Sample midterm 1 Solutions

Q1

$$\int x\cos(-3x^2)\,dx \qquad u=-3x^2, \quad \frac{du}{dx}=-6x$$

$$\int x\cos(u)\frac{dx}{du}\,du = \int x\cos(u)\frac{1}{-6x}\,du = -\frac{1}{6}\int \cos(u)\,du$$

$$= -\frac{1}{6}\sin(u)+c = -\frac{1}{6}\sin(-3x^2)+c$$

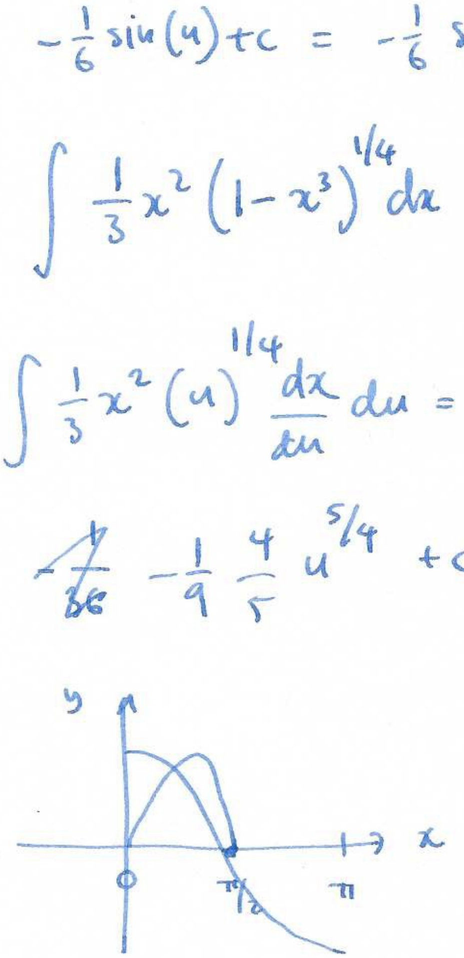
Q2

$$\int \frac{1}{3}x^2(1-x^3)^{1/4}\,dx. \qquad u=1-x^3, \quad \frac{du}{dx}=-3x^2.$$

$$\int \frac{1}{3}x^2(u)^{1/4}\frac{dx}{du}\,du = \int \frac{1}{3}x^2u^{1/4}\frac{1}{-3x^2}\,du = \int -\frac{1}{9}u^{1/4}\,du$$

$$= \cancel{-\frac{1}{36}}\ -\frac{1}{9}\,\frac{4}{5}u^{5/4}+c = -\frac{4}{45}u^{5/4}+c = -\frac{4}{45}(1-x^3)^{5/4}+c.$$

Q3



find intersection: $\sin(2x) = \cos(x)$

$\sin(2x) = 2\sin x\cos x$

$$\underbrace{\cos(x)}_{x=\pi/2}\underbrace{(2\sin x-1)}_{\substack{\sin x = 1/2 \\ x=\pi/6}} = 0$$

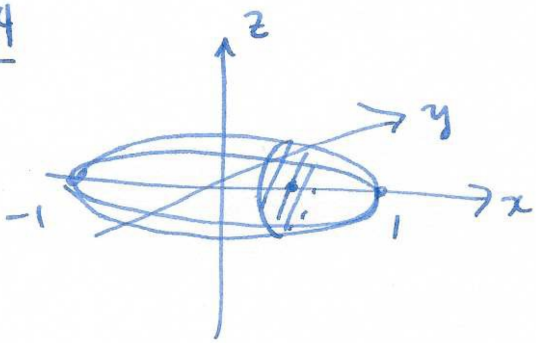
on $[0,\frac{\pi}{2}]$:

$$\int_0^{\pi/6}\cos x-\sin 2x\,dx + \int_{\pi/6}^{\pi/2}\sin 2x-\cos x\,dx$$

$$\left[\sin x+\frac{1}{2}\cos 2x\right]_0^{\pi/6} + \left[-\frac{1}{2}\cos 2x-\sin x\right]_{\pi/6}^{\pi/2}.$$

$$\frac{1}{2}+\frac{1}{4}-\frac{1}{2} \qquad +\frac{1}{2}-1-\left(-\frac{1}{4}-\frac{1}{2}\right) = \frac{1}{4}+\frac{1}{2}-1+\frac{1}{4}+\frac{1}{2} = \frac{1}{2}.$$

Q4



a) at cross section x: $y^2+z^2 = \underbrace{\frac{1-x^2}{16}}_{(\text{radius})^2 \text{ of circle}}$.

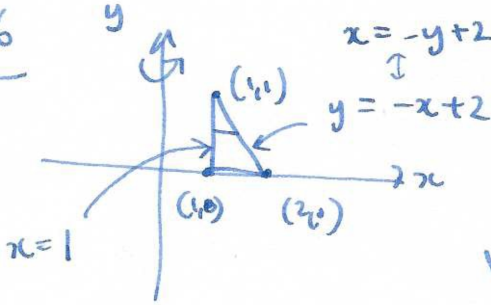
$$\text{area} = \pi r^2 = \pi\left(\frac{1-x^2}{16}\right).$$

b) $V = \int A(x)\,dx = \frac{\pi}{16}\int_{-1}^{1} 1-x^2\,dx = \frac{\pi}{16}\left[x-\frac{1}{3}x^3\right]_{-1}^{1} = \frac{\pi}{16}\left(1-\frac{1}{3}-\left(-1+\frac{1}{3}\right)\right)$

$= \frac{\pi}{16}\cdot\frac{4}{3} = \frac{\pi}{12}.$

Q5 $\int_{-3}^{3} e^{-x/3}\,dx = \left[3e^{-x/3}\right]_{-3}^{3} = 3\left(e^{-1}-e^{1}\right)$

Q6



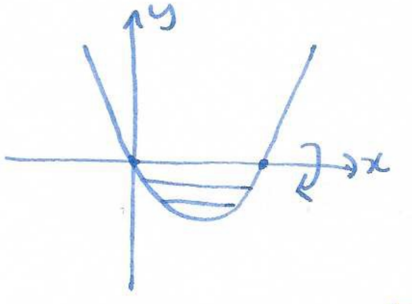
area of disc πr^2  $\pi(r_2^2-r_1^2)$

area of ring: $\pi\left((\cancel{2-x}\ 2-y)^2 - 1^2\right)$.

$V = \int A\,dy = \int_0^1 (2-y)^2-1\,dy = \int_0^1 4-2y+y^2-1\,dy$

$= \int_0^1 3-2y+y^2\,dy = \left[3y-y^2+\frac{1}{3}y^3\right]_0^1 = 3-1+\frac{1}{3} = 2+\frac{1}{3} = \frac{7}{3}.$

Q7



$y = x^2-9x = x(x-9)$ min value at $\left(9/2, -81/4\right)$.

$\int A(y)\,dy$ area of shell $= \underbrace{\text{height}} \times 2\pi \underbrace{r}_{y}$.

$$\frac{9+\sqrt{81+4y}}{2} - \frac{9-\sqrt{81+4y}}{2} = \sqrt{81+4y}$$

$x^2-9x-y=0 \Rightarrow x = \frac{9\pm\sqrt{81+4y}}{2}$

$\int_{-81/4}^{0} 2\pi y\sqrt{81+4y}\,dy$ $u = 81+4y$, $\frac{du}{dy} = 4$ $\int_0^{81} 2\pi\left(\frac{u}{4}-81\right)u^{1/2}\,\frac{1}{4}\,du$

$$\frac{1}{2}\pi \int_0^{81} \frac{1}{4}u^{3/2} - 81u^{1/2}\, du = \frac{\pi}{2}\left[\frac{1}{4}\frac{2}{5}u^{5/2} - \frac{2}{3}81\, u^{3/2}\right]_0^{81}$$

$$= \frac{\pi}{2}\left[\frac{1}{10}u^{5/2} - 54u^{3/2}\right] = \frac{\pi}{2}\left(\frac{1}{10}81^{5/2} - 54\ 81^{3/2}\right) = \frac{\pi}{2}\left(\frac{1}{10}9^5 - 54.9^3\right).$$

Q8 $\int \underbrace{x^2}_{v'} \underbrace{\ln(x)}_{u}\, dx$ $\qquad \int uv'dx = uv - \int u'v\,dx$

$u = \ln(x) \qquad u' = \frac{1}{x}$

$v' = x^2 \qquad v = \frac{1}{3}x^2$

$$\int x^2 \ln(x)\, dx = \frac{1}{3}x^3\ln(x) - \int \frac{1}{3}x^3 \cdot \frac{1}{x}\, dx$$

$$= \frac{1}{3}x^3\ln(x) - \int \frac{1}{3}x^2\, dx = \frac{1}{3}x^3\ln(x) - \frac{1}{9}x^3 + c$$

Q9 $$\int \underbrace{e^{-2x}}_{u} \underbrace{\cos(3x)}_{v'}\, dx = e^{-2x}\frac{1}{3}\sin(3x) - \int -2e^{-2x}\cdot\frac{1}{3}\sin(3x)\, dx$$

$$= \frac{1}{3}e^{-2x}\sin(3x) + \frac{2}{3}\underbrace{\int e^{-2x}\sin(3x)dx}$$

$$= e^{-2x}\cdot -\frac{1}{3}\cos(3x) - \int -2e^{-2x}\cdot -\frac{1}{3}\cos(3x)\, dx$$

so

$$\int e^{-2x}\cos(3x)dx = \frac{1}{3}e^{-2x}\sin(3x) - \frac{2}{9}e^{-2x}\cos(3x) \cancel{+} - \frac{4}{9}\int \cancel{4}e^{-2x}\cos(3x)dx$$

so $$\int e^{-2x}\cos(3x)dx = \frac{9}{13}\left[\frac{1}{3}e^{-2x}\sin(3x) - \frac{2}{3}e^{-2x}\cos(3x)\right] + c.$$

Q10 $\int_0^{\pi/2} \sin^3(x)\cos^2(x)\, dx$ $\qquad u = \cos(x)$

$\frac{du}{dx} = -\sin(x)$

$$\int_1^0 \sin^3(x)\, u^2 \frac{dx}{du}\, du = \int_1^0 \frac{\sin^3(x)\, u^2}{-\sin(x)}\, du = \int_0^1 \underbrace{\sin^2 x}_{1-\cos^2 x = 1-u^2}\, u^2\, du$$

$$\int_0^1 u^2 - u^4 \, du = \left[\tfrac{1}{3}u^3 - \tfrac{1}{5}u^5 \right]_0^1 = \tfrac{1}{3} - \tfrac{1}{5} = \tfrac{2}{15}$$

Q11

$$\int \cos(3x)\cos(5x)\,dx$$

$$\cos(A+B) = \cos A \cos B - \sin A \sin B$$
$$\cos(A-B) = \cos A \cos B + \sin A \sin B$$
$$\cos(A+B) + \cos(A-B) = 2\cos A \cos B$$

$$= \int \cos(8x) + \cos(-2x)\,dx = \left[\frac{1}{8}\sin(8x) + \frac{1}{2}\sin(2x) \right] + C$$

Q12

$$\int \frac{x^2}{\sqrt{x^2+1}}\,dx$$

$$\sin^2 x + \cos^2 x = 1$$
$$\tan^2 x + 1 = \sec^2 x$$

$$x = \tan u$$
$$\frac{dx}{du} = \sec^2 u$$

$$\int \frac{\tan^2 u}{\sqrt{\tan^2 u + 1}} \cdot \frac{dx}{du}\,du = \int \frac{\tan^2 u}{\sec u} \cdot \sec^2 u \,du$$

$$= \int \tan^2 u \sec u \,du = \int \underbrace{\tan u}_{u}\ \underbrace{\tan u \sec u}_{v'}\,du$$

$$= \tan u \sec u - \int \sec^2 u \cdot \sec u \,du = \tan u \sec u - \int (1+\tan^2 u)\sec u \,du$$

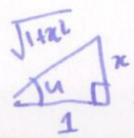
so $\int \tan^2 u \sec u \,du = \tan u \sec u - \int \sec u \,du - \int \tan^2 u \sec u \,du.$

$$2\int \tan^2 u \sec u \,du = \tan u \sec u - \ln|\sec u + \tan u| + c$$

so

$$= \frac{1}{2} x\sqrt{1+x^2} - \frac{1}{2}\ln\left| x + \sqrt{1+x^2} \right| + c$$

$\tan u = x$



$\sec u = \sqrt{1+x^2}$