Reciprocals of Binary Power Series

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A Few Identities

\[
\left( \sum_{n \geq 0} p(n) q^n \right) \left( \sum_{n = -\infty}^{\infty} q^{n(3n-1)/2} \right) \equiv 1 \pmod{2}
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\[
\left( 1 + \sum_{n \geq 0} q^{2n} \right) \left( \sum_{n \geq 0} q^{2n-1} \right) \equiv 1 \pmod{2}
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Let
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\left(1 + \sum_{n \geq 0} q^{2n}\right) \left(\sum_{n \geq 0} q^{2n-1}\right) = \sum_{k \geq 0} R(k) q^k.
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A Proof

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If \( R(k) > 0 \), then

\[ k = 2^n + 2^m - 1, \]

and if \( n \neq m \)

\[ k = (2^n) + (2^m - 1) = (2^m) + (2^n - 1), \]

so \( R(k) = 2 \).
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If \( n = m \), then
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k = (2^n) + (2^m - 1) = (0) + (2^{n+1} - 1),
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(1 + q)(1 + q + q^2 + q^3 + \cdots) \equiv 1 \pmod{2}
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Nonnegative integer sets \( A \) and \( B \) are reciprocals if their generating functions are reciprocals in \( \mathbb{F}_2[[q]] \).

\[
A = \{0, 1\}, \quad B = \{0, 1, 2, 3, \ldots\}
\]

\[
A = \{0, 1, 2, 4, 8, 16, \ldots\}, \quad B = \{0, 1, 3, 7, 15, \ldots\}
\]
Suppose

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The coefficient of \(q^n\) is

\[b_n + b_{n-1} a_1 + b_{n-2} a_2 + \cdots + b_2 a_{n-2} + b_1 a_{n-1} + a_n = 0.\]
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Remark: $\mathcal{F} \in \mathbb{F}_2[[q]]$ is invertible if and only if...
If \( \max A = d \), then

\[
b_n = b_{n-1}a_1 + b_{n-2}a_2 + \cdots + b_{n-d}a_d.
\]

The sequence \((b)\) is a linear recurrence sequence with boundary \(b_0 = 1\), \(b_{-1} = 0\), \(b_{-2} = 0\), \ldots
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- \((b)\) is periodic.

- \((b)\) may have more 0 than 1 (when \(d\) is small)

- If \(q\) generates the multiplicative group of \(\mathbb{F}_2[q]/(A)\), then every binary word of length \(d\) appears in \((b)\) except 0000 \(\cdots\) 000. This is called a reduced de Bruijn cycle.

- Period length = \(2^d - 1\), with \(2^{d-1}\) ones. Density slightly larger than \(1/2\).
The points \((n, \delta(\bar{P}_n))\), where the coeffs of \(P_n\) are the binary expansion of \(n\).
• What are the possible densities of reciprocals of finite sets?
• Is the bias toward $< 1/2$ a law of small numbers?
If $\mathcal{P}(q)$ is a polynomial, then there is another polynomial $\mathcal{P}^*$ and a positive integer $D$ such that $\mathcal{P}\mathcal{P}^* = 1 + q^D$. We call the minimal such $D$ the order of $\mathcal{P}$. 
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**Theorem:** If $\mathcal{P}$ has degree $d$ and order $2^d - 1$, then

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**Proposition:** The reciprocal of an eventually periodic set is one too.
Quadratic Sequences

\[ \Theta(c_1, c_2) := \left\{ c_1 n + c_2 \frac{n(n-1)}{2} : n \in \mathbb{Z} \right\} \]
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**WOLOG:** \( \gcd(c_1, c_2) = 1, \ 0 \leq 2c_1 \leq c_2 \)
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$$\Theta(0, 1) = \left\{ \binom{n}{2} : n \geq 1 \right\}$$

$$\Theta(1, 2) = \left\{ n^2 : n \geq 0 \right\}$$

$$\Theta(1, 3) = \{ \text{pentagonals} \}$$
The Experimental Density of the Inverse of a Quadratic Sequence

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The reciprocal of the set $\Theta(c_1, c_2)$, where $0 \leq 2c_1 \leq c_2$ and $\gcd(c_1, c_2) = 1$, has density 0 if $c_2 \equiv 2 \pmod{4}$, and otherwise has density $1/2$.

More precisely, if $c_2 \equiv 2 \pmod{4}$, then

$$
\lim_{n \to \infty} \frac{|\Theta(c_1, c_2) \cap [0, n]|}{n / \log n} = C,
$$

for some positive constant $C$ depending only on $c_2$. If $c_2 \not\equiv 2 \pmod{4}$, then

$$
\limsup_{n \to \infty} \left| \frac{|\Theta(c_1, c_2) \cap [0, n]| - n/2}{\sqrt{n \log \log(n)/2}} \right| = 1.
$$
Two Modest Conjectures

How many numbers less than $N$ can be written in the form

$$x_0^2 + 2x_1^2 + 4x_2^2 + 8x_3^2 + 16x_4^2 + \cdots,$$

with nonnegative $x_i$, in an odd number of ways?
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Current bests:

$$\# \geq \left( \frac{\pi^2 \sqrt{3}}{2} - o(1) \right) \frac{\sqrt{N}}{\log N} \quad \text{(D. Eichhorn)}$$

$$\lim_{N \to \infty} \frac{N - \#}{\sqrt{N}} = \infty \quad \text{(Serre)}$$
Let $f_1, f_2, \ldots$ be independent binary random variables, with

$$\mathbb{P}[f_n = 0] \mathbb{P}[f_n = 1]$$

bounded away from 0.

Define $\bar{f}_1, \bar{f}_2, \ldots$ by

$$(1 + f_1 q + f_2 q^2 + f_3 q^3 + \cdots)(1 + \bar{f}_1 q + \bar{f}_2 q^2 + \bar{f}_3 q^3 + \cdots) = 1.$$ 

Then the number of $\bar{f}_1, \bar{f}_2, \ldots, \bar{f}_N$ that are 1 is $\sim N/2$ with probability 1.
\[ \bar{f}_n = \sum_{\bar{x}} f_{x_1} f_{x_2} \cdots f_{x_{\ell}} \]

where the summation extends over all tuples \( \bar{x} = (x_1, \ldots, x_{\ell}) \) with \( n = \sum_{i=1}^{\ell} x_i \) and each \( x_i > 0 \) (\( \ell \) is allowed to vary).
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\[ \bar{f}_n = f_n + f_{n-2i} \bar{f}_i + f_{n-4i} \bar{f}_2 + \cdots f_{n/2} \bar{f}_{n/4} + \text{mess} \]

and \text{mess} depends only on \( f_1, f_2, \ldots, f_{n/2-1} \).
Thus,

$$H[f_n|f_1, \ldots, f_{n/2-1}] \geq H[\sum_{i \in A} f_i|A]$$

where $A = \{n - 2i: 0 \leq i < n/4, \bar{f}_i = 1\}$. Since

- $|A| \to \infty$ (requires easy proof),
- this uncertainty goes to 1/2 (requires proof),
- and so $P[f_n = 0] \to 1/2$ (obvious),
- and consequently $\#\{n \leq N: \bar{f}_n = 0\} \sim N/2$ (obscure Borel-Cantelli Lemma)
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Plans for future development:

- Take some interesting set of integers, call it $A$. Find $\bar{A}$.
- Probabilistic argument is not most general possible.
- arXiv:math.NT/0506496
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