i - s_i|\right) dU\, dS\, dT \\
&\le \int_B \int_B \prod_{p=1}^r \\
&\quad \cdot \left(\int \exp\left(-\frac{r}{2(r-1)} \sum_{i=1, i\neq p}^r |u_i - u_{i-1}|^2 |t_i - s_i|\right) dU\right)^{1/r} dS\, dT \\
&\le c \int_B \int_B \prod_{p=1}^r \prod_{i=1, i\neq p}^r |t_i - s_i|^{-1/r}\, dS\, dT \\
&= c \int_B \int_B \prod_{i=1}^r |t_i - s_i|^{-(r-1)/r}\, dS\, dT < \infty.
\end{aligned}
\tag{2.4}
$$

This proves Proposition 1.

PROPOSITION 2. *With probability one, X has a jointly continuous local time on I.*

PROOF. Fix some $\gamma < 1/(r-1)$. We will prove the finiteness, for all k even of

$$
\begin{aligned}
&\int_{R^{2(r-1)k}} \int_{I^k} \prod_{l=1}^k |U^l|^\gamma \\
&\qquad \exp\left(-\frac{1}{2} V\left(\sum_{j=1}^k X(T^j) \cdot U^j\right)\right) dT^1 \cdots dT^k\, dU^1 \cdots dU^k
\end{aligned}
\tag{2.5}
$$

where $T^j = (t_1^j, \cdots, t_r^j)$, $U^j = (u_1, \cdots, u_{r-1}^j)$. According to Section 26 of Geman and Horowitz (1980), as explained in detail in Geman, Horowitz and Rosen (1983) Section 2, this will suffice to prove our proposition.

Let $\pi^1, \cdots, \pi^r$ be r (not necessarily distinct) permutations of $\{1, \cdots, k\}$. Define

$$
\begin{aligned}
&\Delta(\pi', \cdots, \pi^r) \\
&\qquad = \{(T^1, \cdots, T^k) \mid t_i^{\pi^i(j)} < t_i^{\pi^i(j+1)},\ 1 \le i \le r,\ 1 \le j \le k-1\}.
\end{aligned}
\tag{2.6}
$$

Let $u_0^j \doteq u_r^j = 0$, and $t_i^{\pi^i(k+1)} \doteq t_{i+1}^{\pi^{i+1}(1)}$. By (2.1) we see that the order relation in (2.6) is also true if we allow $j = k$. We will now evaluate $V(\sum_{j=1}^k X(T^j) \cdot U^j)$ on $\Delta(\pi^1, \cdots, \pi^r)$. We first rewrite

$$\begin{aligned} \sum_{j=1}^k X(T^j) \cdot U^j &= \sum_{j=1}^k \sum_{i=1}^{r-1} u_i^j \cdot (W_{t_{i+1}^j} - W_{t_i^j}) \\ &= \sum_{1\le i\le r, 1\le j\le k} v_i^j (W_{t_i^{\pi^i(j+1)}} - W_{t_i^{\pi^i(j)}}) \end{aligned} \tag{2.7}$$

where

$$v_i^j = \sum' u_m^l$$

with the sum running over all pairs (l, m) such that $[t_m^l, t_{m+1}^l] \supseteq [t_i^{\pi^i(j)}, t_i^{\pi^i(j+1)}]$. The relation is possible only for $m = i$, $(\pi^i)^{-1}(l) \le j$ and $m = i - 1$, $(\pi^i)^{-1}(l) > j$. Thus

$$v_i^j = \sum_{l\le j} u_i^{\pi^i(l)} + \sum_{l>j} u_{i-1}^{\pi^i(l)}. \tag{2.8}$$

(2.7) and independence show that

$$V(\sum_{j=1}^k X(T^j) \cdot U^j) = \sum_{1\le i\le r, 1\le j\le k} |v_i^j|^2 (\bar{t}_i^{j+1} - \bar{t}_i^j) \tag{2.9}$$

where we have set $\bar{t}_i^j = t_i^{\pi^i(j)}$.

Note from (2.8) that

$$v_i^j - v_i^{j-1} = u_i^{\pi^i(j)} - u_{i-1}^{\pi^i(j)}, \quad (v_i^0 \doteq v_{i-1}^k). \tag{2.10}$$

Also for any $p = 1, \cdots, r$ and some constant c independent of k

$$\begin{aligned} |U^j| &\le c \sum_{i=1, i\ne p}^r |u_i^j - u_{i-1}^j| \le c \prod_{i\ne p} (1 + |u_i^j - u_{i-1}^j|) \\ &\le c \prod_{i=1, i\ne p}^r (1 + |v_i^{\bar{\pi}^i(j)} - v_i^{\bar{\pi}^i(j)-1}|), \quad \text{by (2.10)}, \end{aligned} \tag{2.11}$$

where $\bar{\pi}^i = (\pi^i)^{-1}$, and therefore

$$\begin{aligned} \prod_{j=1}^k |U^j| &\le c^k \prod_{j=1}^k \prod_{p=1}^r (\prod_{i=1, i\ne p}^r (1 + |v_i^{\bar{\pi}^i(j)} - v_i^{\bar{\pi}^i(j)-1}|))^{1/r} \\ &\le c^k \prod_{p=1}^r (\prod_{i=1, i\ne p}^r \prod_{j=1}^k (1 + |v_i^{\bar{\pi}^i(j)}| + |v_i^{\bar{\pi}^i(j)-1}|))^{1/r} \\ &\le c^k \prod_{p=1}^r (\prod_{i,j, i\ne p} (1 + |v_i^j|^2))^{1/r}. \end{aligned} \tag{2.12}$$

Let

$$\begin{aligned} C_p &= \{(i, j) \mid 1 \le i \le r,\ 1 \le j \le k,\ (i, j) \ne (p, 1), \cdots, (p, k-1)\} \\ &= \{(i, j) \mid i \ne p\} \cup \{\{p, k\}\}. \end{aligned}$$

From (2.10) we see that $\{v_i^j, (i, j) \in C_p\}$ is a set of coordinates for $R^{2(r-1)k}$. On

$\Delta(\pi^1, \cdots, \pi^r)$ we now have the bound

$$\int_{R^{2(r-1)k}} \prod_{l=1}^{k} |U^l|^{\gamma} \exp\left(-\frac{1}{2} V(\textstyle\sum_{j=1}^{k} X(T^j) \cdot U^j)\right) dU^1 \cdots dU^k$$

$$\le c^k \int_{R^{2(r-1)k}} \prod_{p=1}^{r} \prod_{(i,j)\in C_p} (1 + |v_i^j|^2)^{\gamma/r}$$

$$\cdot \exp\left(-\frac{1}{2r} \textstyle\sum_{(i,j)\in C_p} |v_i^j|^2(\bar{t}_i^{j+1} - \bar{t}_i^j)\right) dU^1 \cdots dU^k$$

$$\le c^k \prod_{p=1}^{r} \left(\int_{R^{2(r-1)k}} \prod_{C_p} (1 + |v_i^j|^2)^{\gamma} \right. \tag{2.13}$$

$$\left. \cdot \exp\left(-\frac{1}{2} \textstyle\sum_{C_p} |v_i^j|^2(\bar{t}_i^{j+1} - \bar{t}_i^j)\right) dU^1 \cdots dU^k\right)^{1/r}$$

$$\le c^k \prod_{p=1}^{r} \prod_{C_p} (\bar{t}_i^{j+1} - \bar{t}_i^{j+1})^{-(1/r)(1+\gamma)}$$

$$\le c^k \prod_{i,j,j\neq k} (\bar{t}_i^{j+1} - \bar{t}_i^j)^{-((r-1)/r)(1+\gamma)}$$

where we used (2.1) in the form $\bar{t}_i^{k+1} - \bar{t}_i^k = \bar{t}_{i+1}^1 - \bar{t}_i^k \ge a_{i+1} - (a_i + h) > 0$. This immediately shows that (2.5) is finite for $\gamma < 1/(r-1)$, proving Proposition 2.

REMARK. The methods of Section 26, Geman and Horowitz (1980) allow us to conclude that $\alpha(x, B)$ is a Hölder continuous function of x of any order $< 1/(r-1)$. We are, however, more interested in the following "Hölder continuity in the set variable".

PROPOSITION 3. *For each $x \in R^{2(r-1)}$ there exist* a.s. *finite random variables c, δ such that*

$$\alpha(x, B) \le c\,|B|^{1/r}\,|\lg(|\lg|B||)|^{r-1}, \text{ a.s.}$$

for all hypercubes $B \subseteq I$ of edgelength less than δ. Furthermore, for any fixed $T \in I$ we can choose c, δ such that

$$\alpha(x, B) \le c\,|B|^{1/r}\,|\lg(|\lg|B||)|^{r-1}$$

for all hypercubes $B \subseteq I$ of edgelength less than δ with a corner at T.

PROOF. Let B have edgelength h. Using the bound (2.13) with $\gamma = 0$, and

Section 25 of Geman and Horowitz (1980) we have

$$
\begin{aligned}
&E([\alpha(x, B)]^k) \\
&\quad \le c^k \int_{B^k} \int_{R^{2(r-1)k}} \exp\Big(-\frac{1}{2} V(\textstyle\sum_{j=1}^k X(T^j) \cdot U^j)\Big)\, du\, dT \\
&\quad \le c^k \textstyle\sum_{\pi^1,\dots,\pi^r} \displaystyle\int_{B^k \cap \Delta(\pi^1,\dots,\pi^r)} \textstyle\prod_{i,j,j\neq k} (\bar{t}_i^{j+1} - \bar{t}_i^j)^{-(r-1)/r}\, dT \\
(2.15) &\quad \le c^k (k!)^r \Big[\int \cdots \int_{0<t_1<\cdots<t_{k-1}<h} \textstyle\prod_{i=1}^{k-1} (t^{j+1} - t^j)^{-(r-1)/r}\, dt^j \cdots dt^k \Big]^r \\
&\quad \le c^k (k!)^r h \Big[\int \cdots \int_{0<S_1<\cdots<S_{k-1}<h} \textstyle\prod_{i=1}^{k-1} (S_i - S_{i-1})^{-(r-1)/r}\, dS_1 \cdots dS_{k-1} \Big]^r \\
&\quad \le c^k (k!)^r \Big[\frac{h \cdot h^{(1-(r-1)/r)(k-1)} [(k-1)!]^{(r-1)/r}}{(k-1)!} \Big]^r \\
&\quad \le c^k h^k (k!)^{(r-1)}.
\end{aligned}
$$

In the last inequality we have used the lemma of Kono (1977). The methods of that paper now yield our proposition (see also Ehm (1981), and Geman, Horowitz and Rosen (1983), Theorems 2 and 3).

PROOF OF THEOREM 1. We are now ready to carry out the ideas described in the introduction. X has Holder continuous paths of any order $< \frac{1}{2}$. Propositions 2 and 3 now show (see Adler, 1981, Theorems 8.73, 8.74)

$$\dim(X^{-1}(0) \cap I) \le 1$$

with equality if $\alpha(0, I) > 0$.

We next use an idea of Tran (1976). Let

$$I_n = \textstyle\prod_{i=1}^r [2rn + i,\ 2rn + i + \frac{1}{2}].$$

An easy computation shows that $E(\alpha(0, I_0)) > 0$, hence if $A_n = \{\alpha(0, I_n) > 0\}$ we have $P(A_0) = \delta > 0$ for some $\delta > 0$. Since the A_n are independent and $P(A_n) = \delta$ for all n, the Borel-Cantelli lemma says that a.s. we can find an n with

$$\alpha(0, I_n) > 0.$$

Since $R^r_<$ can be covered by a countable number of hypercubes I,

$$\dim(X^{-1}(0) \cap R^r_<) = 1 \quad \text{a.s.}$$

and symmetry now finishes the proof of Theorem 1.

3. The N-parameter Wiener process in R^d. In this section we study the r-multiple times of the N-parameter Wiener process in R^d, $W^{N,d}$ (see (1.2)). We will see how to modify the arguments of Section 2. Note first that for reasons of symmetry, in proving Theorem 1 it sufficed to consider hypercubes $I \subseteq R^r_<$. However, with $W^{N,d}$, each of the N-parameters may be ordered differently. The following observation will be helpful in handling such situations. Let p be a permutation of $\{1, \cdots, r\}$. We then have

$$\begin{aligned} \textstyle\sum_{i=1}^{r-1} u_i(a_{p(i+1)} - a_{p(i)}) &= \textstyle\sum_{i=1}^{r-1} \pm\, u_i(\sum_{l:[l,l+1]\subseteq[p(i),p(i+1)]} a_{l+1} - a_l) \\ &= \textstyle\sum_{l=1}^{r-1} (\sum_{i:[p(i),p(i+1)]\supseteq[l,l+1]} \pm\, u_i)(a_{l+1} - a_l) \qquad (3.1) \\ &= \textstyle\sum_{l=1}^{r-1} \bar{u}_l(a_{l+1} - a_l), \quad \text{with} \end{aligned}$$

$$\bar{u}_l = \textstyle\sum_{i:[p(i),p(i+1)]\supseteq[l,l+1]} \pm\, u_i. \qquad (3.2)$$

In these formulae $[p(i), p(i+1)]$ is the interval between $p(i)$ and $p(i+1)$ (we allow $p(i) > p(i+1)$), and the sign in $\pm\, u_i$ is positive or negative depending on whether or not $p(i) < p(i+1)$.

We now show that

$$\bar{u}_{p(l)} - \bar{u}_{p(l)-1} = u_l - u_{l-1}, \quad (\bar{u}_0 \doteq u_0 \doteq 0). \qquad (3.3)$$

This will prove that $\{u_l\} \to \{\bar{u}_l\}$ is a nonsingular linear change of coordinates. To establish (3.3) write out

$$\bar{u}_{p(l)} - \bar{u}_{p(l)-1} = \textstyle\sum_{i:[p(i),p(i+1)]\supseteq[p(l),p(l)+1]} \pm\, u_i - \sum_{i:[p(i),p(i+1)]\supseteq[p(l)-1,p(l)]} \pm\, u_i. \qquad (3.4)$$

The only u_i's that survive correspond to those i's such that $[p(i), p(i+1)]$ does not contain both $[p(l)-1, p(l)]$ and $[p(l), p(l)+1]$. This means that $p(l)$ is an endpoint of $[p(i), p(i+1)]$. If $p(l) < p(l+1)$, we get a contribution $+\, u_l$ from the first sum, while if $p(l) > p(l+1)$ we get a contribution from the second sum, which is also (recall our convention on signs) $+\, u_l$. Similarly we find that we always have a contribution $-u_{l-1}$. This proves (3.3).

With these preliminaries aside, let I denote a hypercube in $(R^r_{\neq})^N$. For ease of notation we sometimes write X for $X^{N,d}$.

Proposition 4. *Let $Nr > d(r-1)/2$; then for any Borel set, $B \subseteq I$, $X^{N,d}$ has a local time $\alpha(x, B)$, and $\alpha(x, B) \in L^2(R^{d(r-1)}, dx)$* a.s.

Proof. Since I is a hypercube in $(R^r_{\neq})^N$, we can find N permutations of $\{1, \cdots, r\}, p^l, 1 \le l \le N$, such that for any $T \in I$ with

$$T = (t_1, \cdots, t_r), \quad t_i = (t_{i,1}, t_{i,2}, \cdots, t_{i,N})$$

we have

$$t_{p^l(1),l} < t_{p^l(2),l} < \cdots < t_{p^l(r),l}, \quad 1 \le l \le N. \qquad (3.5)$$

Let B denote d-dimensional Brownian motion. We first bound the variance $V(u \cdot (X(T) - X(S)))$ by a sum along the axes (see Rosen, 1981, or Ehm, 1981) and then use (3.1) with $p = (p^l)^{-1}$:

$$
\begin{aligned}
&V(u \cdot (X(T) - X(S))) \\
(3.6) \quad &\geq \varepsilon \textstyle\sum_{l=1}^{n} V(\sum_{i=1}^{r-1} u_i \cdot ([B_{t_{i+1,l}} - B_{s_{i+1,l}}] - [B_{t_{i,l}} - B_{s_{i,l}}])) \\
&= \varepsilon \textstyle\sum_{l=1}^{N} V(\sum_{i=1}^{r-1} \bar{u}_{i,l} \cdot ([B_{t_{p^l(i+1),l}} - B_{s_{p^l(i+1),l}}] - [B_{t_{p^l(i),l}} - B_{s_{p^l(i),l}}])) \\
&= \varepsilon \textstyle\sum_{l=1}^{N} \sum_{i=1}^{r} |\bar{u}_{i,l} - \bar{u}_{i-1,l}|^2 |\bar{t}_{i,l} - \bar{s}_{i,l}|
\end{aligned}
$$

where $\bar{t}_{i,l} = t_{p^l(i),l}$, $\bar{s}_{i,l} = s_{p^l(i),l}$.

We now use the generalized Hölder inequality and the calculations of (2.4) to obtain the bound

$$
\begin{aligned}
&\int_B \int_B \int_{R^{d(r-1)}} \exp\left(-\frac{1}{2} V(u \cdot (X(T) - X(S))\right) du\, ds\, dT \\
&\leq \int_B \int_B \int_{R^{d(r-1)}} \textstyle\prod_{l=1}^{N} \\
&\quad \cdot \exp\left(-\frac{\varepsilon}{2} \textstyle\sum_{i=1}^{r} |\bar{u}_{i,l} - \bar{u}_{i-1,l}|^2 |\bar{t}_{i,l} - \bar{s}_{i,l}|\right) du\, ds\, dT \\
(3.7) \quad &\leq \int_B \int_B \textstyle\prod_{l=1}^{N} \\
&\quad \cdot \left(\int \exp\left(-\frac{N\varepsilon}{2} \textstyle\sum_{i=1}^{r} |\bar{u}_{i,l} - \bar{u}_{i-1,l}|^2 |\bar{t}_{i,l} - \bar{s}_{i,l}|\right) du\right)^{1/N} ds\, dT \\
&\leq c \int_B \int_B \textstyle\prod_{l=1}^{N} \prod_{i=1}^{r} |\bar{t}_{i,l} - \bar{s}_{i,l}|^{-d(r-1)/2Nr}\, ds\, dT \\
&< \infty \quad \text{if} \quad Nr > d(r-1)/2
\end{aligned}
$$

as in the proof of Proposition 1. This proves our proposition.

PROPOSITION 5. *If $Nr > r(d-1)/2$, then $X^{N,d}$ has a jointly continuous local time on I, with probability one.*

PROOF. As in the previous proof, with the notation used there

$$
\begin{aligned}
&V(\textstyle\sum_{j=1}^{k} X(T^j) \cdot u^j) \geq \varepsilon \sum_{l=1}^{N} V(\sum_{j=1}^{k} \sum_{i=1}^{r-1} u_i^j \cdot (B_{t^j_{i+1,l}} - B_{t^j_{i,l}})) \\
(3.8) \quad &= \varepsilon \textstyle\sum_{l=1}^{N} V(\sum_{j=1}^{k} \sum_{i=1}^{r-1} \bar{u}^j_{i,l} \cdot (B_{t^j_{p^l(i+1),l}} - B_{t^j_{p^l(i),l}})) \\
&= \varepsilon \textstyle\sum_{l=1}^{N} \sum_{i,j} |v^j_{i,l}|^2 (\bar{t}^{j+1}_{i,l} - \bar{t}^j_{i,l})
\end{aligned}
$$

on

$$\Delta(\pi_1^1, \cdots, \pi_N^r) = \{(T^1, \cdots, T^k) \mid t_{i,l}^{\pi_l^i(j)} < t_{i,l}^{\pi_l^i(j+1)}, \forall i, j, l\}$$

where $\bar{t}^j_{i,l} = t^{\pi_l^{p^l(i)}(j)}_{p^l(i),l}$ and for each fixed l, $l = 1, \cdots, N$, $v^j_{i,l}$ is formed from the $\bar{u}^j_{i,l}$

as in (2.8) (see (2.7)). We now use the generalized Hölder's inequality

$$\int_{I^k\cap\Delta(\pi_1^{\prime},\cdots,\pi_N^{\prime})} \int_{R^{dk(r-1)}} \prod_{p=1}^{k} |u^p|^{\gamma} \exp\left(-\frac{1}{2} V\left(\textstyle\sum_{j=1}^{k} X(T^j)\cdot u^j\right)\right) du\, dT$$

$$= \int_{I^k\cap\Delta} \int_{R^{dk(r-1)}} \prod_{l=1}^{N} \left(\prod_{p=1}^{k} |u^p|^{\gamma/N}\right)$$

$$\cdot \exp\left(-\frac{\varepsilon}{2} \textstyle\sum_{i,j} |v_{i,l}^j|^2 (\bar{t}_{i,l}^{j+1} - \bar{t}_{i,l}^j)\right) du\, dT \tag{3.9}$$

$$\leq \int_{I^k\cap\Delta} \prod_{l=1}^{N}$$

$$\cdot \left(\left(\int_{R^{dk(r-1)}} \prod_{p=1}^{k} |u^p|^{\gamma} \exp\left(-N\frac{\varepsilon}{2} \textstyle\sum_{i,j} |v_{i,l}^j|^2 (\bar{t}_{i,l}^{j+1} - \bar{t}_{i,l}^j)\, du\right)\right)\right)^{1/N} dT$$

$$\leq ck \int_{I^k\cap\Delta} \prod_{l=1}^{N} \prod_{i,j,j\neq k} |\bar{t}_{i,l}^{j+1} - \bar{t}_{i,l}^j|^{-((r-1)/rN)(d/2+\gamma)}\, dT$$

as in (2.13). As in the proof of Proposition 2, this establishes our proposition.

Similarly, our last estimate yields as before

$$\alpha(x, B) \leq c\,|B|^{(1-d(r-1)/2Nr)}(|\lg|B||)^{(d/2)(r-1)}$$

for all hypercubes B, I sufficiently small.

PROOF OF THEOREM 2. We need only recall that $W^{N,d}$ is Holder continuous of any order $< 1/2$ and possesses independent increments (see Orey and Pruitt, 1973), and then we can take over the proof of Theorem 1.

4. Index-β processes. In this section we study the r-multiple times of the index-β process W^{β} (see (1.5)). W is known to be locally non-deterministic on $C_\varepsilon = \{t \mid \varepsilon \leq |t| \leq \varepsilon^{-1}\}$ for any $\varepsilon > 0$. This means that there exists a constant $d > 0$ such that for any $t_1, \cdots, t_m \in C_\varepsilon$ satisfying

$$|t_{l+1} - t_l| \leq |t_{l+1} - t_i|, \quad \forall\ 1 \leq i \leq l \leq m, \tag{4.1}$$

we have

$$V\left(\textstyle\sum_{l=1}^{m} u_l \cdot (W_{t_l}^{\beta} - W_{t_{l-1}}^{\beta})\right) \geq d \textstyle\sum_{l=1}^{m} |t_l - t_{l-1}|^{2\beta} |u_l|^2, \tag{4.2}$$

(see Pitt, 1978).

PROOF OF THEOREM 3. If $t_1, \cdots, t_l \in R^N$ we say they are spatially ordered if (4.1) holds, and we write this as

$$t_1 \preceq t_2 \preceq \cdots \preceq t_l.$$

Let $T \in R_{a,b}^r$. We can always find a hypercube $T \subseteq C_\varepsilon$ for some $\varepsilon > 0$, such that

$T \in I$, and I is so small that for any

$$T^1, \cdots, T^k \in I, \quad T^j = (t_1^j, \cdots, t_r^j), \quad t_i^j \in R^N$$

there is a spatial ordering of the points t_i^j such that the sets $\{t_i^j, i \text{ fixed}\}$ occur in blocks. More precisely, there is a permutation p of $\{1, \cdots, r\}$ such that

$$t_{p(i)}^j \leq t_{p(i+1)}^{\bar{j}}, \quad \forall i, j, \bar{j}. \tag{4.3}$$

Whenever we estimate an integral of the form considered in Section 2, we may restrict our attention to the set of $T^1, \cdots, T^k$ satisfying (4.3) for a fixed permutation p. On this set the considerations of (3.1) show that after a change of variables we can assume p is the identity. We are then formally in the situation of Section 2 where $I \subseteq R_<^r$, and (4.2) provides an adequate substitute for independence. The methods of that section show that for $Nr > \beta d(r - 1)$

$$\dim(L_\beta^r \cap I) \leq Nr - \beta d(r - 1)$$

with equality holding on a set of positive probability. Since $R_{a,b}^r$ can be covered by a countable number of such I's, and W^β is Holder contininous of any order $< \beta$, Theorem 3 is proven.

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