Sturmian Words and the Permutation that Orders Fractional Parts

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Abstract

A Sturmian word is a map $W : \mathbb{N} \to \{0,1\}$ for which the set of $\{0,1\}$-vectors $F_n(W) := \{(W(i), W(i+1), \dots, W(i+n-1))^T : i \in \mathbb{N}\}$ has cardinality exactly $n+1$ for each positive integer n. Our main result is that the volume of the simplex whose $n+1$ vertices are the $n+1$ points in $F_n(W)$ does not depend on W. Our proof of this motivates studying algebraic properties of the permutation $\pi_{\alpha,n}$ (where α is any irrational and n is any positive integer) that orders the fractional parts $\{\alpha\}, \{2\alpha\}, \dots, \{n\alpha\}$, i.e., $0 < \{\pi_{\alpha,n}(1)\alpha\} < \{\pi_{\alpha,n}(2)\alpha\} < \cdots < \{\pi_{\alpha,n}(n)\alpha\} < 1$. We give a formula for the sign of $\pi_{\alpha,n}$, and prove that for every irrational α there are infinitely many n such that the order of $\pi_{\alpha,n}$ (as an element of the symmetric group S_n) is less than n.

1 Introduction

A *binary word* is a map from the nonnegative integers into $\{0,1\}$. The *factors of* W are the column vectors $(W(i), W(i+1), \dots, W(i+n-1))^T$, where $i \geq 0$ and $n \geq 1$. In particular, the set of factors of length n of a binary word W is defined by

$$F_n(W) := \left\{ (W(i), W(i+1), \dots, W(i+n-1))^T : i \geq 0 \right\}.$$

Obviously, $|F_n(W)| \leq 2^n$ for any binary word W. It is known [Lot02, Theorem 1.3.13] that if $|F_n(W)| < n+1$ for any n, then W is eventually periodic. If $|F_n(W)| = n+1$ for every n—the most simple non-periodic case—then W is called a *Sturmian word*. Sturmian words arise in many fields, including computer graphics, game theory, signal analysis, diophantine approximation, automata, and quasi-crystallography. The new book of Lothaire [Lot02] provides an excellent introduction to combinatorics on words; the second chapter is devoted to Sturmian words.

Throughout this paper, W is always a Sturmian word, n is always a positive integer, and α is always an irrational between 0 and 1. A typical example of a Sturmian word is given by $c_\alpha(i) := \lfloor (i+2)\alpha \rfloor - \lfloor (i+1)\alpha \rfloor$, the so-called *characteristic word with slope* α. By routine manipulation, one finds that $c_\alpha(i) = 1$ if and only if $i+1 \in \{\lfloor k\alpha \rfloor : k \in \mathbb{Z}^+\}$. The integer sequences $(\lfloor k\alpha + \beta \rfloor)_{k=1}^{\infty}$ are called *Beatty sequences*. The study of Beatty sequences is intimately related to the study of Sturmian words, and the interested reader can locate most of the literature through the bibliographies of [Sto76], [Bro93], and [Tij00].

In this paper, we consider the $n+1$ factors in $F_n(W)$ to be the vertices of a simplex in $\mathbb{R}^n$. Our main result is

Theorem 1.1. *If W is a Sturmian word, then the volume of the simplex $F_n(W)$ is $\frac{1}{n!}$.*

The remarkable aspect of Theorem 1.1 is that the volume of the simplex $F_n(W)$ is independent of W. The key to the proof of Theorem 1.1 is to study $F_n(W)$ for all Sturmian words W simultaneously. The primary tool is the representation theory of finite groups.

Sturmian words are examples of one-dimensional quasicrystals, at least with respect to some of the 'working definitions' currently in use. In contrast to the study of crystals, group theory has not been found very useful in the study of quasicrystals. According to M. Senechal [Sen95], "The one-dimensional case suggests that symmetry may be a relatively unimportant feature of aperiodic crystals." Thus, the prominent role of symmetric groups in the proof of Theorem 1.1 comes as a surprise.

The proof of Theorem 1.1 reveals a deep connection between the simplex $F_n(c_\alpha)$ and algebraic properties of the permutation $\pi_{\alpha,n}$ of $1, 2, \ldots, n$ that orders the fractional parts $\{\alpha\}, \{2\alpha\}, \ldots, \{n\alpha\}$, i.e.,

$$0 < \{\pi_{\alpha,n}(1)\alpha\} < \{\pi_{\alpha,n}(2)\alpha\} < \cdots < \{\pi_{\alpha,n}(n)\alpha\} < 1.$$

The definition of $\pi_{\alpha,n}$ has a combinatorial flavor, and accordingly some attention has been given to its combinatorial qualities. Using the geometric theory of continued fractions, Sós [Sós57] gives a formula for $\pi_{\alpha,n}$ in terms of n, $\pi_{\alpha,n}(n)$, and $\pi_{\alpha,n}(1)$ (see Lemma 3.1.1). Boyd & Steele [BS79] reduce the problem of finding the longest increasing subsequence in $\pi_{\alpha,n}$ to a linear programming problem, which they then solve explicitly. Schoißengeier [Sch84] used Dedekind eta sums to study $\pi_{\alpha,n}$ and give his formula for the star-discrepancy of $n\alpha$-sequences.

Here, motivated by the appearance of $\pi_{\alpha,n}$ in our study of the simplex $F_n(W)$, we initiate the study of algebraic properties of $\pi_{\alpha,n}$. If σ is an element of a group (with identity element $\mathtt{id}$), we let $\operatorname{ord}(\sigma)$ be the least positive integer t such that $\sigma^t = \mathtt{id}$, or ∞ if no such integer exists. We use this notation with permutations, matrices, and congruence classes (the class will always be relatively prime to the modulus). For any permutation σ, let $\operatorname{sgn}(\sigma)$ be the sign of σ, i.e., $\operatorname{sgn}(\sigma) = 1$ if σ is an even permutation and $\operatorname{sgn}(\sigma) = -1$ if σ is an odd permutation. Our main results concerning $\pi_{\alpha,n}$ are stated in Theorems 1.2 and 1.3.

Theorem 1.2. *For every irrational α, there are infinitely many positive integers n such that $\operatorname{ord}(\pi_{\alpha,n}) < n$.*

Theorem 1.3. *For every irrational α and positive integer n,*

$$\operatorname{sgn}(\pi_{\alpha,2n}) = \operatorname{sgn}(\pi_{\alpha,2n+1}) = \prod_{\ell=1}^{n} (-1)^{\lfloor 2\ell\alpha \rfloor}.$$

In particular, although $\pi_{\alpha,n}$ is "quasi-random" in the sense of [Coo02], it is highly structured in an algebraic sense.

Sections 2 and 3 are logically independent and may be read in either order. In Section 2, we consider Sturmian words and the simplex $F_n(W)$. Section 3 is devoted to proving Theorems 1.2 and 1.3. Section 4 is a list of questions raised by the results of Sections 2 and 3 that we have been unable to answer. A *Mathematica* notebook containing code for generating the functions and examples in this paper is available from the author.

2 Sturmian Words

2.1 Introduction to Sturmian Words

An excellent introduction to the theory of Sturmian words is given in [Lot02, Chapter 2]. We restate the results needed in this paper in this subsection.

If $\alpha \in (0,1)$ is irrational and β is any real number, then the words $s_{\alpha,\beta}$ and $s'_{\alpha,\beta}$ defined by

$$s_{\alpha,\beta}(i) := \lfloor (i+1)\alpha + \beta \rfloor - \lfloor i\alpha + \beta \rfloor$$
$$s'_{\alpha,\beta}(i) := \lceil (i+1)\alpha + \beta \rceil - \lceil i\alpha + \beta \rceil$$

are Sturmian, and every Sturmian word arises in this way [Lot02, Theorem 2.3.13]. The irrational number α is called the slope of the word, and the word $c_\alpha := s_{\alpha,\alpha}$ is called the characteristic word of slope α. It is easily shown [Lot02, Proposition 2.1.18] that $F_n(W)$ depends only on the slope of W, and so it is consistent to write $F_n(\alpha)$ in place of $F_n(W)$. In fact, we shall use the equation $F_n(\alpha) = F_n(s_{\alpha,\beta})$ for all β. It is often easier to think in terms of 'where the 1s are'; elementary manipulation reveals that

$$c_\alpha(i) = \begin{cases} 1 & i+1 \in \{\lfloor \frac{k}{\alpha} \rfloor : k \geq 1\} \\ 0 & \text{otherwise.} \end{cases}$$

The $n+1$ elements of $F_n(\alpha)$ are n-dimensional vectors, naturally defining a simplex in $\mathbb{R}^n$. Whenever a family of simplices arises, there are several questions that must be asked. Can the simplex $F_n(\alpha)$ be degenerate? If $F_n(\alpha)$ is not degenerate, can one express its volume as a function of n and α? Under what conditions on α, β, n is $F_n(\alpha) \cong F_n(\beta)$?

The first and second questions are answered by Theorem 1.1, which we prove in Subsection 2.5 below. Computer calculation suggests a simple answer to the third question, which we state as a conjecture in Section 4.

Example: The characteristic word with slope $e^{-1} \approx 0.368$ begins

$$(c_{e^{-1}}(0), c_{e^{-1}}(1), c_{e^{-1}}(2), \dots) = (0,1,0,0,1,0,0,1,0,1,0,0,1,0,0,1,0,0,1,0,1,\dots).$$

Note that

$$c_{e^{-1}}(i) = \begin{cases} 1 & i+1 \in \{\lfloor ne \rfloor : n \geq 1\} = \{2,5,8,10,\dots\} \\ 0 & \text{otherwise.} \end{cases}$$

The set of factors of $c_{e^{-1}}$ of length 6, arranged in anti-lexicographic order, is

$$F_6(c_{e^{-1}}) = F_6(e^{-1}) = \left\{ \begin{pmatrix}1\\0\\1\\0\\0\\1\end{pmatrix}, \begin{pmatrix}1\\0\\0\\1\\0\\1\end{pmatrix}, \begin{pmatrix}1\\0\\0\\1\\0\\0\end{pmatrix}, \begin{pmatrix}0\\1\\0\\1\\0\\0\end{pmatrix}, \begin{pmatrix}0\\1\\0\\0\\1\\0\end{pmatrix}, \begin{pmatrix}0\\0\\1\\0\\1\\0\end{pmatrix}, \begin{pmatrix}0\\0\\1\\0\\0\\1\end{pmatrix} \right\}.$$

□

2.2 Definitions

To analyze a simplex, one first orders the vertices (we order them anti-lexicographically). Then, one translates the simplex so that one vertex is at the origin (we move the last factor to $\vec{0}$). Finally, one writes the coordinates of the other vertices as the columns of a matrix. If this matrix is non-singular, then the simplex is not degenerate. In fact, the volume of the simplex is the absolute value of the determinant divided by $n!$. We are thus led to define the matrix $\mathcal{M}_n(\alpha)$, whose j-th column is $\vec{v}_j - \vec{v}_{n+1}$, where $F_n(\alpha) = \{\vec{v}_1, \vec{v}_2, \dots, \vec{v}_{n+1}\}$ ordered anti-lexicographically.

Example:

$$\mathcal{M}_6(e^{-1}) = \begin{pmatrix} 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & -1 & -1 & -1 & -1 & 0 \\ 0 & 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & -1 & -1 & -1 & -1 \end{pmatrix}.$$

$\square$

When a list of vectors is enclosed by parentheses, it denotes a matrix whose first column is the first vector, second column the second vector, and so on. For example,

$$\mathcal{M}_n(\alpha) := (\vec{v}_1 - \vec{v}_{n+1}, \vec{v}_2 - \vec{v}_{n+1}, \ldots \vec{v}_n - \vec{v}_{n+1}).$$

We also define $\mathcal{V}_k$ to be the $n \times n$ matrix all of whose entries are 0, save the k-th column, which is $\vec{e}_{k-1} - 2\vec{e}_k + \vec{e}_{k+1}$.

We shall make frequent use of Knuth's notation:

$$[\![Q]\!] = \begin{cases} 1 & Q \text{ is true;} \\ 0 & Q \text{ is false.} \end{cases}$$

We denote the symmetric group on the symbols $1, 2, \ldots, n$ by S_n. We use several notations for permutations interchangeably. We use standard cycle notation when convenient, and frequently use *one-line notation* for permutations, i.e.,

$$\sigma = [\sigma(1), \ldots, \sigma(n)].$$

Thus, if a list of distinct numbers is surrounded by parentheses then it is a permutation in cycle notation, and if the numbers $1, 2, \ldots, n$ are in any order and surrounded by brackets then it is a permutation in one-line notation. We multiply permutations from right to left. Also, set $\mathcal{P}_\sigma = (p_{ij})$, with $p_{ij} = [\![j = \sigma(i)]\!]$. This is the familiar representation of S_n as permutation matrices.

One permutation we have already defined is $\pi_{\alpha,n}$. For notational convenience we set $\pi_{\alpha,n}(0) := 0$ and $\pi_{\alpha,n}(n+1) := n+1$. Also, set $P_j := \{j\alpha\}$ for $0 \le j \le n$, and set $P_{n+1} := 1$. Thus

$$0 = P_{\pi_{\alpha,n}(0)} < P_{\pi_{\alpha,n}(1)} < P_{\pi_{\alpha,n}(2)} < \cdots < P_{\pi_{\alpha,n}(n)} < P_{\pi_{\alpha,n}(n+1)} = 1.$$

We write $\vec{e}_i$ $(1 \le i \le n)$ be the n-dimensional column vector with every component 0 *except* the i-th component, which is 1. We set $\vec{e}_{n+1} = \vec{0}$, the n-dimensional 0 vector. We denote the identity matrix as $\mathcal{I} := (\vec{e}_1, \vec{e}_2, \ldots, \vec{e}_n)$. Let $\vec{\delta}_i := \vec{e}_{i+1} - \vec{e}_i$ $(1 \le i \le n)$. In particular, $\vec{\delta}_n = -\vec{e}_n$.

We will also use the notation $h(\vec{v})$ for the Hamming weight of the $\{0,1\}$-vector $\vec{v}$, i.e., the number of 1's.

Set

$$D(\sigma) := \{1\} \cup \{k\colon \sigma^{-1}(k-1) > \sigma^{-1}(k)\}.$$

In other words, $D(\sigma)$ consists of those k for which $k-1$ does not occur before k in $[\sigma(1), \sigma(2), \ldots, \sigma(n)]$. For example, $D([1,3,5,4,2,6]) = \{1,3,5\}$.

Set

$$\vec{w}_1^\sigma := \sum_{i \in D(\sigma)} \vec{e}_i$$

and for $1 \le j \le n$, set

$$\vec{w}_{j+1}^\sigma := \vec{w}_j^\sigma + \vec{\delta}_{\sigma(j)}.$$

We now define two matrices: the $n \times (n+1)$ matrix

$$\mathcal{L}_\sigma := (\vec{w}_1^\sigma, \vec{w}_2^\sigma, \ldots, \vec{w}_n^\sigma, \vec{w}_{n+1}^\sigma),$$

and the square $n \times n$ matrix

$$\mathcal{M}_\sigma := (\vec{w}_1^\sigma - \vec{w}_{n+1}^\sigma, \vec{w}_2^\sigma - \vec{w}_{n+1}^\sigma, \ldots, \vec{w}_n^\sigma - \vec{w}_{n+1}^\sigma).$$

Proposition 2.3.1 below shows that $\mathcal{M}_n(\alpha) = \mathcal{M}_{\pi_{\alpha,n}}$, justifying our definitions.

Example: Set $n = 5$ and $\sigma = [5,2,3,1,4] = (1,5,4)(2)(3)$. We find that $D(\sigma) = \{1,2,5\}$, and so $\vec{w}_1^\sigma = \vec{e}_1 + \vec{e}_2 + \vec{e}_5$. By definition $\vec{w}_2^\sigma = \vec{w}_1^\sigma + \vec{\delta}_{\sigma(1)} = \vec{w}_1^\sigma + \vec{e}_6 - \vec{e}_5 = \vec{e}_1 + \vec{e}_2$, and so on. Thus

$$\mathcal{L}_\sigma = \begin{pmatrix} 1 & 1 & 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \quad \text{and} \quad \mathcal{M}_\sigma = \begin{pmatrix} 1 & 1 & 1 & 1 & 0 \\ 0 & 0 & -1 & -1 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & -1 & -1 & -1 & -1 \end{pmatrix}.$$

Note that the first column of $\mathcal{M}_\sigma$ is $\vec{e}_1$; that this is always the case is proven in Lemma 2.4.2. Further, the second column of $\mathcal{M}_\sigma$ is $\vec{e}_1 + \vec{\delta}_{\sigma(1)}$, the third is $\vec{e}_1 + \vec{\delta}_{\sigma(1)} + \vec{\delta}_{\sigma(2)}$, and so forth. This pattern holds in general and is proved in Lemma 2.4.3 below. It is not immediate from the definitions that $\mathcal{L}_\sigma$ is always a $\{0,1\}$-matrix or that $\mathcal{M}_\sigma$ is a $\{-1,0,1\}$-matrix; we prove this in Lemma 2.4.4.

In Lemma 2.4.5 we prove that if $\sigma \neq \tau$ then $\mathcal{M}_\sigma \neq \mathcal{M}_\tau$. The proof relies on reconstructing σ and $\mathcal{L}_\sigma$ from $\mathcal{M}_\sigma$. This reconstruction proceeds as follows. The '-1' entries of $\mathcal{M}_\sigma$ are in the second and fifth rows; this gives $\vec{w}_6^\sigma = \vec{e}_2 + \vec{e}_5$, which is the last column of $\mathcal{L}_\sigma$. In fact, the j-th column of $\mathcal{L}_\sigma$ is the j-th column of $\mathcal{M}_\sigma$ plus $\vec{e}_2 + \vec{e}_5$. Once we know the columns of $\mathcal{L}_\sigma := (\vec{w}_1^\sigma, \vec{w}_2^\sigma, \ldots, \vec{w}_6^\sigma)$, we can use the definition of $\vec{w}_{j+1}^\sigma$ to find $\sigma(j)$. For example, $\vec{\delta}_{\sigma(4)} = \vec{w}_5^\sigma - \vec{w}_4^\sigma = \vec{e}_2 - \vec{e}_1 = \vec{\delta}_1$, and so $\sigma(4) = 1$.

Lemma 2.4.6 generalizes the observation that

$$\mathcal{M}_{[1,2,4,3,5]} = \mathcal{M}_{(4,3)} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 & 1 \end{pmatrix} = \mathcal{I} + \mathcal{V}_4.$$

With $\phi = \frac{\sqrt{5}-1}{2}$, we compute that $\pi_{\phi,5} = [5,2,4,1,3] = (1,5,3,4)(2)$, and one may directly verify that $\mathcal{M}_{\pi_{\phi,5}} = \mathcal{M}_\phi(5)$. This is no accident, by Proposition 2.3.1 below $\mathcal{M}_\alpha(n) = \mathcal{M}_{\pi_{\alpha,n}}$ for all α and n. The equation

$$\mathcal{M}_{(4,3)}\mathcal{M}_{\pi_{\phi,5}} = \mathcal{M}_{(4,3)(1,5,3,4)(2)} = \mathcal{M}_{(1,5,4)(2)(3)},$$

which is the same as

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 & 1 & 1 & 0 \\ 0 & 0 & -1 & -1 & 0 \\ 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & -1 & -1 \\ 0 & -1 & -1 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1 & 1 & 0 \\ 0 & 0 & -1 & -1 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & -1 & -1 & -1 & -1 \end{pmatrix},$$

is an example of the isomorphism of Proposition 2.4.1. □

2.3 The Matrices $\mathcal{M}_n(\alpha)$ and $\mathcal{M}_{\pi_{\alpha,n}}$

Proposition 2.3.1. $\mathcal{M}_n(\alpha) = \mathcal{M}_{\pi_{\alpha,n}}$.

Proof. For brevity, we write π in place of $\pi_{\alpha,n}$. First observe that $\vec{w}_1^\pi, \vec{w}_2^\pi, \ldots, \vec{w}_{n+1}^\pi$ are in anti-lexicographic order by definition, and so for $1 \le i < j \le n+1$ we have $\vec{w}_i^\pi \ne \vec{w}_j^\pi$. We know from [Lot02, Proposition 2.1.18] that $F_n(\alpha) = F_n(s_{\alpha,\beta})$ for every β, and from [Lot02, Theorem 2.1.13] that $|F_n(\alpha)| = n+1$. Thus, it suffices to show that $\vec{w}_j^\pi \in F_n(s_{\alpha,\beta})$ for some β. In fact, we shall show that

$$\big(s_{\alpha,\beta_j}(1), s_{\alpha,\beta_j}(2), \ldots, s_{\alpha,\beta_j}(n)\big) = \vec{w}_j^\pi$$

with $\beta_j := -P_1 - P_{\pi(j)}$.

Using the identities $\lfloor x \rfloor = x - \{x\}$ and $\{x - y\} = \{x\} - \{y\} + [\![\{x\} < \{y\}]\!]$, we have

$$\begin{aligned} s_{\alpha,\beta_j}(i) &= \big\lfloor (i+1)\alpha - P_1 - P_{\pi(j)} \big\rfloor - \big\lfloor i\alpha - P_1 - P_{\pi(j)} \big\rfloor \\ &= \alpha - P_i + P_{i-1} - [\![P_i < P_{\pi(j)}]\!] + [\![P_{i-1} < P_{\pi(j)}]\!] \\ &= [\![P_i < P_{i-1}]\!] - [\![P_i < P_{\pi(j)}]\!] + [\![P_{i-1} < P_{\pi(j)}]\!]. \end{aligned} \tag{1}$$

The last equality follows from the knowledge that $s_{\alpha,\beta_j}(i) \in \mathbb{Z}$, and consequently if $P_i < P_{i-1}$ then $\alpha - P_i + P_{i-1} > \alpha > 0$ must in fact be 1, and if $P_i > P_{i-1}$ then $\alpha - P_i + P_{i-1} < \alpha < 1$ must in fact be 0.

We first consider $j = 1$. We have $P_1 > P_0$, $P_1 \ge P_{\pi(1)}$, and $P_0 < P_{\pi(1)}$, whence $s_{\alpha,\beta_1}(1) = 1 = [\![1 \in D(\pi)]\!]$. For $2 \le i \le n$, we have $P_i \ge P_{\pi(1)}$ and $P_{i-1} \ge P_{\pi(1)}$, whence

$$s_{\alpha,\beta_1}(i) = [\![P_i < P_{i-1}]\!] = [\![\pi^{-1}(i) < \pi^{-1}(i-1)]\!] = [\![i \in D(\pi)]\!].$$

Therefore, $(s_{\alpha,\beta_1}(1), s_{\alpha,\beta_1}(2), \ldots, s_{\alpha,\beta_1}(n)) = \vec{w}_1^\pi$.

Now suppose that $2 \le j \le n+1$. Since $\vec{w}_j^\pi$ is defined by $\vec{w}_j^\pi - \vec{w}_{j-1}^\pi = \vec{\delta}_{\pi(j-1)} = \vec{e}_{\pi(j-1)+1} - \vec{e}_{\pi(j-1)}$, we need to show that $s_{\alpha,\beta_j}(i) - s_{\alpha,\beta_{j-1}}(i) = [\![i = \pi(j-1)+1]\!] - [\![i = \pi(j-1)]\!]$. By Eq. (1), we have

$$\begin{aligned} s_{\alpha,\beta_j}(i) - s_{\alpha,\beta_{j-1}}(i) &= -[\![P_i < P_{\pi(j)}]\!] + [\![P_{i-1} < P_{\pi(j)}]\!] + [\![P_i < P_{\pi(j-1)}]\!] - [\![P_{i-1} < P_{\pi(j-1)}]\!] \\ &= \big([\![P_{i-1} < P_{\pi(j)}]\!] - [\![P_{i-1} < P_{\pi(j-1)}]\!]\big) - \big([\![P_i < P_{\pi(j)}]\!] - [\![P_i < P_{\pi(j-1)}]\!]\big) \\ &= [\![i - 1 = \pi(j-1)]\!] - [\![i = \pi(j-1)]\!]. \end{aligned}$$

■

We remark that in the same manner one may prove that if

$$1 - \{\pi_{\alpha,n}(j)\alpha\} \le \{i\alpha + \beta\} < 1 - \{\pi_{\alpha,n}(j-1)\alpha\},$$

then

$$(s_{\alpha,\beta}(i), s_{\alpha,\beta}(i+1), \ldots, s_{\alpha,\beta}(i+n-1)) = \vec{w}_j^{\pi_{\alpha,n}}.$$

From this it is easy to prove that $F_n(s_{\alpha,\beta})$ does not depend on β and has cardinality $n+1$, the two results of [Lot02] that we used. There is another fact that can be proved in this manner that we do not use explicitly but which may help the reader develop an intuitive understanding of the simplices $F_n(W)$. If r, r' are consecutive Farey fractions of order $n+1$ and $r < \alpha < \gamma < r'$, then $F_n(\alpha) = F_n(\beta)$.

2.4 A Curious Representation

Proposition 2.4.1. *The map $\sigma \mapsto \mathcal{M}_\sigma$ is an isomorphism.*

The proof we give of this is more a verification than an explanation. We remark that there are several anti-isomorphisms involved in our choices. We have chosen to multiply permutations right-to-left rather than left-to-right. We have chosen to consider the list $[a, b, c, \dots]$ as the permutation taking 1 to a, 2 to b, 3 to c, etc., rather than the permutation taking a to 1, b to 2, c to 3, etc. Finally, we have chosen to define the vectors $\vec{w}^\sigma$ to be columns rather than rows. If we were to change any two of these conventions, then we would still get an isomorphism.

We begin with some simple observations about $\mathcal{M}_\sigma$ before proving Proposition 2.4.1. We defined $\vec{w}^\sigma_{j+1} := \vec{w}^\sigma_j + \vec{\delta}_{\sigma(j)}$. An easy inductive consequence of this definition is that $\vec{w}^\sigma_{j+1} = \vec{w}^\sigma_1 + \sum_{i=1}^{j} \vec{\delta}_{\sigma(i)}$ for $1 \le j \le n$; we use this repeatedly and without further fanfare.

Lemma 2.4.2. $\vec{w}^\sigma_1 - \vec{w}^\sigma_{n+1} = \vec{e}_1$.

Proof. Since $\vec{w}^\sigma_{n+1} = \vec{w}^\sigma_1 + \sum_{i=1}^{n} \vec{\delta}_{\sigma(i)}$, all we need to show is that $\sum_{i=1}^{n} \vec{\delta}_{\sigma(i)} = -\vec{e}_1$. As $\{1, 2, \dots, n\} = \{\sigma(1), \sigma(2), \dots, \sigma(n)\}$, we have

$$\vec{w}^\sigma_{n+1} - \vec{w}^\sigma_1 = \sum_{i=1}^{n} \vec{\delta}_{\sigma(i)} = \sum_{i=1}^{n} \vec{\delta}_i = \sum_{i=1}^{n} (\vec{e}_{i+1} - \vec{e}_i) = \vec{e}_{n+1} - \vec{e}_1 = -\vec{e}_1. \quad \blacksquare$$

Lemma 2.4.3. *The j-th column of $\mathcal{M}_\sigma$ is $\vec{e}_1 + \sum_{i=1}^{j-1} \vec{\delta}_{\sigma(i)}$.*

Proof. The j-th column of $\mathcal{M}_\sigma$ is defined as $\vec{w}^\sigma_j - \vec{w}^\sigma_{n+1}$. If $j = 1$, then this lemma reduces to Lemma 2.4.2. If $j > 1$, then Lemma 2.4.2 gives

$$\vec{w}^\sigma_j - \vec{w}^\sigma_{n+1} = \left(\vec{w}^\sigma_1 + \sum_{i=1}^{j-1} \vec{\delta}_{\sigma(i)} \right) - (\vec{w}^\sigma_1 - \vec{e}_1) = \vec{e}_1 + \sum_{i=1}^{j-1} \vec{\delta}_{\sigma(i)}. \quad \blacksquare$$

Lemma 2.4.4. *The entries of $\mathcal{L}_\sigma$ are 0 and 1. The entries of $\mathcal{M}_\sigma$ are -1, 0, and 1.*

Proof. It is easily seen from Lemma 2.4.3 that $\mathcal{M}_\sigma$ is a $\{-1, 0, 1\}$-matrix, and this also follows from the more subtle observation that $\mathcal{L}_\sigma$ is a $\{0, 1\}$-matrix. To prove that $\mathcal{L}_\sigma$ is a $\{0, 1\}$-matrix is the same as showing that each $\vec{w}^\sigma_j$ $(1 \le j \le n+1)$ is a $\{0, 1\}$-vector. This is obvious for $\vec{w}^\sigma_1 := \sum_{i \in D(\sigma)} \vec{e}_i$. We have $\vec{w}^\sigma_{j+1} = \vec{w}^\sigma_1 + \sum_{i=1}^{j} \vec{\delta}_{\sigma(i)}$; define $\vec{v}, c_i$ by $\vec{v} := \sum_{i=1}^{j} \vec{\delta}_{\sigma(i)} = \sum_{i=1}^{n} c_i \vec{e}_i$.

The i-th component of $\vec{v}$ can only be affected by $\vec{\delta}_{i-1}$ (which adds 1 to the i-th component) and $\vec{\delta}_i$ (which subtracts 1). It is thus clear that c_i is (-1), (0), (0), or (1) depending, respectively, on whether (only i), ($i-1$ and i), (neither $i-1$ nor i), or (only $i-1$) is among $\{\sigma(1), \sigma(2), \dots, \sigma(j)\}$. To show that $\vec{w}^\sigma_{j+1} = \vec{w}^\sigma_1 + \vec{v}$ is a $\{0, 1\}$-vector, we need to show two things. First, if $c_i = -1$ then $i \in D(\sigma)$ (and so the i-th component of $\vec{w}^\sigma_1$ is 1). Second, if $c_i = 1$ then $i \notin D(\sigma)$ (and so the i-th component of $\vec{w}^\sigma_1$ is 0).

If $c_i = -1$, then i is and $i-1$ is not among $\{\sigma(1), \dots, \sigma(j)\}$. This means that $\sigma^{-1}(i-1) > j \ge \sigma^{-1}(i)$, and so by the definition of D, $i \in D(\sigma)$. If, on the other hand, $c_i = 1$, then $i-1$ is and i is not among $\{\sigma(1), \dots, \sigma(j)\}$. This means that $\sigma^{-1}(i-1) \le j < \sigma^{-1}(i)$, and so by the definition of D, $i \notin D(\sigma)$. $\blacksquare$

Lemma 2.4.5. *The map $\sigma \mapsto \mathcal{M}_\sigma$ is 1-1.*

Proof. Given $\mathcal{M}_\sigma$, we find σ. We first note that moving from $\mathcal{L}_\sigma$ to σ is easy since $\vec{w}^\sigma_{j+1} - \vec{w}^\sigma_j = \vec{\delta}_{\sigma(j)}$. Some effort is involved in finding $\mathcal{L}_\sigma$ from $\mathcal{M}_\sigma$. We need to find $\vec{w}^\sigma_{n+1}$.

Every row of $\mathcal{L}_\sigma$ contains at least one 0 (explanation below). Thus $\mathcal{M}_\sigma$ will contain a -1 in exactly those rows in which $\vec{w}^\sigma_{n+1}$ contains a 1, and we are done. We have $\vec{w}^\sigma_{j+1} = \vec{w}^\sigma_j + \vec{\delta}_{\sigma(j)} = \vec{w}^\sigma_j + \vec{e}_{\sigma(j)+1} - \vec{e}_{\sigma(j)}$. The $\sigma(j)$-th row of $\vec{e}_{\sigma(j)+1}$ is 0, and $\vec{w}^\sigma_j, \vec{e}_{\sigma(j)}$ are $\{0, 1\}$-vectors, so the $\sigma(j)$-th row of $\vec{w}^\sigma_{j+1}$ is either 0 or -1. But by Lemma 2.4.4, $\mathcal{L}_\sigma$ is a $\{0, 1\}$-matrix, whence the $\sigma(j)$-th row of the $(j+1)$-st column of $\mathcal{L}_\sigma$ is 0. $\blacksquare$

Recall that $\mathcal{V}_k$ is defined to be the matrix all of whose entries are 0, save the k-th column, which is $\vec{e}_{k-1} - 2\vec{e}_k + \vec{e}_{k+1}$.

Lemma 2.4.6. *$\mathcal{M}_{(k,k-1)} = \mathcal{I} + \mathcal{V}_k$, for $2 \le k \le n$.*

Proof. Follow the definitions. With $\sigma = (k, k-1)$, we have $D(\sigma) = \{1, k\}$, $\vec{w}_j^{(k,k-1)} = \vec{e}_j + \vec{e}_k$ for $n \ge j \ne k$, $\vec{w}_k^{(k,k-1)} = \vec{e}_{k-1} + \vec{e}_{k+1}$ and $\vec{w}_{n+1}^{(k,k-1)} = \vec{e}_k$. ∎

We have laid the necessary groundwork, and turn now to proving Proposition 2.4.1.

Proof. We have already seen in Lemma 2.4.5 that the map $\sigma \mapsto \mathcal{M}_\sigma$ is 1-1; all that remains is to show that this map respects multiplication, i.e., for any $\sigma, \tau \in S_n$, $\mathcal{M}_\tau \mathcal{M}_\sigma = \mathcal{M}_{\tau\sigma}$. Since we may write τ as a product of transpositions of the form $(k, k-1)$ it is sufficient to show that $\mathcal{M}_{(k,k-1)}\mathcal{M}_\sigma = \mathcal{M}_{(k,k-1)\sigma}$ for every k ($2 \le k \le n$) and $\sigma \in S_n$.

We need to split the work into two cases: $\sigma^{-1}(k-1) < \sigma^{-1}(k)$ and $\sigma^{-1}(k-1) > \sigma^{-1}(k)$. In each case we first describe the rows of $\mathcal{M}_{(k,k-1)}\mathcal{M}_\sigma - \mathcal{M}_\sigma$ using Lemma 2.4.6, and then compute the columns of $\mathcal{M}_{(k,k-1)\sigma} - \mathcal{M}_\sigma$ from the definition. We will find that $\mathcal{M}_{(k,k-1)}\mathcal{M}_\sigma - \mathcal{M}_\sigma = \mathcal{M}_{(k,k-1)\sigma} - \mathcal{M}_\sigma$ in each case, which concludes the proof. Since the two cases are handled similarly, we present only the first case.

Suppose that $\sigma^{-1}(k-1) < \sigma^{-1}(k)$. By Lemma 2.4.6, $\mathcal{M}_{(k,k-1)}\mathcal{M}_\sigma - \mathcal{M}_\sigma = \mathcal{V}_k\mathcal{M}_\sigma$. The matrix $\mathcal{V}_k$ is zero except in the $(k-1, k)$, (k, k), and $(k+1, k)$ positions (note: we sometimes refer to positions which do not exist for $k = n$; the reader may safely ignore this detail), where it has value 1, -2, 1, respectively. Thus $\mathcal{V}_k\mathcal{M}_\sigma$ is zero except in the $(k-1)$-st and $(k+1)$-st rows (which are the same as the k-th row of $\mathcal{M}_\sigma$), and the k-th row (which is -2 times the k-th row of $\mathcal{M}_\sigma$).

We now describe the k-th row of $\mathcal{L}_\sigma$. By the hypothesis of this case, $k \notin D(\sigma)$, so the k-th row of $\vec{w}_1^\sigma$ is 0. Since $\vec{w}_{j+1}^\sigma = \vec{w}_1^\sigma + \sum_{i=1}^j \vec{\delta}_{\sigma(i)}$, the k-th row of $\vec{w}_j^\sigma$ is 0 for $1 \le j \le \sigma^{-1}(k-1)$, is 1 for $\sigma^{-1}(k-1) < j \le \sigma^{-1}(k)$, and is 0 for $\sigma^{-1}(k) < j \le n+1$. This gives the k-th row of $\mathcal{M}_\sigma$ as $\sigma^{-1}(k-1)$ '0's followed by $\sigma^{-1}(k) - \sigma^{-1}(k-1)$ '1's, followed by $n - \sigma^{-1}(k)$ '0's.

We now compute $\mathcal{M}_{(k,k-1)\sigma} - \mathcal{M}_\sigma$. The columns of $\mathcal{L}_{(k,k-1)\sigma}$ are given by $\vec{w}_{j+1}^{(k,k-1)\sigma} = \vec{w}_1^{(k,k-1)\sigma} + \sum_{i=1}^j \vec{\delta}_{(k,k-1)\sigma(i)}$ (except the first, but the first column of $\mathcal{M}_\tau$ is $\vec{e}_1$, independent of τ). Now $(k, k-1)\sigma(i) = \sigma(i)$ for $i \notin \{\sigma^{-1}(k), \sigma^{-1}(k-1)\}$, so that $\sum_{i=1}^j \vec{\delta}_{(k,k-1)\sigma(i)} = \sum_{i=1}^j \vec{\delta}_{\sigma(i)}$ for $j < \sigma^{-1}(k-1)$ and for $j \ge \sigma^{-1}(k)$. For $\sigma^{-1}(k-1) \le j < \sigma^{-1}(k)$,

$$\sum_{i=1}^j \vec{\delta}_{(k,k-1)\sigma(i)} = \left(\sum_{i=1}^j \vec{\delta}_{\sigma(i)}\right) - \vec{\delta}_{k-1} + \vec{\delta}_k.$$

Thus the $(j+1)$-st column of $\mathcal{M}_{(k,k-1)\sigma} - \mathcal{M}_\sigma$ is

$$\left(\vec{w}_{j+1}^{(k,k-1)\sigma} - \vec{w}_{n+1}^{(k,k-1)\sigma}\right) - \left(\vec{w}_{j+1}^\sigma - \vec{w}_{n+1}^\sigma\right) = \sum_{i=1}^j \vec{\delta}_{(k,k-1)\sigma(i)} - \sum_{i=1}^j \vec{\delta}_{\sigma(i)},$$

which is $\vec{0}$ for $j < \sigma^{-1}(k-1)$ and for $j \ge \sigma^{-1}(k)$, and $-\vec{\delta}_{k-1} + \vec{\delta}_k = \vec{e}_{k-1} - 2\vec{e}_k + \vec{e}_{k+1}$ for $\sigma^{-1}(k-1) \le j < \sigma^{-1}(k)$. We have shown that $\mathcal{M}_{(k,k-1)}\mathcal{M}_\sigma - \mathcal{M}_\sigma = \mathcal{M}_{(k,k-1)\sigma} - \mathcal{M}_\sigma$ in the case $\sigma^{-1}(k-1) < \sigma^{-1}(k)$. ∎

2.5 Proof of Theorem 1.1

Proof of Theorem 1.1. The volume of the n-dimensional simplex whose vertices have coordinates $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_{n+1}$ is $\frac{1}{n!}$ times the absolute value of the determinant of the matrix

$$(\vec{v}_1 - \vec{v}_{n+1}, \vec{v}_2 - \vec{v}_{n+1}, \ldots, \vec{v}_n - \vec{v}_{n+1}).$$

In our case, this means that the volume of the simplex $F_n(\alpha)$ is $\frac{1}{n!}\,|\det(\mathcal{M}_n(\alpha))|$. We will show that $\det(\mathcal{M}_n(\alpha)) = \pm 1$.

Now $\mathcal{M}_n(\alpha) = \mathcal{M}_{\pi_{\alpha,n}}$ by Proposition 2.3.1, and for any integer t we have $\left(\mathcal{M}_{\pi_{\alpha,n}}\right)^t = \mathcal{M}_{\pi^t_{\alpha,n}}$ by Proposition 2.4.1. Since $\pi_{\alpha,n} \in S_n$, a finite group, there is a positive integer t such that $\pi^t_{\alpha,n}$ is the identity permutation (which we denote by `id`). Thus

$$(\det \mathcal{M}_n(\alpha))^t = \left(\det \mathcal{M}_{\pi_{\alpha,n}}\right)^t = \det\left(\mathcal{M}^t_{\pi_{\alpha,n}}\right) = \det\left(\mathcal{M}_{\pi^t_{\alpha,n}}\right) = \det\left(\mathcal{M}_{\mathtt{id}}\right) = \det \mathcal{I} = 1.$$

Consequently, $\det \mathcal{M}_n(\alpha)$ is a t-th root of unity, and since the entries of $\mathcal{M}_n(\alpha)$ are integers, $\det \mathcal{M}_n(\alpha) = \pm 1$. ■

2.6 The Character of the Representation

We review the needed facts and definitions from the representation theory of finite groups (an excellent introduction is [Sag01]). A *representation* of a finite group G is a homomorphism $\mathcal{R} : G \to SL_m(\mathbb{C})$ for some $m \geq 1$. The representation is said to be *faithful* if the homomorphism is in fact an isomorphism. Thus, Proposition 2.4.1 implies that $\{\mathcal{M}_\sigma \colon \sigma \in S_n\}$ is a faithful representation of S_n. The *character* of $\mathcal{R}$ is the map $g \mapsto tr(\mathcal{R}(g))$, with tr being the trace function. We will use Corollary 1.9.4(5) of [Sag01], which states that if two representations $\mathcal{R}_1, \mathcal{R}_2$ of G have the same character, then they are similar, i.e., there is a matrix $\mathcal{Q}$ such that $\forall g \in G \left(\mathcal{Q}^{-1}\mathcal{R}_1(g)\mathcal{Q} = \mathcal{R}_2(g)\right)$. Any such matrix $\mathcal{Q}$ is called an *intertwining matrix* for the representations $\mathcal{R}_1, \mathcal{R}_2$.

Proposition 2.6.1. *The representations $\{\mathcal{P}_\sigma \colon \sigma \in S_n\}$ and $\{\mathcal{M}_\sigma \colon \sigma \in S_n\}$ are similar.*

Proof. The character of $\{\mathcal{P}_\sigma \colon \sigma \in S_n\}$ is obviously given by $tr(\mathcal{P}_\sigma) = \#\{i \colon \sigma(i) = i\}$. We will show that $tr(\mathcal{M}_\sigma) = \#\{i \colon \sigma(i) = i\}$ also, thereby establishing that the representations are similar.

We first note that $tr(\mathcal{M}_\sigma) = tr(\mathcal{L}_\sigma) - h(\vec{w}^\sigma_{n+1})$, since every "1" in $\vec{w}^\sigma_{n+1}$ is subtracted from exactly one diagonal position when we form $\mathcal{M}_\sigma$ from $\mathcal{L}_\sigma$ by subtracting $\vec{w}^\sigma_{n+1}$ from each column. We show below that

$$tr(\mathcal{L}_\sigma) = h(\vec{w}^\sigma_1) - 1 + \#\{i \colon \sigma(i) = i\}, \tag{2}$$

so by Lemma 2.4.2 we have

$$tr(\mathcal{M}_\sigma) = h(\vec{w}^\sigma_1) - 1 + \#\{i \colon \sigma(i) = i\} - h(\vec{w}^\sigma_{n+1}) = \#\{i \colon \sigma(i) = i\},$$

which will conclude the proof.

Until this point we have found it convenient to think of $\mathcal{L}_\sigma$ in terms of its columns. That is not the natural viewpoint to take in proving Eq. (2), however. The difference between the $(j+1)$-st and j-th columns of $\mathcal{L}_\sigma$ is $\vec{\delta}_{\sigma(j)} = \vec{e}_{i+1} - \vec{e}_i$; we think of this relationship as a "1" moving down from the i-th row to the $(i+1)$-st row. In looking at the matrix $\mathcal{L}_\sigma$ we see each "1" in the first column continues across to the east, occasionally moving down a row (southeast), or even 'off' the bottom of the matrix. We call the path a "1" takes a *snake*.

Example: With $\sigma = [1,4,5,8,2,3,9,6,10,7]$, we find

$$\mathcal{L}_\sigma = \begin{pmatrix} \mathbf{1} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \mathbf{1} & \mathbf{1} & \mathbf{1} & \mathbf{1} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \mathbf{1} & 0 & 0 & 0 & 0 & 0 \\ \mathbf{1} & \mathbf{1} & 0 & 0 & 0 & 0 & \mathbf{1} & \mathbf{1} & \mathbf{1} & \mathbf{1} & \mathbf{1} \\ 0 & 0 & \mathbf{1} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \mathbf{1} & \mathbf{1} & \mathbf{1} & \mathbf{1} & \mathbf{1} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \mathbf{1} & \mathbf{1} & 0 \\ \mathbf{1} & \mathbf{1} & \mathbf{1} & \mathbf{1} & 0 & 0 & 0 & 0 & 0 & 0 & \mathbf{1} \\ 0 & 0 & 0 & 0 & \mathbf{1} & \mathbf{1} & \mathbf{1} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \mathbf{1} & \mathbf{1} & 0 & 0 \end{pmatrix}$$

The matrix $\mathcal{L}_\sigma$ has 3 snakes beginning in positions $(1,1)$, $(4,1)$, and $(8,1)$. The first snake occupies the positions $(1,1)$, $(2,2)$, $(2,3)$, $(2,4)$, $(2,5)$, $(3,6)$, $(4,7)$, $(4,8)$, $(4,9)$, $(4,10)$, and $(4,11)$. $\square$

Only the last snake moves off the bottom of $\mathcal{L}_\sigma$; after all, n only occurs once in a permutation. The other snakes, of which there are $h(\vec{w}_1^\sigma)-1$, begin on or below the diagonal and end on or above the diagonal. Thus each must intersect the diagonal at least once. Moreover, each fixed point of σ will keep a snake on a diagonal for an extra row. Thus, $tr(\mathcal{L}_\sigma) = h(\vec{w}_1^\sigma) - 1 + \#\{i \colon \sigma(i) = i\}$. $\blacksquare$

We turn now to identifying the intertwining matrices, i.e., the matrices $\mathcal{Q}$ such that $\mathcal{Q}^{-1}\mathcal{M}_\sigma\mathcal{Q} = \mathcal{P}_\sigma$ for every $\sigma \in S_n$.

Proposition 2.6.2. *The $n \times n$ matrix $\mathcal{Q} = (q_{ij})$ satisfies $\mathcal{Q}^{-1}\mathcal{M}_\sigma\mathcal{Q} = \mathcal{P}_\sigma$ for every $\sigma \in S_n$ iff there are complex numbers a, b with $(na+b)b^{n-1} \neq 0$ and $q_{11} = a+b$, $q_{1k} = a$, $q_{kk} = b$, $q_{k,k-1} = -b$ $(2 \le k \le n)$.*

Proof. We first note that it is sufficient to restrict σ to a generating set of S_n. To see this, let $S_n = \langle \sigma_1, \ldots, \sigma_r \rangle$. If $\mathcal{Q}$ satisfies $\mathcal{Q}^{-1}\mathcal{M}_\sigma\mathcal{Q} = \mathcal{P}_\sigma$ for every $\sigma \in S_n$, then clearly $\mathcal{Q}^{-1}\mathcal{M}_{\sigma_i}\mathcal{Q} = \mathcal{P}_{\sigma_i}$ $(1 \le i \le r)$. In the other direction, if $\mathcal{Q}$ satisfies $\mathcal{Q}^{-1}\mathcal{M}_{\sigma_i}\mathcal{Q} = \mathcal{P}_{\sigma_i}$ $(1 \le i \le r)$ and $\sigma = \sigma_{i_1}\sigma_{i_2}\ldots\sigma_{i_s}$, then

$$\mathcal{Q}^{-1}\mathcal{M}_\sigma\mathcal{Q} = \mathcal{Q}^{-1}\left(\mathcal{M}_{\sigma_{i_1}}\mathcal{M}_{\sigma_{i_2}}\ldots\mathcal{M}_{\sigma_{i_s}}\right)\mathcal{Q} = \prod_{j=1}^{s}\left(\mathcal{Q}^{-1}\mathcal{M}_{\sigma_{i_j}}\mathcal{Q}\right)$$

$$= \prod_{j=1}^{s}\mathcal{P}_{\sigma_{i_j}} = \mathcal{P}_{\prod_{j=1}^{s}\sigma_{i_j}} = \mathcal{P}_\sigma.$$

Thus we can restrict our attention to the transpositions $(k, k-1)$ $(2 \le k \le n)$. We identified $\mathcal{M}_{(k,k-1)}$ in Lemma 2.4.6 as $\mathcal{M}_{(k,k-1)} = \mathcal{I} + \mathcal{V}_k$, where $\mathcal{V}_k$ is the matrix all of whose entries are zero, save the k-th column, which is $\vec{e}_{k-1} - 2\vec{e}_k + \vec{e}_{k+1}$.

We suppose that $\mathcal{Q} = (q_{ij})$ satisfies $\mathcal{M}_{(k,k-1)}\mathcal{Q} = \mathcal{Q}\mathcal{P}_{(k,k-1)}$ to find linear constraints on the unknowns q_{ij}. We will find that these constraints (for $2 \le k \le n$) are equivalent to $q_{11} = a+b$, $q_{1k} = a$, $q_{kk} = b$, $q_{k,k-1} = -b$ $(2 \le k \le n)$. The determinant of $\mathcal{Q}$ is easily seen to be $(na+b)b^{n-1}$, so that as long as this is nonzero, $\mathcal{Q}^{-1}\mathcal{M}_\sigma\mathcal{Q} = \mathcal{P}_\sigma$ for every $\sigma \in S_n$.

We have $\mathcal{M}_{(k,k-1)}\mathcal{Q} = \mathcal{I}\mathcal{Q} + \mathcal{V}_k\mathcal{Q}$, so that $\mathcal{M}_{(k,k-1)}\mathcal{Q} = \mathcal{Q}\mathcal{P}_{(k,k-1)}$ is equivalent to $\mathcal{V}_k\mathcal{Q} = \mathcal{Q}(\mathcal{P}_{(k,k-1)} - \mathcal{I})$. This is a convenient form since most entries in the matrices $\mathcal{V}_k$ and $\mathcal{P}_{(k,k-1)} - \mathcal{I}$ are zero. The entries of the product $\mathcal{V}_k\mathcal{Q}$ are 0 except for the $(k-1)$-st, k-th, and $(k+1)$-st rows which are equal to the k-th row of $\mathcal{Q}$, to -2 times the k-th row of $\mathcal{Q}$, and to the k-th row of $\mathcal{Q}$, respectively. The entries of the product $\mathcal{Q}(\mathcal{P}_{(k,k-1)} - \mathcal{I})$ are 0 except for the $(k-1)$-st and k-th columns, which are equal to the k-th column of $\mathcal{Q}$

minus the $(k-1)$-st column of $\mathcal{Q}$, and to the $(k-1)$-st column of $\mathcal{Q}$ minus the k-th column of $\mathcal{Q}$, respectively. The entries which are zero in one matrix or other lead to the families of equations $q_{kj}=0$ (for $j\notin\{k-1,k\}$) and $q_{jk}=q_{j,k-1}$ (for $j\notin\{k-1,k,k+1\}$). The entries which are non-zero in both products give the six equations

$$\begin{pmatrix} q_{k,k-1} & q_{kk} \\ -2q_{k,k-1} & -2q_{kk} \\ q_{k,k-1} & q_{kk} \end{pmatrix} = \begin{pmatrix} q_{k-1,k}-q_{k-1,k-1} & q_{k-1,k-1}-q_{k-1,k} \\ q_{kk}-q_{k,k-1} & q_{k,k-1}-q_{kk} \\ q_{k+1,k}-q_{k+1,k-1} & q_{k+1,k-1}-q_{k+1,k} \end{pmatrix},$$

which are equivalent to $q_{k-1,k-1}=q_{k-1,k}+q_{k,k}$, $q_{k,k-1}=-q_{kk}$, and $q_{k+1,k-1}=q_{k+1,k}+q_{kk}$. Taking $q_{nn}=b$ and $q_{1n}=a$, the result follows. ■

Corollary 2.6.3.

$$\mathcal{M}_\sigma = \begin{pmatrix} 1 & & & & & \\ -1 & 1 & & & \mathbf{0} & \\ & -1 & 1 & & & \\ & & & \ddots & & \\ & \mathbf{0} & & & \ddots & \\ & & & & -1 & 1 \end{pmatrix} \cdot \mathcal{P}_\sigma \cdot \begin{pmatrix} 1 & & & & & \\ 1 & 1 & & & \mathbf{0} & \\ 1 & 1 & 1 & & & \\ \vdots & & & \ddots & & \\ \vdots & \mathbf{1} & & & \ddots & \\ & \dots & \dots & \dots & 1 & 1 \end{pmatrix}.$$

Proof. Set $a=0$, $b=1$ in Theorem 2.6.2. All that needs to be checked is that $\mathcal{Q}^{-1}$ is as claimed, i.e., the $n\times n$ matrix with "1"s on and below and the diagonal and "0"s above the diagonal. ■

We remark that, since it is easy to recover $\mathcal{L}_\sigma$ from $\mathcal{M}_\sigma$ (see the proof of Lemma 2.4.4) this Corollary gives a simple method for computing the factors of length n of a Sturmian word with slope α given only the permutation ordering $\{\alpha\},\dots,\{n\alpha\}$ (we don't even need α). Also, we note that the matrix with "1"s on and below the diagonal is a summation operator, and its inverse is a difference operator. If one could prove Corollary 2.6.3 directly, this would provide a second proof of Theorem 1.1.

2.7 The Simplex $F_n(W)$

Stolarsky & Porta [personal communication] observed experimentally that $\mathcal{M}_n(\alpha)$ has determinant ±1, and moreover that the roots of its characteristic polynomial are roots of unity. The first observation was proved in the course of the proof of Theorem 1.1 in Subsection 2.5. The second observation also follows from the fact that $\mathcal{M}_n(\alpha)$ lies in a finite group.

We now summarize the results of this paper as they relate to $\mathcal{M}_n(\alpha)$ and $F_n(\alpha)$.

Theorem 2.1. *Let $\alpha\in(0,1)$ be irrational, and $n\geq1$ an integer.*

i. *The volume of the simplex $F_n(\alpha)$ is $\frac{1}{n!}$.*

ii. $\det(\mathcal{M}_n(\alpha))=\pm1$.

iii. $\mathcal{M}_n(\alpha)=\mathcal{M}_{\pi_{\alpha,n}}=\mathcal{Q}\mathcal{P}_{\pi_{\alpha,n}}\mathcal{Q}^{-1}$ *(see Theorem 2.6.2 for the definition of $\mathcal{Q}$);*

iv. $\det(\mathcal{M}_{2n}(\alpha))=\det(\mathcal{M}_{2n+1}(\alpha))=\prod_{\ell=1}^{n}(-1)^{\lfloor 2\ell\alpha\rfloor}$.

v. *If $\pi_{\alpha,n}(n)=n$, then*

$$\operatorname{ord}(\mathcal{M}_{n-1}(\alpha))=\operatorname{ord}(\mathcal{M}_n(\alpha))=\operatorname{ord}(\pi_{\alpha,n}(1) \bmod n).$$

vi. If $\pi_{\alpha,n}(1) = n$*, then*

$$\operatorname{ord}(\mathcal{M}_{n-1}(\alpha)) = \operatorname{ord}(-\pi_{\alpha,n}(n) \bmod n)$$

and

$$\operatorname{ord}(\mathcal{M}_n(\alpha)) = \operatorname{ord}(-\pi_{\alpha,n}(n) \bmod gn),$$

where g *is the smallest positive integer such that* $\gcd\left(n, \frac{\pi_{\alpha,n}(n)+1}{g}\right) = 1$.

Items (i) and (ii) are proved in Subsection 2.5. Item (iii) is a combination of Theorem 2.6.2 and Proposition 2.3.1. Items (iv), (v) and (vi) are immediate consequences of Proposition 2.3.1, the facts $\det(\mathcal{M}_\sigma) = \operatorname{sgn}(\sigma)$ and $\operatorname{ord}(\mathcal{M}_\sigma) = \operatorname{ord}(\sigma)$, and Theorem 3.1. They are included here for the purpose of listing everything known about $\mathcal{M}_n(\alpha)$ in one place.

3 Ordering Fractional Parts

3.1 Statement of Results

Table 1 gives the sign and multiplicative order of $\pi_{e,n}$ for $2 \le n \le 136$. Visually inspecting the table, one quickly notices that $\operatorname{sgn}(\pi_{e,2n}) = \operatorname{sgn}(\pi_{e,2n+1})$ for all n, and that $\operatorname{ord}(\pi_{e,n})$ is surprisingly small for $n = 70, 71, 109, 110$. The first several convergents to e are $2, 3, \frac{8}{3}, \frac{11}{4}, \frac{19}{7}, \frac{87}{32}, \frac{106}{39}, \frac{193}{71}$; the value 71 is the denominator of a convergent, and $110 = 71 + 39$ is the sum of two denominators. Note also that for some values of n, $\operatorname{ord}(\pi_{e,n})$ is extraordinarily large, e.g., $\operatorname{ord}(\pi_{e,123}) = 22383900$. None of these observations are peculiar to the irrational e. Some of these observations are explained by Theorem 3.1 below, and the others remain conjectural.

As in Section 2, we make frequent use of Knuth's notation:

$$[\![Q]\!] = \begin{cases} 1 & Q \text{ is true;} \\ 0 & Q \text{ is false.} \end{cases}$$

Lemma 3.1.1, giving $\pi_{\alpha,n}$ in terms of only $\pi_{\alpha,n}(1)$, $\pi_{\alpha,n}(n)$, and n, is proved by V. T. Sós in [Sós57]. Her method of proof is similar to our proof of Lemma 3.3.1 below. The lemma is also proved—in terse English—in [Sla67]. We will derive Theorem 3.1 from Sós's Lemma.

Lemma 3.1.1 (Sós). *Let* α *be irrational,* n *a positive integer, and* $\pi = \pi_{\alpha,n}$*. Then*

$$\pi(k+1) = \pi(k) + \pi(1)\,[\![\pi(k) \le \pi(n)]\!] - \pi(n)\,[\![n < \pi(1) + \pi(k)]\!]$$

for $1 \le k < n$.

The surprising Three-Distance Theorem is an easy corollary: If α is irrational, the $n+2$ points $0, \{\alpha\}, \{2\alpha\}, \ldots, \{n\alpha\}, 1$ divide the interval $[0,1]$ into $n+1$ subintervals which have at most 3 distinct lengths. Alessandri & Berthé [AB98] give an excellent and up-to-date survey of generalizations of the Three-Distance Theorem.

The primary goal of this section is to prove Theorem 3.1, which refines Theorem 1.2, and to prove Theorem 1.3. Corollary 3.2.1 is of independent interest.

Theorem 3.1. *Let* $\alpha \notin \mathbb{Q}$ *and* $n \in \mathbb{Z}^+$.

i. If $\pi_{\alpha,n}(n) = n$*, then* $\operatorname{ord}(\pi_{\alpha,n-1}) = \operatorname{ord}(\pi_{\alpha,n}) = \operatorname{ord}(\pi_{\alpha,n}(1) \bmod n)$.

ii. If $\pi_{\alpha,n}(1) = n$*, then* $\operatorname{ord}(\pi_{\alpha,n-1}) = \operatorname{ord}(-\pi_{\alpha,n}(n) \bmod n)$*, and*

$$\operatorname{ord}(\pi_{\alpha,n}) = \operatorname{ord}(-\pi_{\alpha,n}(n) \bmod gn),$$

where g *is the least positive integer such that* $\gcd\left(n, \frac{\pi_{\alpha,n}(n)+1}{g}\right) = 1$.

3.2 The Multiplicative Order of the Permutation

Theorem 3.1 follows from Sós's Lemma.

Proof of Theorem 3.1(i). Suppose that $\pi_{\alpha,n}(n) = n$. We have

$$\pi_{\alpha,n}(k) = \begin{cases} \pi_{\alpha,n-1}(k) & 1 \le k \le n-1; \\ k & k = n, \end{cases}$$

and so obviously $\operatorname{ord}(\pi_{\alpha,n-1}) = \operatorname{ord}(\pi_{\alpha,n})$. We show that the length of every orbit divides $\operatorname{ord}(\pi_{\alpha,n}(1) \bmod n)$, and that the length of the orbit of 1 is exactly $\operatorname{ord}(\pi_{\alpha,n}(1) \bmod n)$. From Sós's Lemma (Lemma 3.1.1), we have in this case for $1 \le k < n$ the congruence $\pi_{\alpha,n}(k+1) \equiv \pi_{\alpha,n}(k) + \pi_{\alpha,n}(1) \pmod n$. By induction, this gives $\pi_{\alpha,n}(k) \equiv k\pi_{\alpha,n}(1) \pmod n$ for $1 \le k \le n$. Thus, the orbit of the point k is

$$k,\ k\pi_{\alpha,n}(1),\ k\pi_{\alpha,n}(1)^2,\ k\pi_{\alpha,n}(1)^3,\ \ldots$$

The length of the orbit of k divides $\operatorname{ord}(\pi_{\alpha,n}(1) \bmod n)$, and in particular the orbit of 1 has length equal to $\operatorname{ord}(\pi_{\alpha,n}(1) \bmod n)$. ■

Proof of Theorem 3.1(ii). Suppose that $\pi_{\alpha,n}(1) = n$. Sós's Lemma gives

$$\pi_{\alpha,n}(k) \equiv (1-k)\pi_{\alpha,n}(n) \pmod n.$$

Table 1: Algebraic properties of $\pi_{\alpha,n}$ with $\alpha = e$ and $2 \le n \le 136$

n	$\operatorname{sgn}(\pi_{\alpha,n})$	$\operatorname{ord}(\pi_{\alpha,n})$	n	$\operatorname{sgn}(\pi_{\alpha,n})$	$\operatorname{ord}(\pi_{\alpha,n})$	n	$\operatorname{sgn}(\pi_{\alpha,n})$	$\operatorname{ord}(\pi_{\alpha,n})$
2	-1	2	47	-1	44	92	1	2107
3	-1	2	48	-1	540	93	1	13244
4	-1	2	49	-1	120	94	-1	18810
5	-1	4	50	1	680	95	-1	20034
6	-1	6	51	1	1848	96	-1	3348
7	-1	6	52	-1	50	97	-1	11256
8	1	7	53	-1	90	98	-1	1702
9	1	6	54	-1	962	99	-1	188
10	-1	10	55	-1	1848	100	1	957
11	-1	10	56	-1	588	101	1	2100
12	-1	12	57	-1	276	102	-1	102
13	-1	36	58	1	165	103	-1	2052
14	-1	40	59	1	1260	104	-1	1950
15	-1	14	60	-1	1848	105	-1	1260
16	1	15	61	-1	2040	106	-1	5964
17	1	3	62	-1	62	107	-1	13860
18	1	3	63	-1	15640	108	1	54366
19	1	15	64	1	2040	109	1	10
20	1	77	65	1	424	110	-1	10
21	1	12	66	-1	966	111	-1	2310
22	-1	12	67	-1	1476	112	-1	720
23	-1	12	68	-1	56	113	-1	3738
24	1	4	69	-1	232	114	1	1938
25	1	4	70	-1	14	115	1	112
26	1	36	71	-1	14	116	-1	92820
27	1	48	72	1	6840	117	-1	11220
28	1	6	73	1	406	118	-1	5520
29	1	24	74	-1	390	119	-1	60060
30	-1	210	75	-1	780	120	-1	14280
31	-1	4	76	-1	192	121	-1	1680
32	-1	4	77	-1	228	122	1	6240
33	-1	180	78	-1	32130	123	1	22383900
34	-1	210	79	-1	390	124	-1	820820
35	-1	420	80	1	630	125	-1	215460
36	1	120	81	1	72	126	-1	9360
37	1	37	82	1	2728	127	-1	17160
38	-1	12	83	1	6138	128	1	68640
39	-1	12	84	1	152	129	1	47888
40	-1	40	85	1	6669	130	-1	7276
41	-1	1980	86	-1	31920	131	-1	508
42	-1	414	87	-1	400	132	-1	6720
43	-1	42	88	1	192	133	-1	4914
44	1	580	89	1	14616	134	-1	1560
45	1	168	90	1	18585	135	-1	11752
46	-1	1120	91	1	25080	136	1	3045

For $1 \le k < n$ we have

$$\pi_{\alpha,n-1}(k) = \pi_{\alpha,n}(k+1) \equiv (1-(k+1))\pi_{\alpha,n}(n) = -k\pi_{\alpha,n}(n) \pmod{n}.$$

Thus for $r \ge 1$ we have $\pi^r_{\alpha,n-1}(k) \equiv k(-\pi_{\alpha,n}(n))^r \pmod{n}$. The length of the orbit of k under $\pi_{\alpha,n-1}$ divides $\mathrm{ord}(-\pi_{\alpha,n}(n) \bmod n)$, and in particular the orbit of 1 has length equal to $\mathrm{ord}(-\pi_{\alpha,n}(n) \bmod n)$. This proves that $\mathrm{ord}(\pi_{\alpha,n-1}) = \mathrm{ord}(-\pi_{\alpha,n}(n) \bmod n)$.

For notational convenience, set $x = -\pi_{\alpha,n}(n)$. Now from

$$\pi_{\alpha,n}(k) \equiv (1-k)\pi_{\alpha,n}(n) = (k-1)x \pmod{n}$$

it is readily seen by induction that for $R \ge 0$ we have

$$\pi^R_{\alpha,n}(k) \equiv kx^R - \left(x + x^2 + \cdots + x^R\right) \pmod{n}. \tag{3}$$

Let r be the least positive integer for which $\forall k \left(\pi^r_{\alpha,n}(k) = k\right)$, i.e., let r be the least common multiple of the length of the orbits of $\pi_{\alpha,n}$. We must show that $r = \mathrm{ord}(x \bmod gn)$.

Define the integer γ by $g\gamma = x - 1$, and note that $\gcd(\gamma, n) = 1$. We may rearrange Eq. (3), setting $R = r$, as

$$\frac{x^r - 1}{x - 1} \equiv (k-1)(x^r - 1) \pmod{n}, \tag{4}$$

the division being real, not modular. With $k = 1$, Eq. (4) becomes $0 \equiv \frac{x^r-1}{x-1} = \frac{x^r-1}{g\gamma} \pmod{n}$, which holds iff $\frac{x^r-1}{g} \equiv 0 \pmod{n}$. This, in turn, holds iff there is an integer β with $\beta n = \frac{x^r-1}{g}$, i.e., $\beta n g = x^r - 1$. Thus r is a multiple of $\mathrm{ord}(x \bmod gn)$, and in particular $r \ge \mathrm{ord}(x \bmod gn)$.

We claim that $\forall k \left(\pi^{\mathrm{ord}(x \bmod gn)}_{\alpha,n}(k) = k\right)$, so that $r \le \mathrm{ord}(x \bmod gn)$, which will conclude the proof. We have

$$\frac{x^{\mathrm{ord}(x \bmod gn)} - 1}{x - 1} = \frac{\beta gn}{g\gamma} = \frac{\beta n}{\gamma}$$

and

$$(k-1)(x^{\mathrm{ord}(x \bmod gn)} - 1) \equiv 0 \pmod{n},$$

so that substituting $r = \mathrm{ord}(x \bmod gn)$ into Eq. (4) we write $\frac{\beta n}{\gamma} \equiv 0 \pmod{n}$, which holds for all k since $\gcd(\gamma, n) = 1$. Thus $r \le \mathrm{ord}(x \bmod gn)$. ■

For quadratic irrationals one can easily identify the convergents and intermediate fractions and, if the denominators have enough structure, explicitly compute $\mathrm{ord}(\pi_{\alpha,n})$ when n or $n+1$ is such a denominator. We have for example

Corollary 3.2.1. *Let $\phi = \frac{\sqrt{5}-1}{2}$, and f_n be the n-th Fibonacci number. Then for $n \ge 2$,*

$$\mathrm{ord}(\pi_{\phi,-1+f_{2n}}) = \mathrm{ord}(\pi_{\phi,f_{2n}}) = 2$$

and

$$\mathrm{ord}(\pi_{\phi,-1+f_{2n+1}}) = \mathrm{ord}(\pi_{\phi,f_{2n+1}}) = 4.$$

Proof. From the continued fraction expansion of ϕ we know that $\pi_{\phi,f_{2n}}(f_{2n}) = f_{2n}$ and $\pi_{\phi,f_{2n}}(1) = f_{2n-1}$. The identity $f^2_{2n-1} - f_{2n}f_{2n-2} = 1$ shows that $\mathrm{ord}(f_{2n-1} \bmod f_{2n}) = 2$ for $n \ge 2$, whence by Theorem 3.1(i) we have $\mathrm{ord}(\pi_{\phi,-1+f_{2n}}) = \mathrm{ord}(\pi_{\phi,f_{2n}}) = 2$.

The continued fraction expansion of ϕ also tells us $\pi_{\phi,f_{2n+1}}(1) = f_{2n+1}$ and $\pi_{\phi,f_{2n+1}}(f_{2n+1}) = f_{2n}$. The identity $f^2_{2n} - f_{2n+1}f_{2n-1} = -1$ shows $\mathrm{ord}(-f_{2n} \bmod f_{2n+1}) = 4$ for $n \ge 2$, whence by Theorem 3.1(ii), $\mathrm{ord}(\pi_{\phi,-1+f_{2n+1}}) = 4$.

We defined g to be the least positive integer such that $\gcd\left(f_{2n+1}, \frac{f_{2n}+1}{g}\right) = 1$. In particular, $g | f_{2n}+1$, and so the identity

$$(-f_{2n})^4 - 1 = f_{2n-1}(f_{2n}-1)f_{2n+1}(f_{2n}+1) \equiv 0 \pmod{g f_{2n+1}}$$

shows that $\mathrm{ord}(-f_{2n} \bmod g f_{2n+1})$ divides 4, and the identity $(-f_{2n})^2 - f_{2n+1}f_{2n-1} = -1$ shows that

$$\mathrm{ord}(-f_{2n} \bmod g f_{2n+1}) \geq \mathrm{ord}(-f_{2n} \bmod f_{2n+1}) = 4,$$

whence $\mathrm{ord}(-f_{2n} \bmod g f_{2n+1}) = 4$ and by Theorem 3.1(ii), $\mathrm{ord}(\pi_{\phi, f_{2n+1}}) = 4$. ■

3.3 The Sign of the Permutation

Recall that a k-cycle is even exactly if k is odd. We define $\rho(n,k)$ to be the $(n-k)$-cycle

$$\rho(n,k) := (n, n-1, \ldots, k+1) = (n, n-1)(n-1, n-2) \cdots (k+2, k+1).$$

Define also

$$B_\alpha(k) := \#\{q \colon 1 \leq q < k,\ \{q\alpha\} < \{k\alpha\}\},$$

which counts the integers in $[1,k)$ that are 'better' denominators for approximating α from below. Clearly,

$$\begin{aligned} \pi_{\alpha,n} &= \pi_{\alpha,n-1}\, \rho(n, B_\alpha(n)) \\ &= \rho(1, B_\alpha(1))\, \rho(2, B_\alpha(2)) \ \ldots\ \rho(n, B_\alpha(n)) \\ &= \prod_{k=1}^{n} \rho(k, B_\alpha(k)) \end{aligned}$$

so that $\pi_{\alpha,n}$ is the product of $\sum_{k=1}^{n}(k - B_\alpha(k) - 1)$ transpositions. We will show that for k odd, $B_\alpha(k) \equiv 0 \pmod 2$, which will be used to demonstrate that $\mathrm{sgn}(\pi_{\alpha,2n}) = \mathrm{sgn}(\pi_{\alpha,2n+1})$.

Our proof of Lemma 3.3.1 is similar in spirit to Sós's proof of Lemma 3.1.1.

Lemma 3.3.1. *For $k \geq 3$ and $0 < \alpha < 1/2$, α irrational, $B_\alpha(k) + B_{1-\alpha}(k) = k-1$, and*

$$B_\alpha(k) - 2B_\alpha(k-1) + B_\alpha(k-2) = \begin{cases} 1-k, & \{k\alpha\} \in [0, \alpha); \\ k-1, & \{k\alpha\} \in [\alpha, 2\alpha); \\ 0, & \{k\alpha\} \in [2\alpha, 1). \end{cases}$$

Proof. Observe that $0 < \{q\alpha\} < \{k\alpha\}$ iff $\{k(1-\alpha)\} < \{q(1-\alpha)\} < 1$, so that q with $1 \leq q < k$ is in either the set $\{q : 1 \leq q < k, \{q\alpha\} < \{k\alpha\}\}$ or in the set $\{q : 1 \leq q < k, \{q(1-\alpha)\} < \{k(1-\alpha)\}\}$, and is not in both. Thus, $B_\alpha(k) + B_{1-\alpha}(k) = k-1$.

We think of the points $0, \{\alpha\}, \ldots, \{k\alpha\}$ as lying on a circle with circumference 1, and labeled $P_0, P_1, \ldots, P_k$, respectively, i.e., $P_j := \frac{1}{2\pi} e^{2\pi j\alpha\sqrt{-1}} = \frac{1}{2\pi} e^{2\pi\{j\alpha\}\sqrt{-1}}$. "The arc $\overline{P_i P_j}$" refers to the half-open counterclockwise arc from P_i to P_j, containing P_i but not P_j. We say that three distinct points A, B, C are *in order* if $B \notin \overline{CA}$. We say that A, B, C, D are in order if both A, B, C and C, D, A are in order. Essentially, if when moving counter-clockwise around the circle starting from A, we encounter first the point B, then C, then D, and finally A (again), then A, B, C, D are in order.

By rotating the circle so that $P_i \mapsto P_{i+1}$ $(0 \leq i \leq k)$, we find that each P on the arc $\overline{P_{k-2}P_{k-1}}$ is rotated onto a P on the arc $\overline{P_{k-1}P_k}$. Specifically, the number of $P_0, P_1, \ldots, P_{k-2}$ on $\overline{P_{k-2}P_{k-1}}$ is the same as the number of $P_1, P_2, \ldots, P_{k-1}$ on $\overline{P_{k-1}P_k}$. Set

$$X := \{P_0, P_1, \ldots, P_{k-2}\} \quad \text{and} \quad Y := \{P_1, P_2, \ldots, P_{k-1}\},$$

so that what we have observed is

$$\left|X \cap \overline{P_{k-2}P_{k-1}}\right| = \left|Y \cap \overline{P_{k-1}P_k}\right|. \tag{5}$$

Also, we will use

$$B_\alpha(k) = \left|Y \cap \overline{P_0P_k}\right|.$$

Now, first, suppose that $\{k\alpha\} \in [0, \alpha)$, so that the points $P_0, P_k, P_{k-2}, P_{k-1}$ are in order on the circle. We have

$$\begin{aligned}
X \cap \overline{P_{k-2}P_{k-1}} &= X \cap \left(\overline{P_0P_{k-1}} \setminus \overline{P_0P_{k-2}}\right) \\
&= \left(X \cap \overline{P_0P_{k-1}}\right) \setminus \left(X \cap \overline{P_0P_{k-2}}\right) \\
\left|X \cap \overline{P_{k-2}P_{k-1}}\right| &= \left|\left(X \cap \overline{P_0P_{k-1}}\right)\right| - \left|\left(X \cap \overline{P_0P_{k-2}}\right)\right| \\
&= (B_\alpha(k-1)+1) - (B_\alpha(k-2)+1) \\
&= B_\alpha(k-1) - B_\alpha(k-2),
\end{aligned}$$

and similarly

$$\begin{aligned}
Y \cap \overline{P_{k-1}P_k} &= \left(Y \cap \overline{P_{k-1}P_0}\right) \cup \left(Y \cap \overline{P_0P_k}\right) \\
&= \left(Y \setminus \left(Y \cap \overline{P_0P_{k-1}}\right)\right) \cup \left(Y \cap \overline{P_0P_k}\right) \\
\left|Y \cap \overline{P_{k-1}P_k}\right| &= \left(|Y| - \left|Y \cap \overline{P_0P_{k-1}}\right|\right) + \left|Y \cap \overline{P_0P_k}\right| \\
&= (k-1-B_\alpha(k-1)) + B_\alpha(k) \\
&= B_\alpha(k) - B_\alpha(k-1) - (1-k),
\end{aligned}$$

so that Eq. (5) becomes $B_\alpha(k-1) - B_\alpha(k-2) = B_\alpha(k) - B_\alpha(k-1) - (1-k)$, as claimed in the statement of the theorem.

Now suppose that $\{k\alpha\} \in [\alpha, 2\alpha)$, so that the points $P_0, P_{k-1}, P_k, P_{k-2}$ are in order. By arguing as in the above case, we find

$$X \cap \overline{P_{k-2}P_{k-1}} = \left(X \setminus \left(X \cap \overline{P_0P_{k-2}}\right)\right) \cup \left(X \cap \overline{P_0P_{k-1}}\right),$$

and so $\left|X \cap \overline{P_{k-2}P_{k-1}}\right| = B_\alpha(k-1) - B_\alpha(k-2) + (k-1)$. Likewise,

$$Y \cap \overline{P_{k-1}P_k} = \left(Y \cap \overline{P_0P_k}\right) \setminus \left(Y \cap \overline{P_0P_{k-1}}\right)$$

so that $\left|Y \cap \overline{P_{k-1}P_k}\right| = B_\alpha(k) - B_\alpha(k-1)$. Thus, in this case Eq. (5) becomes $B_\alpha(k-1) - B_\alpha(k-2) + (k-1) = B_\alpha(k) - B_\alpha(k-1)$, as claimed in the statement of the theorem.

Finally, suppose that $\{k\alpha\} \in [2\alpha, 1)$, so that the points $P_0, P_{k-2}, P_{k-1}, P_k$ are in order. We find

$$X \cap \overline{P_{k-2}P_{k-1}} = \left(X \cap \overline{P_0P_{k-1}}\right) \setminus \left(X \cap \overline{P_0P_{k-2}}\right)$$

and $\left|X \cap \overline{P_{k-2}P_{k-1}}\right| = B_\alpha(k-1) - B_\alpha(k-2)$. Also,

$$Y \cap \overline{P_{k-1}P_k} = \left(Y \cap \overline{P_0P_k}\right) \setminus \left(Y \cap \overline{P_0P_{k-1}}\right)$$

and so $\left|Y \cap \overline{P_{k-1}P_k}\right| = B_\alpha(k) - B_\alpha(k-1)$. As claimed, we have $B_\alpha(k-1) - B_\alpha(k-2) = B_\alpha(k) - B_\alpha(k-1)$. ■

Lemma 3.3.2 makes explicit the connection between Lemma 3.3.1, arithmetic properties of B_α, and the permutation $\pi_{\alpha,n}$.

Lemma 3.3.2. *Let $\alpha \in (0,1)$ be irrational and $n \in \mathbb{Z}^+$. If k is odd, then $B_\alpha(k)$ is even. If k is even, then $B_\alpha(k) \equiv \lfloor k\alpha \rfloor + 1 \pmod 2$.*

Proof. By Lemma 3.3.1, $B_\alpha(k) + B_{1-\alpha}(k) = k - 1$. Thus for odd k, $B_\alpha(k) + B_{1-\alpha}(k)$ is even, and so either both $B_\alpha(k)$ and $B_{1-\alpha}(k)$ are even or both are odd. This means that, for odd k, without loss of generality we may assume that $0 < \alpha < \frac{1}{2}$.

Reducing the recurrence relation in Lemma 3.3.1 modulo 2, under the hypothesis that k is odd, we find that $B_\alpha(k) \equiv B_\alpha(k-2) \pmod 2$. Since $B_\alpha(1) = 0$, we see that $B_\alpha(k) \equiv 0 \pmod 2$ for all odd k.

Now suppose that k is even and $0 < \alpha < \frac{1}{2}$. The recurrence relation in Lemma 3.3.1 reduces to

$$B_\alpha(k) + B_\alpha(k-2) \equiv [\![\{k\alpha\} \in [0, 2\alpha)]\!] \pmod 2.$$

Set $\beta = 2\alpha$, $k = 2\ell$ and $B'(i) = B_\alpha(2i)$. We have

$$\begin{aligned} B_\alpha(k) = B_\alpha(2\ell) &= B'(\ell) \\ &\equiv B'(\ell - 1) + [\![\{\ell\beta\} \in [0,\beta)]\!] \pmod 2 \\ &\equiv B'(1) + \sum_{i=2}^{\ell} [\![\{i\beta\} \in [0,\beta)]\!] \pmod 2 \\ &= B_\alpha(2) + \lfloor \ell\beta \rfloor \\ &= 1 + \lfloor 2\ell\alpha \rfloor = 1 + \lfloor k\alpha \rfloor, \end{aligned}$$

since $\sum_{i=2}^{\ell} [\![\{i\beta\} \in [0,\beta)]\!]$ (with $\beta \in (0,1)$) counts the integers in the interval $(\beta, \ell\beta]$. This proves the lemma for $0 < \alpha < \frac{1}{2}$ and k even.

Now suppose that k is even and $\frac{1}{2} < \alpha < 1$. By Lemma 3.3.1, $B_\alpha(k) = k - 1 - B_{1-\alpha}(k) \equiv 1 + B_{1-\alpha}(k) \pmod 2$. By the argument (for $0 < \alpha < \frac{1}{2}$) given above, $B_{1-\alpha}(k) \equiv 1 + \lfloor k(1-\alpha) \rfloor \pmod 2$. We have

$$\begin{aligned} B_\alpha(k) &\equiv \lfloor k(1-\alpha) \rfloor \pmod 2 \\ &= k - \lfloor k\alpha \rfloor - \{k(1-\alpha)\} - \{k\alpha\} \\ &\equiv \lfloor k\alpha \rfloor + 1 \pmod 2, \end{aligned}$$

where we have used $\{k(1-\alpha)\} + \{k\alpha\} = 1$ (since α is irrational). ■

Proof of Theorem 1.3. We have $\pi_{\alpha,2n+1} = \pi_{\alpha,2n}\, \rho(2n+1, B_\alpha(2n+1))$. The permutation $\rho(2n+1, B_\alpha(2n+1))$ is the product of $2n + 1 - B_\alpha(2n+1) - 1$ transpositions, which is an even number by Lemma 3.3.2. Thus $\operatorname{sgn}(\pi_{\alpha,2n+1}) = \operatorname{sgn}(\pi_{\alpha,2n})$.

By Lemma 3.3.2, for all integers k

$$k - B_\alpha(k) + 1 \equiv \begin{cases} 0, & k \text{ odd;} \\ \lfloor k\alpha \rfloor, & k \text{ even.} \end{cases}$$

Since the sign of the permutation $\rho(k, B_\alpha(k))$ is $(-1)^{k - B_\alpha(k)}$ and $(-1)^{2n} = 1$, we have

$$\begin{aligned} \operatorname{sgn}(\pi_{\alpha,2n}) = (-1)^{2n} \operatorname{sgn}\left(\prod_{k=1}^{2n} \rho(k, B_\alpha(k)) \right) &= \prod_{k=1}^{2n} (-1) \operatorname{sgn}\left(\rho(k, B_\alpha(k))\right) \\ &= \prod_{k=1}^{2n} (-1)(-1)^{k - B_\alpha(k)} = \prod_{k=1}^{2n} (-1)^{k - B_\alpha(k) + 1} = \prod_{\ell=1}^{n} (-1)^{\lfloor 2\ell\alpha \rfloor}. \end{aligned}$$

■

4 Unanswered Questions

The most significant question we have been unable to answer is *why* the matrices formed from Sturmian words in Section 2 lie in a common representation of S_n. We made two choices with little motivation: we ordered the factors anti-lexicographically; and we subtracted the last factor from the others. What happens if we order the factors differently, or subtract the second factor from the others? Understanding why the structure revealed in Section 2 exists might allow us to predict other phenomena.

Lucas Wiman [personal communication] has proved that

$$\left\{\mathcal{M}_{n-1}\left(\tfrac{c}{n}\right) : \gcd(c,n)=1\right\}$$

is isomorphic to the multiplicative group modulo n, and asks if this is the largest subset of $\{\mathcal{M}_{n-1}(\alpha) : 0 < \alpha < 1\}$ that is a group.

We have shown that the volume of the simplex $F_n(\alpha)$ is independent of α. When are two such simplices actually congruent?

Conjecture: For $\alpha, \beta \in (0,1)$ be irrational, $F_n(\alpha) \cong F_n(\beta)$ (as simplices) iff $F_n(\alpha) = F_n(\beta)$ or $F_n(\alpha) = F_n(1-\beta)$.

We have verified this conjecture for $n \leq 20$ by direct computation. At least, can any such simplex be cut and reassembled (in the sense of Hilbert's 3rd Problem: see Eves [Eve72] for the basic theory and Sydler [Syd65] for a complete characterization) into the shape of another?

While we have identified the matrix $\mathcal{M}_n(\alpha)$, there are many interesting questions that we remain unable to answer. We don't believe that there is a bound on $\sum_{n=1}^{N} \det(\mathcal{M}_n(\alpha)) = \sum_{n=1}^{N} \operatorname{sgn}(\pi_{\alpha,n})$ that is independent of N, but this sum must grow very slowly. We suspect that

$$\left|\sum_{n=1}^{N} \operatorname{sgn}(\pi_{\alpha,n})\right| \ll \log N$$

for almost all α. For example, with $\alpha = \frac{\sqrt{5}-1}{2}$ and $N < e^{13} \approx 442413$,

$$\left|\sum_{n=1}^{N} \operatorname{sgn}(\pi_{\alpha,n})\right| < 10.$$

In Section 3.1 we showed that $\operatorname{ord}(\pi_{\alpha,n})$ is regularly extremely small relative to the average order of a permutation on n symbols. For each irrational α, are there infinitely many values of n for which $\operatorname{ord}(\pi_{\alpha,n})$ is exceptionally large?

One might hope for an explicit formula for $\operatorname{ord}(\pi_{\alpha,n})$ in terms of the base-α Ostrowski expansion of n, but this seems to be extremely difficult. Are metric results more approachable? Specifically, what is the distribution of $\operatorname{ord}(\pi_{\alpha,n})$ and $\operatorname{sgn}(\pi_{\alpha,n})$ for α taken uniformly from $(0,1)$?

Since $\operatorname{ord}(\pi_{\alpha,n})$ appears to vary wildly, it may be advantageous to consider its average behavior. Can one give an asymptotic expansion of $\sum_{n=1}^{N} \operatorname{ord}(\pi_{\alpha,n})$? What can be said about $I(n) := \int_0^1 \operatorname{ord}(\pi_{\alpha,n})\, d\alpha$? Surprisingly, although $I(n)$ seems to be rapidly increasing, it is not monotonic, e.g., $I(35) > I(36) > I(37)$. Is this the 'law of small numbers,' or are there infinitely many values of n for which $I(n+1) > I(n)$? We are not aware of any non-trivial bounds, upper or lower, on $I(n)$. Figure 1 shows $I(n)$ for $1 \leq n \leq 60$.

B_α is an interesting function in its own right. We gave a formula for $B_\alpha(n) \pmod 2$ in Lemma 3.3.2. Is it possible to give a nice formula for $B_\alpha(n)$ for other moduli? It seems likely that there are arbitrarily large integers which are not in the range of B_α, although we have been unable to show that it does not contain

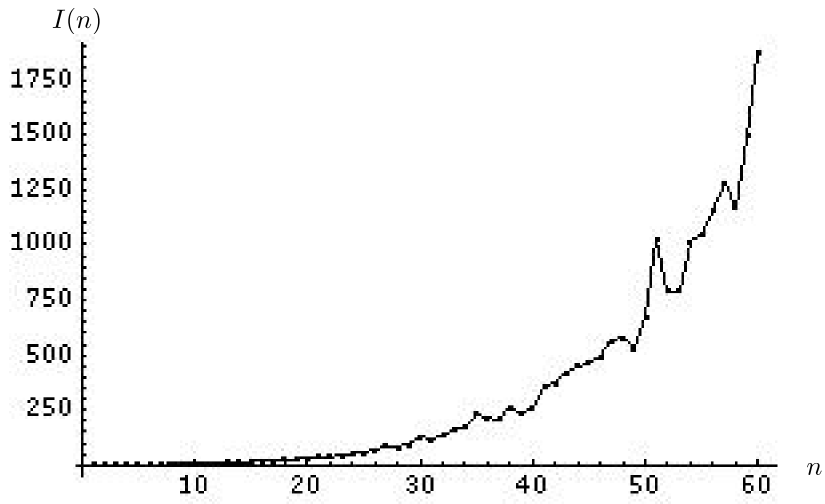

Figure 1: $I(n)$ for $1 \leq n \leq 60$

all nonnegative integers. For example, $B_{(\sqrt{5}-3)/2}(k) \neq 3, B_{\sqrt{2}}(k) \neq 7$ and $B_{e^{-1}}(k) \neq 23$ for $k \leq 10^7$. It is perhaps noteworthy that the least k for which $B_{e^{-1}}(k) = 25$ is $k = 22154$, a reminder that B_α can take new, small values even at large k. From the theory of continued fractions we know that there are infinitely many k for which $B_\alpha(k) = 0$. Is there an $x \neq 0$ and irrational α for which there are infinitely many k such that $B_\alpha(k) = x$?

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References

[AB98] Pascal Alessandri and Valérie Berthé. Three distance theorems and combinatorics on words. *Enseign. Math. (2)*, 44(1-2):103–132, 1998. 3.1

[Bro93] Tom C. Brown. Descriptions of the characteristic sequence of an irrational. *Canad. Math. Bull.*, 36(1):15–21, 1993. 1

[BS79] David W. Boyd and J. Michael Steele. Monotone subsequences in the sequence of fractional parts of multiples of an irrational. *J. Reine Angew. Math.*, 306:49–59, 1979. 1

[Coo02] Joshua N. Cooper. Quasirandom permutations. `arXiv:math.CO/0211001`, 2002. 1

[Eve72] Howard Eves. *A survey of geometry*. Allyn and Bacon Inc., Boston, Mass., revised edition, 1972. 4

[Lot02] M. Lothaire. *Algebraic Combinatorics on Words*. Cambridge University Press, Cambridge, U.K., first edition, 2002. Currently available online at `http://www-igm.univ-mlv.fr/~berstel/Lothaire/`. 1, 2.1, 2.3, 2.3

[Sag01] Bruce E. Sagan. *The symmetric group.* Springer-Verlag, New York, second edition, 2001. Representations, combinatorial algorithms, and symmetric functions. 2.6

[Sch84] Johannes Schoißengeier. On the discrepancy of $(n\alpha)$. *Acta Arith.*, 44(3):241–279, 1984. 1

[Sen95] Marjorie Senechal. *Quasicrystals and geometry.* Cambridge University Press, Cambridge, 1995. 1

[Sla67] Noel B. Slater. Gaps and steps for the sequence $n\theta$ mod 1. *Proc. Cambridge Philos. Soc.*, 63:1115–1123, 1967. 3.1

[Sós57] Vera T. Sós. A lánctörtek egy geometriai interpretációja és alkalmazásai [On a geometrical theory of continued fractions]. *Mat. Lapok*, 8:248–263, 1957. 1, 3.1

[Sto76] Kenneth B. Stolarsky. Beatty sequences, continued fractions, and certain shift operators. *Canad. Math. Bull.*, 19(4):473–482, 1976. 1

[Syd65] J.-P. Sydler. Conditions nécessaires et suffisantes pour l'équivalence des polyèdres de l'espace euclidien à trois dimensions. *Comment. Math. Helv.*, 40:43–80, 1965. 4

[Tij00] R. Tijdeman. Exact covers of balanced sequences and Fraenkel's conjecture. In *Algebraic number theory and Diophantine analysis (Graz, 1998)*, pages 467–483. de Gruyter, Berlin, 2000. 1