Math 330: Exam #1 Review Sheet

Exam #1: Friday, February 21, 2020

(1) For each of the following first order equations, construct approximate graphical solutions by plotting the direction field (the slope) and several solution curves.

a. $\frac{dy}{dx} = y(y^2 - 9)$

b. $\frac{dy}{dx} = -x/y$

c. $\frac{dy}{dx} = y^3 - y^2 - 2y$

Think about the last example. Suppose some system is modeled by the equation:

$$\frac{dy}{dt} = y^3 - y^2 - 2y$$

Someone's life (or job) depends on knowing what the *long-time behavior* of the solution, $y(t)$ is for several different initial conditions: (a) $y(t=0) = 1$, (b) $y(t=0) = 2$, (c) $y(t=0) = 3$.

Use your graphical solution to the problem to determine:

$$\lim_{t\to\infty} y(t)$$

for each of these initial value problems.

(2) Classify the following ODE's as linear, exact, homogeneous or seperable and find the general solution.

a) $e^x y^2 + 2x\,(e^y + 1) + \left(2y\,(e^x + 1) + x^2 e^y\right) y' = 0$

b) $y' + \frac{2}{x} y = \frac{\cos x}{x^2}$

c) $y' = \frac{x - e^{-x}}{y + e^y}$

d) $y' = \frac{y^2 + 2xy}{x^2}$

(2a) Find an implicit solution to the folllowing initial value problem:

$$e^x y^2 + 2x\,(e^y + 1) + \Big(2y\,(e^x + 1) + x^2 e^y\Big)\, y' = 0; \quad y(0) = 2$$

(3) Consider the following equation: $y'' - 2y' + y = 0$

a) Find the general solution.

b) Find the specific solution when $y(0) = y'(0) = 1$.

c) Now consider the non-homogeneous equation:

$$y'' - 2y' + y = e^{-x}$$

Find the general solution.

(4) Consider the differential equation:
$y'' - 5y' + 4y = f(x)$

a) Find the solution to the homogeneous problem, $f(x) = 0$.
b) Find the general solution when $f(x) = x^2 + 1$.

(5) Verify that $y = x$ is a homogeneous solution to the differential equation:

$$x^2 y'' - x\,(x+2)\,y' + (x+2)\,y = 2x^3 \quad x > 0$$

a) Use reduction of order to show that a second homogeneous solution is $y = xe^x$.
b) Find the general solution to the non-homogeneous problem. (Hint: Remember the form of the ODE used to derive the Variation of Parameters formula!)

(6) Consider the differential equation:
$y'' - 6y' + 9y = f(x)$

a) Find the solution to the homogeneous problem, $f(x) = 0$.
b) Find the general solution when $f(x) = e^{2x}$.
c) Find the general solution when $f(x) = e^{3x}$
d) Find the general solution when $f(x) = xe^{3x}$ (???)

(7) Consider the differential equation:
$y'' + 4y = f(x)$

a) Find the solution to the homogeneous problem, $f(x) = 0$.
b) Find the general solution when $f(x) = \cos x$.
c) Find the general solution when $f(x) = \cos 2x$.

(8) Consider the forced mechanical system described by

$$y'' + \omega_0^2 y = \cos(5x)$$

where ω_0 is the natural frequency of the unforced system.

a) Write down the general solution to the homogeneous problem for any value of ω_0.
b) Describe how the long time behavior of the forced system changes as ω_0 changes.
c) For what value of the natural frequency, ω_0, is the forcing function, $f(x) = \cos(5x)$, in RESONANCE with the homogeneous solution?