Sample midterm 2 Solutions

Q1 $\displaystyle\int \frac{x}{\sqrt{4x^2+1}}\,dx$

$\sin^2 x + \cos^2 x = 1$

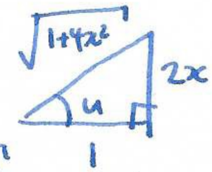
$\tan^2 x + 1 = \sec^2 x$

try: $x = \frac{1}{2}\tan u \qquad \frac{dx}{du} = \frac{1}{2}\sec^2 u$

$$\int \frac{\frac{1}{2}\tan u}{\sqrt{\tan^2 u + 1}}\frac{dx}{du}\,du = \int \frac{\frac{1}{2}\tan u}{\sec u}\,\frac{1}{2}\sec^2 u\,du$$

$$= \frac{1}{4}\int \tan u \sec u\,du = \frac{1}{4}\sec u + C$$

$2x = \tan u$



$\sec u = \sqrt{1+4x^2}$

$$= \frac{1}{4}\sqrt{1+4x^2} + C$$

Q2 $$\frac{x^2 - x + 6}{(x-1)^2(x+1)} = \frac{A}{x-1} + \frac{B}{(x-1)^2} + \frac{C}{x+1}$$

$$= \frac{A(x-1)(x+1) + B(x+1) + C(x-1)^2}{(x-1)^2(x+1)}$$

$x = 1$: $\quad 6 = 2B \qquad B = 3.$

$x = -1$: $\quad 8 = 4C \qquad C = 2$

$x = 0$: $\quad 6 = -A + B + C \qquad A = -1$

$$\int \frac{-1}{x-1} + \frac{3}{(x-1)^2} + \frac{2}{x+1}\,dx = -\ln|x-1| + \frac{3}{x-1} + 2\ln|x+1| + C$$

Q3 $$\lim_{R\to 0}\int_R^1 x^2 \ln x\,dx = \lim_{R\to 0}\left[\tfrac{1}{3}x^3\ln x\right]_R^1 - \int_R^1 \tfrac{1}{3}x^2\,dx$$

$$= \lim_{R\to 0} -\tfrac{1}{3}R^3\ln(R) - \left[\tfrac{1}{9}x^3\right]_R^1 = \lim_{R\to 0} -\tfrac{1}{3}R^3\ln(R) - \tfrac{1}{9}\cancel{R} + \tfrac{1}{9}R$$

l'H: $\lim_{R \to 0} \frac{\ln(R)}{R^{-3}} = \lim_{R \to 0} \frac{1/R}{-3R^{-4}} = \lim_{R \to 0} -\frac{1}{3}R^3 = 0$.

so $\int_0^1 x^2 \ln(x)\,dx = -\frac{1}{9}$.

Q4 $\int_0^\infty \frac{1}{9+x^2}\,dx = \lim_{R \to \infty} \int_0^R \frac{1}{9+x^2}\,dx = \lim_{R \to \infty} \frac{1}{9} \int_0^R \frac{1}{1+(x/3)^2}\,dx$

$u = \frac{x}{3}$ $\frac{du}{dx} = \frac{1}{3}$. $\lim_{R \to \infty} \frac{1}{9} \int_0^R \frac{1}{1+u^2}\,3\,du = \lim_{R \to \infty} \frac{1}{3}\left[\tan^{-1}(u)\right]_0^R$.

$= \frac{\pi}{6}$.

Q5 $f(x) = \sqrt{x^2+2} = (x^2+2)^{1/2}$ $f(1) = \sqrt{3}$.

$f'(x) = \frac{1}{2}(x^2+2)^{-1/2} \cdot 2x$ $f'(1) = \frac{1}{2\sqrt{3}} \cdot 2 = \frac{1}{\sqrt{3}}$.

$f''(x) = -\frac{1}{4}(x^2+2)^{-3/2} \cdot 2x + \frac{1}{2}(x^2+2)^{-1/2} \cdot 2$ $f''(1) = -\frac{1}{4 \cdot 3\sqrt{3}} \cdot 6 + \frac{1}{\sqrt{3}} = \frac{1}{2\sqrt{3}}$.

$f^{(3)}(x) = +\frac{3}{8}(x^2+2)^{-5/2} \cdot 2x + -\frac{1}{4}(x^2+2)^{-3/2} \cdot 2 + -\frac{1}{2}(x^2+2)^{-3/2} \cdot 2x$.

$f^{(3)}(1) = \frac{3}{4}\frac{1}{9\sqrt{3}} - \frac{1}{2 \cdot 3\sqrt{3}} - \frac{1}{3\sqrt{3}} = \frac{-5}{12\sqrt{3}}$.

$T_3(x) = \sqrt{3} + \frac{1}{\sqrt{3}}(x-1) + \frac{1}{2\sqrt{3}}\frac{(x-1)^2}{2!} - \frac{5}{12\sqrt{3}}\frac{(x-1)^3}{3!}$

Q6 $\lim_{n \to \infty} \frac{1}{n} = 0$ $\cos(0) = 1$, diverges.

$\left(\text{so } \lim_{n \to \infty} \cos\left(\tfrac{1}{n}\right) = 1\right)$.

Q7 $\ln(n) < n$ for $n \geq 1$ so $\frac{\ln(n)^2}{n^4} \leq \frac{1}{n^2}$.

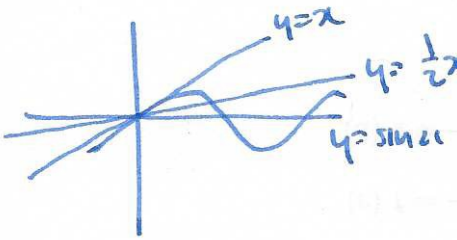
$\sum_{n=1}^\infty \frac{1}{n^2}$ converges $\Rightarrow$ $\sum_{n=1}^\infty \frac{\ln(n)^2}{n^4}$ converges

p-test or integral test. (comparison test).

Q8 ratio test: $\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \frac{2^{n+1}}{(n+1)!} \cdot \frac{n!}{2^n} = \lim_{n\to\infty} \frac{2}{n+1} = 0 < 1$

so $\sum_{n=1}^{\infty} a_n$ converges.

Q9 intuition: $\sin\left(\frac{1}{n}\right) \sim \frac{1}{n}$ for n large.



use: $\frac{1}{2}x < \sin x < x$

comparison test: $\sum_{n=1}^{\infty} \frac{1}{2n}$ diverges

$\Rightarrow \sum_{n=1}^{\infty} \sin\left(\frac{1}{n}\right)$ diverges.

Q10 $\sin\left(\frac{1}{n^2}\right) < \frac{1}{n^2}$ $\quad \sum_{n=1}^{\infty} \frac{1}{n^2}$ converges $\underset{\text{comparison test}}{\Rightarrow}$ $\sum_{n=1}^{\infty} \sin\left(\frac{1}{n^2}\right)$ converges.