Sample midterm 1 solutions

Q1 $\int -2x^2\cos(-2x^3)\,dx$ $\quad u = -2x^3$, $\quad \frac{du}{dx} = -6x^2$

$$\int -2x^2\cos(u)\frac{dx}{du}\,du = \int -2x^2\cos(u)\frac{1}{-6x^2}\,du = \int \frac{1}{3}\cos(u)\,du$$

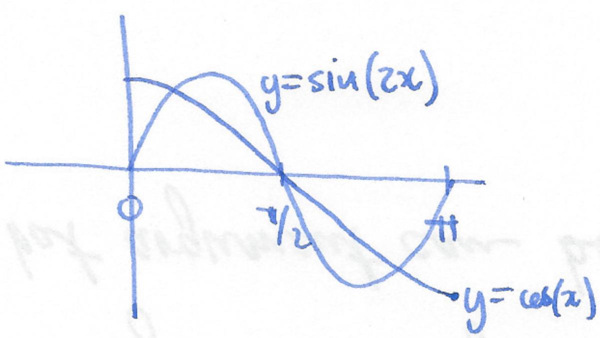
$$= \frac{1}{3}\sin(u) + c = \frac{1}{3}\sin(-2x^3) + c$$

Q2 $\int 3x^3(1-x^4)^{1/5}\,dx$ $\quad u = 1-x^4$, $\quad \frac{du}{dx} = -4x^3$

$$\int 3x^3 u^{1/5}\frac{dx}{du}\,du = \int 3x^3 u^{1/5}\frac{1}{-4x^3}\,du = \int -\frac{3}{4}u^{1/5}\,du = -\frac{3}{4}\frac{5}{6}u^{6/5} + c$$

$$= -\frac{5}{12}(1-x^4)^{6/5} + c$$

Q3



solve: $\sin(2x) = \cos(x)$

$2\sin x\cos x = \cos x$.

$\cos x(2\sin x - 1) = 0$

$\cos x = 0 \iff x = \pi/2 \pm n\pi$. $\quad x = \frac{\pi}{2}$

$\sin x = \frac{1}{2} \iff \pm\frac{\pi}{6} \pm \pi n$ $\quad x = \frac{5\pi}{6}$

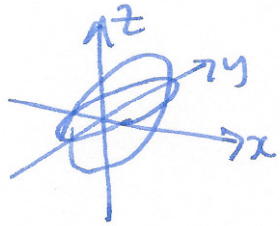
$$\int_{\pi/2}^{5\pi/6} \cos x - \sin 2x\,dx + \int_{5\pi/6}^{\pi} \sin 2x - \cos x\,dx$$

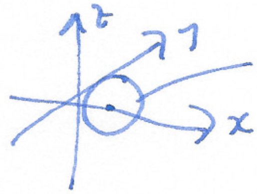
$$= \left[\sin x + \frac{1}{2}\cos 2x\right]_{\pi/2}^{5\pi/6} + \left[-\frac{1}{2}\cos 2x - \sin x\right]_{\frac{5\pi}{6}}^{\pi}.$$

$$= \frac{1}{2} + \frac{1}{2}\cdot\frac{1}{2} - \left(1 + \frac{1}{2}(-1)\right) + \left(-\frac{1}{2} - 0\right) - \left(-\frac{1}{2}\cdot\frac{1}{2} - \frac{1}{2}\right)$$

$$1/4 \quad + \quad 1/4 \quad = 1/2$$

Q4





$y^2+z^2 = 1-16x^2$.

circle of radius $r^2 = 1-16x^2$

a) area $\pi r^2 = \pi(1-16x^2)$.
of vertical cross section.

b) $V = \int A(x)\,dx = \int_{-1/4}^{1/4} \pi(1-16x^2)\,dx = \pi\left[\cancel{16}x - \frac{16}{3}x^3\right]_{-1/4}^{1/4}$.

$= \pi\left(\left(\frac{1}{4} - \frac{16}{3}\cdot\frac{1}{16\cdot 4}\right) - \left(-\frac{1}{4} + \frac{16}{3}\cdot\left(\frac{1}{16\cdot 4}\right)\right)\right)$.

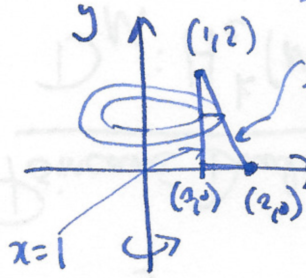
$= \pi\left(\frac{1}{2} - \frac{1}{12}\right) = \frac{5\pi}{12}$.

Q5

$\frac{1}{4}\int_{-2}^{2} e^{-x/4}\,dx = \frac{1}{4}\left[-4e^{-x/4}\right]_{-2}^{2} = \frac{1}{4}\left(-4e^{-1/2} + 4e^{1/2}\right)$

$= e^{1/2} - e^{-1/2}$.

Q6



$\frac{y-4}{-2} = x$

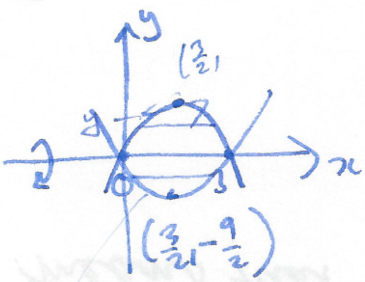
$y = -2x+4$

$x+2y=$

$\pi\int_0^2 \left(2-\frac{1}{2}y\right)^2 - 1^2\,dy = \pi\int_0^2 3 - 2y + \frac{1}{4}y^2\,dy$

$= \pi\left[3y - y^2 + \frac{1}{12}y^3\right]_0^2 = \pi\left(6 - 4 + \frac{8}{12}\right) = \pi\frac{8}{3}$.

Q7



$2\pi\int_{-\frac{9}{2}}^{0} y\cdot\left(\frac{6+\sqrt{36-8y}}{4} - \frac{6-\sqrt{36-8y}}{4}\right)dy$

$y = 2x^2-6x$

$2x^2-6x-y=0$

$x = \frac{6\pm\sqrt{36+8y}}{4}$.

$= 2\pi\int_{-\frac{9}{2}}^{0} y\,\frac{\sqrt{36+8y}}{2}\,dy = 2\pi\int_{-\frac{9}{2}}^{0} y\sqrt{9+2y}\,dy$

$u = 9 + 2y$

$\frac{du}{dy} = +2$

$$2\pi \int_{\cancel{18}0}^{9} (u-9)\frac{1}{+2} \sqrt{u} \frac{1}{+2} du = \frac{2\pi}{4} \int_{\cancel{18}0}^{9} u^{3/2} - 9u^{1/2} du$$

$$= \frac{2\pi}{4} \left[\frac{2u^{5/2}}{5} - 9 \cdot \frac{2}{3} u^{3/2} \right]_{\cancel{18}0}^{9} = \frac{2\pi}{4} \left[\left(\frac{2}{5} \cdot 9^{5/2} - 6 \cdot 9^{3/2} \right) - \cancel{\left(\frac{2 \cdot 18^{5/2}}{5} - 6 \cdot 18^{3/2} \right)} \right]$$

$\frac{2}{5} 3^5 - 6 \cdot 27$ $\quad$ 0.

$= 32.4\pi$.

Q8 $\int \underset{u'}{x} \, \underset{v}{\ln(x+1)} \, dx$ $\qquad$ $\int u'v \, dx = uv - \int uv' dx$

$u' = x \quad u = \frac{1}{2}x^2$ $\qquad = \frac{1}{2}x^2 \ln(x+1) - \int \frac{1}{2}x^2 \cdot \frac{1}{x+1} dx$.

$v = \ln(x+1) \quad v' = \frac{1}{x+1}$

$z = x+1$

$\frac{dz}{dx} = 1$.

$$= \frac{1}{2}x^2 \ln(x+1) - \frac{1}{2} \int (z-1)^2 \frac{1}{z} dz = \frac{1}{2}x^2 \ln(x+1) - \frac{1}{2} \int z - 2 + \frac{1}{z} dz.$$

$$= \frac{1}{2}x^2 \ln(x+1) - \frac{1}{4}z^2 + z - \frac{1}{2}\ln(z) + c = \frac{1}{2}x^2 \ln(x+1) - \frac{1}{4}(x+1)^2 + (x+1) - \frac{1}{2}\ln(x+1) + c.$$

Q9 $\int \underbrace{e^{-3x}}_{u'} \underbrace{\cos(2x)}_{v} dx = -\frac{1}{3}e^{-3x}\cos(2x) - \int -\frac{1}{3}e^{-3x} \cdot -2\sin(2x) \, dx$

$u' = e^{-3x} \quad u = -\frac{1}{3}e^{-3x}$

$v = \cos(2x) \quad v' = -2\sin(2x)$

$$\int e^{-3x}\cos(2x) dx = -\frac{1}{3}e^{-3x}\cos(2x) - \int \underbrace{\frac{2}{3}e^{-3x}}_{u'} \underbrace{\sin(2x)}_{v} dx$$

$u' = \frac{2}{3}e^{-3x} \quad u = -\frac{2}{9}e^{-3x}$

$v = \sin(2x) \quad v' = 2\cos(2x)$

$$\int e^{-3x}\cos(2x) dx = -\frac{1}{3}e^{-3x}\cos(2x) - -\frac{2}{9}e^{-3x}\sin(2x) + \int -\frac{2}{9}e^{-3x} \cdot 2\cos(2x) dx$$

$$\frac{13}{9}\int e^{-3x}\cos(2x)\,dx = -\frac{1}{3}e^{-3x}\cos(2x) + \frac{2}{9}e^{-3x}\sin(2x) + c.$$

$$\int e^{-3x}\cos(2x)\,dx = -\frac{3}{13}e^{-3x}\cos(2x) + \frac{2}{13}e^{-3x}\sin(2x) + c.$$

Q10 $\displaystyle\int \underbrace{x}_{v}\underbrace{e^{x}\sin x}_{u'}\,dx$ $\qquad \displaystyle\int u'v\,dx = uv - \int uv'\,dx.$

$u' = e^{x}\sin x \qquad u = ?$ (*)

$v = x \qquad v' = 1$

$$\int \underbrace{e^{x}}_{u'}\underbrace{\sin x}_{v}\,dx = e^{x}\sin x - \int \underbrace{e^{x}}_{u'}\underbrace{\cos x}_{v}$$

$u' = e^{x} \qquad u = e^{x}$

$v = \sin x \qquad v' = \cos x$

$u' = e^{x} \qquad u = e^{x}$

$v = \cos x \qquad v' = \cancel{\cos x}\ -\sin x$

$$\int e^{x}\sin x\,dx = e^{x}\sin x - e^{x}\cos x\ \bar{+} \int e^{x}\sin x\,\cancel{\cos x}\,dx$$

$$\int e^{x}\sin x\,dx = \frac{1}{2}e^{x}\sin x - \frac{1}{2}e^{x}\cos x \ (*).$$

$$\int x e^{x}\sin x\,dx = x\frac{1}{2}e^{x}(\sin x - \cos x) - \frac{1}{2}\int \underbrace{e^{x}\sin x}_{\text{done this already}} - e^{x}\cos x\,dx.$$

$$\int \underbrace{e^{x}}_{u'}\underbrace{\cos x}_{v}\,dx = +\underbrace{e^{x}}_{u}\underbrace{\cancel{\sin x}\,\cos x}_{v} + \int \underbrace{e^{x}}_{u'}\underbrace{\sin x}_{v}\,dx$$

$$= e^{x}\cos x + e^{x}\sin x - \int e^{x}\cos x\,dx$$

$$\int x e^{x}\sin x\,dx = \frac{1}{2}xe^{x}(\sin x - \cos x) - \frac{1}{2}\left(\frac{1}{2}\left(e^{x}(\sin x - \cos x)\right) + \frac{1}{2}\cdot\frac{1}{2}e^{x}(\cos x + \sin x)\right).$$

Q11 $\displaystyle\int_{-\frac{\pi}{2}}^{0} \sin^{2}(x)\cos^{3}(x)\,dx = \int_{-\frac{\pi}{2}}^{0} \sin^{2}x\,(1 - \sin^{2}x)\cos x\,dx$ $\qquad$ sub $u = \sin x\ \cancel{\cos x}$, $\dfrac{du}{dx} = \cos x.$

$$= \int_{\cancel{-\pi}\ -1}^{0} u^{2}(1 - u^{2})\cos x\,\frac{1}{\cos x}\,du = \left[\frac{u^{3}}{3} - \frac{u^{5}}{5}\right]_{-1}^{0} = -\left(-\frac{1}{3} + \frac{1}{5}\right) = +\frac{2}{15}.$$

Q42 $\int \sin(3x)\cos(7x)\,dx = \frac{1}{2}\int \sin(10x) - \sin(-4x)\,dx$

use $\left.\begin{aligned} \sin(A+B) &= \sin A\cos B + \cos A\sin B \\ \overset{\cancel{\cos}}{\sin}(A-B) &= \sin A\cos B - \cos A\sin B \end{aligned}\right\}$ $\sin A\cos B = \frac{1}{2}\left(\sin(A+B) - \sin(A-B)\right)$

$= \frac{1}{20}\cos(10x) + \frac{1}{8}\sin(4x) + c$.