Math 232 Calculus 2 Spring 15 Midterm 2a

Name: Solutions

- I will count your best 8 of the following 10 questions.
- You may use a calculator, and a 3×5 index card of notes, but no phones or other notes.

1	10	
2	10	
3	10	
4	10	
5	10	
6	10	
7	10	
8	10	
9	10	
10	10	
	80	

Midterm 2	
Overall	

(1) Find $\int \sqrt{1-4x^2}\, dx$.

$\sin^2 u + \cos^2 u = 1$

$\cos^2 u = 1 - \sin^2 u$

$x = \frac{1}{2}\sin u$

$\frac{dx}{du} = \frac{1}{2}\cos u$

$\int \sqrt{1-\sin^2 u}\, \frac{dx}{du}\, du$

$\int \cos u \cdot \frac{1}{2}\cos u\, du$

$\frac{1}{2}\int \cos^2 u\, du$

$\cos 2\theta = \cos^2\theta - \sin^2\theta$

$= 2\cos^2\theta - 1$

$\frac{1}{2}\int \frac{1}{2}\cos 2u + \frac{1}{2}\, du$

$\frac{1}{8}\sin 2u + \frac{1}{4}u + C$

$\sin 2\theta = 2\sin\theta\cos\theta$

$\frac{1}{4}\sin u \cos u + \frac{1}{4}\sin^{-1}(2x) + C$

$\frac{1}{4}2x\sqrt{1-4x^2} + \frac{1}{4}\sin^{-1}(2x) + C$

$\frac{1}{2}x\sqrt{1-4x^2} + \frac{1}{4}\sin^{-1}(2x) + C$

(2) Find $\int \frac{x}{(x-1)^2}\,dx$.

$$\frac{x}{(x-1)^2} = \frac{A}{x-1} + \frac{B}{(x-1)^2} = \frac{A(x-1)+B}{(x-1)^2}$$

$x=1$: $1 = B$

$x=0$: $0 = -A+B \Rightarrow A = 1$

$$\int \frac{1}{x-1} + \frac{1}{(x-1)^2}\,dx = \ln|x-1| - (x-1)^{-1} + C$$

$$= \ln|x-1| - \frac{1}{x-1} + C$$

(3) Find $\int_1^3 \frac{1}{x \ln x}\, dx$. $= \lim_{R \to 1} \int_R^3 \frac{1}{x \ln x}\, dx$

try $u = \ln x$

$\frac{du}{dx} = \frac{1}{x}$

$= \lim_{R \to 0} \int_R^{\ln(3)} \frac{1}{xu} \frac{dx}{du}\, du$

$= \lim_{R \to 0} \int_R^{\ln(3)} \frac{1}{u}\, du$

$= \lim_{R \to 0} \Big[\ln(u) \Big]_R^{\ln(3)}$

$= \lim_{R \to 0} \ln(\ln(3)) - \ln(R)$

diverges.

(4) Find $\int_0^\infty xe^{-3x}\,dx$.

$$\int uv'\,dx = uv - \int u'v\,dx$$

$u = x \quad u' = 1$

$v' = e^{-3x} \quad v = -\frac{1}{3}e^{-3x}$

$$= \lim_{R\to\infty} \left[x \cdot -\frac{1}{3}e^{-3x} \right]_0^R - \int_0^R -\frac{1}{3}e^{-3x}\,dx$$

$$= \lim_{R\to\infty} \underbrace{-\frac{R}{3}e^{-3R}}_{\substack{\to 0 \\ \text{as } R\to\infty}} + \frac{1}{3}\left[-\frac{1}{3}e^{-3x}\right]_0^R$$

$$= \lim_{R\to\infty} \underbrace{-\frac{1}{9}e^{-3R}}_{\substack{\to 0 \\ \text{as } R\to\infty}} + \frac{1}{9} = \frac{1}{9}$$

(5) Find the degree three Taylor polynomial centered at $x = 0$ for the function $f(x) = \sin(x^2)$.

$f(x) = \sin(x^2)$ $\qquad f(0) = 0$

$f'(x) = \cos(x^2).2x$ $\qquad f'(0) = 0$

$f''(x) = -\sin(x^2).4x^2 + 2\cos(x^2)$ $\qquad f''(0) = 2$

$f^{(3)}(x) = -\cos(x^2).8x^3 - \sin(x^2)\,8x - 2\sin(x^2).2x$ $\qquad f^{(3)}(0) = 0$

$$T_3(x) = \frac{2x^2}{2!} = x^2$$

(6) Does the series $\sum_{n=1}^{\infty} \frac{(-1)^n n}{2n+1}$ converge or diverge?

$\lim_{n\to\infty} \frac{(-1)^n n}{2n+1} \neq 0 \Rightarrow$ does not converge.

(e.g. $\lim_{2n\to\infty} \frac{(-1)^n n}{2n+1} = \lim_{n\to\infty} \frac{2n}{4n+1} = \frac{1}{2} \Rightarrow \lim_{n\to\infty} \frac{(-1)^n n}{2n+1} \neq 0$.)

(7) Does the series $\sum_{n=1}^{\infty} \frac{1}{n}$ converge or diverge? Explain your answer.

diverges.

$$1 + \frac{1}{2} + \underbrace{\frac{1}{3} + \frac{1}{4}}_{> \frac{1}{2}.} + \underbrace{\frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8}}_{> \frac{1}{2}} + \underbrace{\frac{1}{9} + \cdots + \frac{1}{16}}_{> \frac{1}{2}}$$

$\Rightarrow$ partial sums $\to \infty$.

(8) (a) Use partial fractions to find an explicit formula for the partial sum
$$s_N = \sum_{n=1}^{N} \frac{1}{4n^2-1}.$$
(b) Use your answer to (a) to show that the sum converges, by finding $\lim_{N\to\infty} s_N$.

a)
$$\frac{1}{4n^2-1} = \frac{A}{2n-1} + \frac{B}{2n+1} = \frac{A(2n+1)+B(2n-1)}{4n^2-1}$$

$n=\frac{1}{2}$: $1 = 2A$

$n=-\frac{1}{2}$: $1 = -2B$

$$\frac{1}{4n^2-1} = \frac{1/2}{2n-1} + \frac{-1/2}{2n+1}$$

$$s_N = \left(\frac{1/2}{1} - \frac{1/2}{3}\right) + \left(\cancel{\frac{1/2}{3}} - \cancel{\frac{1/2}{5}}\right) + \cancel{\frac{1/2}{5}} - \cdots + \cancel{\frac{1/2}{2N-1}} - \frac{1/2}{2N+1}$$
$$= \frac{1}{2} - \frac{1}{2(2N+1)}$$

b) $\lim_{N\to\infty} \frac{1}{2} - \frac{1}{2(2N+1)} = \frac{1}{2}$ so sum converges to $\frac{1}{2}$.

(9) (a) Find a formula for the partial sum $s_n = \frac{1}{2} + \frac{1}{4} + \frac{1}{8} \cdots + \frac{1}{2^n}$, for example by comparing s_n with $\frac{1}{2}s_n$.

(b) Use your answer to (a) to show that the sum converges, by finding $\lim_{N\to\infty} s_N$.

a)

$$S_n = \frac{1}{2} + \frac{1}{4} + \cdots, + \frac{1}{2^n}$$

$$\frac{1}{2}S_n = \frac{1}{4} + \frac{1}{8} + \cdots + \frac{1}{2^{n+1}}$$

$$S_n - \frac{1}{2}S_n = \frac{1}{2} - \frac{1}{2^{n+1}} \Rightarrow \frac{1}{2}S_n = \frac{1}{2} - \frac{1}{2^{n+1}} \Rightarrow S_n = 1 - \frac{1}{2^n}.$$

b)

$$\lim_{n\to\infty} \cancel{\frac{1}{2} / \frac{1}{2^{n+1}}} = \cancel{\frac{4}{2}}$$

$$1 - \frac{1}{2^n} = 1$$

so sum converges to ~~4/2~~ 1

(10) Show the series $\sum_{n=1}^{\infty} \frac{1+2^n}{4^n}$ converges, by any method.

$$\sum_{n=1}^{\infty} \frac{1+2^n}{4^n} = \sum_{n=1}^{\infty} \frac{1}{4^n} + \frac{1}{2^n}$$

$$a + ar + ar^2 + \dots = \frac{a}{1-r}$$

$$= \frac{1/4}{1-1/4} + \frac{1/2}{1-1/2} = \frac{1}{3} + 1 = \frac{4}{3}.$$