Math 232 Calculus 2 Fall 17 Midterm 2a

Name: Solutions

- I will count your best 8 of the following 10 questions.
- You may use a calculator, and a 3×5 index card of notes, but no phones or other notes.
- You must show your work to receive credit for a question.

1	10	
2	10	
3	10	
4	10	
5	10	
6	10	
7	10	
8	10	
9	10	
10	10	
	80	

Midterm 2	
Overall	

(1) Find $\displaystyle\int \frac{1}{x \ln x}\, dx$.

try: $u = \ln x$

$\dfrac{du}{dx} = \dfrac{1}{x}$

$$\int \frac{1}{xu} \cdot \frac{dx}{du}\, du = \int \frac{1}{xu} \cdot x\, du$$

$$= \int \frac{1}{u}\, du = \ln u + c = \ln(\ln x) + c$$

(2) Find $\displaystyle\int \frac{x+1}{(x-1)^2}\,dx$.

$$\frac{x+1}{(x-1)^2} = \frac{A}{x-1} + \frac{B}{(x-1)^2} = \frac{A(x-1)+B}{(x-1)^2}$$

$x=1: \quad 2 = B$

$x=0: \quad 1 = -A+B \quad A=1$

$$\int \frac{1}{x-1} + \frac{2}{(x-1)^2}\,dx = \ln|x-1| \cancel{+} - 2(x-1)^{-1} + C$$

$$\int uv'\,dx = uv - \int u'v\,dx$$

(3) Find $\int_1^{\infty} xe^{-3x}\,dx$.

$$\int xe^{3x}\,dx = -\frac{1}{3}e^{-3x}\cdot x + \int \frac{1}{3}e^{-3x}\,dx = -\frac{1}{3}xe^{-3x} - \frac{1}{9}e^{-3x} + c$$

$$\int_1^{\infty} xe^{-3x}\,dx = \lim_{R\to\infty}\int_1^R xe^{-3x}\,dx = \lim_{R\to\infty}\left[-\frac{1}{3}xe^{-3x} - \frac{1}{9}e^{-3x}\right]_1^R$$

$$= \lim_{R\to\infty} \underbrace{-\frac{1}{3}Re^{-3R} - \frac{1}{9}e^{-3R}}_{\to 0} + \frac{1}{3}e^{-3} + \frac{1}{9}e^{-3} = \cancel{\frac{4}{3}}\ \frac{4}{9}e^{-3}$$

$\sin^2 u + \cos^2 u = 1 \iff \cos^2 u = 1 - \sin^2 u$

(4) Find $\displaystyle\int_0^2 \frac{1}{\sqrt{4-x^2}}\,dx$.

try: $x = 2\sin u$

$\dfrac{dx}{du} = 2\cos u$

$$\int \frac{1}{\sqrt{4-x^2}}\,dx = \int \frac{1}{\sqrt{4-4\sin^2 u}}\,\frac{dx}{du} = \int \frac{1}{2\sqrt{\cos^2 u}}\,2\cos u\,du$$

$$= \int 1\,du = u + c = \sin^{-1}\left(\frac{x}{2}\right) + c$$

$$\int_0^2 \frac{1}{\sqrt{4-x^2}}\,dx = \lim_{R\to 2} \int_0^R \frac{1}{\sqrt{4-x^2}}\,dx = \lim_{R\to 2} \left[\sin^{-1}\left(\frac{x}{2}\right)\right]_0^R$$

$$= \lim_{R\to 2} \sin^{-1}\left(\frac{R}{2}\right) - 0 = \sin^{-1}(1) = \frac{\pi}{2}.$$

(5) Find the degree three Taylor polynomial centered at $x = 0$ for the function $f(x) = e^{\sin(x)}$.

$f(x) = e^{\sin x}$

$f'(x) = e^{\sin x} \cdot \cos x$

$f''(x) = e^{\sin x} \cdot \cos^2 x + e^{\sin x} \cdot -\sin x$

$f^{(3)}(x) = e^{\sin x} \cdot \cos^3 x + e^{\sin x} \cdot 2\cos x \sin x - e^{\sin x} \cdot \sin x \cos x - e^{\sin x} \cdot \cos x \cdot$

$f(0) = 1$

$f'(0) = 1$

$f''(0) = 1$

$f^{(3)}(0) = 0$

$T(x) = 1 + x + \frac{x^2}{2!} + 0 \cdot \frac{x^3}{3!}$

(6) Does the series $\sum_{n=1}^{\infty} \frac{(-1)^n n}{n+2}$ converge or diverge?

Diverges as $\lim_{n\to\infty} a_n \neq 0$ $\quad \lim_{n\to\infty} |a_n| = \lim_{n\to\infty} \frac{n}{n+2} = \lim_{n\to\infty} \frac{1}{1+2/n} = 1.$

(7) Does the series $\sum_{n=1}^{\infty} \frac{1}{n}$ converge or diverge? Explain your answer.

Diverges:

$$1 + \frac{1}{2} + \underbrace{\frac{1}{3} + \frac{1}{4}}_{\geq \frac{1}{2}} + \underbrace{\frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8}}_{\geq \frac{1}{2}} + \underbrace{\frac{1}{9} + \cdots + \frac{1}{16}}_{\geq \frac{1}{2}} + \cdots$$

etc.

(8) (a) Use partial fractions to find an explicit formula for the partial sum

$$s_N = \sum_{n=1}^{N} \frac{1}{n^2 + 5n + 6}.$$

(b) Use your answer to (a) to show that the sum converges, by finding $\lim_{N\to\infty} s_N$.

a)

$$\frac{1}{(n+3)(n+2)} = \frac{A}{n+3} + \frac{B}{n+2} = \frac{A(n+2) + B(n+3)}{(n+2)(n+3)}$$

$n = -2: \quad 1 = B$

$n = -3: \quad 1 = -A$

$$= \frac{1}{n+2} - \frac{1}{n+3}.$$

$$s_N = \frac{1}{3} - \frac{1}{4} + \frac{1}{4} - \frac{1}{5} + \frac{1}{5} - \frac{1}{6} + \cdots + \frac{1}{N+2} - \frac{1}{N+3} = \frac{1}{3} - \frac{1}{N+3}.$$

b) $$\lim_{N\to\infty} \frac{1}{3} - \frac{1}{N+3} = \frac{1}{3}$$

(9) (a) Find a formula for the partial sum $s_N = \frac{1}{2} - \frac{1}{4} + \frac{1}{8} - \frac{1}{16} + \cdots + \frac{(-1)^N}{2^N}$, for example by comparing s_N with $-\frac{1}{2}s_N$.

(b) Use your answer to (a) to show that the sum converges, by finding $\lim_{N\to\infty} s_N$.

$$s_N = \frac{1}{2} - \frac{1}{4} + \frac{1}{8} + \cdots + \frac{(-1)^N}{2^N}$$

$$-\frac{1}{2}s_N = -\frac{1}{4} + \frac{1}{8} + \cdots + \frac{(-1)^N}{2^N} - \frac{(-1)^{N+1}}{2^{N+1}}$$

$$s_N + \frac{1}{2}s_N = \frac{1}{2} - \frac{(-1)^{N+1}}{2^{N+1}}$$

$$s_N\left(1+\frac{1}{2}\right) = \frac{1}{2} - \frac{(-1)^{N+1}}{2^{N+1}}$$

$$s_N = \frac{\frac{1}{2} - \frac{(-1)^{N+1}}{2^{N+1}}}{3/2} \qquad \lim_{N\to\infty} \frac{\frac{1}{2} - \frac{(-1)^{N+1}}{2^{N+1}}}{3/2} = \frac{1}{2}\cdot\frac{2}{3} = \frac{1}{3}$$

(10) Show the series $\sum_{n=1}^{\infty} \frac{2^n}{4^n - 1}$ converges, by any method.

limit comparison test: $b_n = \frac{1}{2^n}$

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{2^n}{4^n-1} \cdot \frac{2^n}{1} = \lim_{n\to\infty} \frac{4^n}{4^n-1} = \lim_{n\to\infty} \frac{1}{1-1/4^n} = 1$$

so Σa_n converges iff Σb_n converges.

↑ geometric series with $r = \frac{1}{2} < 1$ so

$\sum_{n=1}^{\infty} \frac{2^n}{4^n-1}$ converges.