Math 232 Calculus 2 Spring 15 Sample Final b

Name: Solutions

- I will count your best 8 of the following 10 questions.
- You may use a calculator, and a 3×5 index card of notes, but no phones or other notes.

1	10	
2	10	
3	10	
4	10	
5	10	
6	10	
7	10	
8	10	
9	10	
10	10	
	80	

Final	
Overall	

(1) Find the following integrals.

(a) $\int x\sin(3x)\ dx.$

$$\int uv'\,dx = uv - \int u'v\,dx$$

$u = x \qquad u' = 1$

$v' = \sin 3x \qquad v = -\frac{1}{3}\cos 3x$

$$= -\frac{1}{3}x\cos 3x + \frac{1}{3}\int \cos 3x\ dx$$

$$= -\frac{1}{3}x\cos 3x + \frac{1}{9}\sin 3x + c$$

(b) $\int \frac{1}{x\ln(x)}\ dx.$

$u = \ln(x)$

$\frac{du}{dx} = \frac{1}{x}$

$$\int \frac{1}{x.u}\frac{dx}{du}\,du = \int \frac{1}{x.u}\,x\,du = \int \frac{1}{u}\,du$$

$$= \ln(u) + c = \ln(\ln(x)) + c$$

(2) Find $\int \sin^3 x \, dx$. $= \int (1-\cos^2 x)\sin x \, dx$ $\quad u = \cos x$ $\quad \frac{du}{dx} = -\sin x$

$\int (1-u^2) \sin x \frac{dx}{du} \, du = \int (1-u^2) \sin x \frac{1}{-\sin x} \, du = \int u^2 - 1 \, du$

$= \frac{1}{3}u^3 - u + c = \frac{1}{3}\cos^3 x - \cos x + c$

(3) Find the volume of revolution obtained by rotating the curve $y = xe^{-2x}$ around the x-axis on the interval $[0, \infty)$.

$$\int_0^\infty \pi x^2 e^{-4x}\, dx \qquad \int uv'\,dx = uv - \int u'v\,dx$$

$$= \lim_{R\to\infty} \underline{\int_0^R \pi x^2 e^{-4x}\,dx} \qquad u = x^2 \quad u' = 2x \qquad v' = e^{-4x} \quad v = -\frac{1}{4}e^{-4x}$$

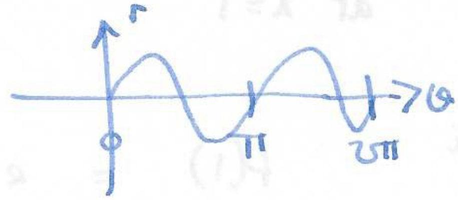
$$\pi \lim_{R\to\infty} \underbrace{\left[-\frac{1}{4}x^2e^{-4x}\right]_0^R}_{0.} + \frac{1}{4}\int_0^R 2xe^{-4x}\,dx$$

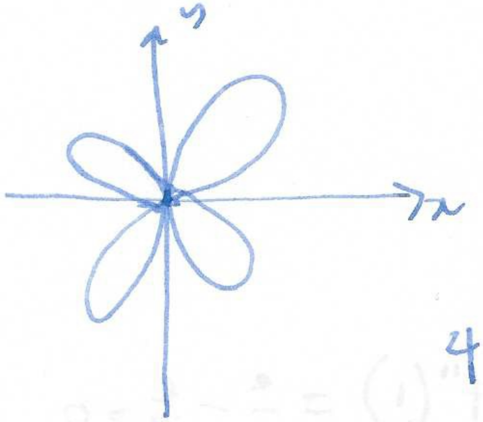
$$u = x \quad u' = 1 \qquad v' = e^{-4x} \quad v = -\frac{1}{4}e^{-4x}$$

$$\pi \lim_{R\to\infty} \frac{1}{2}\underbrace{\left[-\frac{1}{4}xe^{-4x}\right]_0^R}_{0.} - \frac{1}{2}\int_0^R -\frac{1}{4}e^{-4x}\,dx$$

$$\pi \lim_{R\to\infty} \frac{1}{8}\left[-\frac{1}{4}e^{-4x}\right]_0^R = \frac{\pi}{32}.$$

(4) Sketch the polar coordinate graph $r = \sin(2\theta)$ and find the area bounded by the curve.





symmetric, so area is

$$4\int_0^{\pi/2} \frac{1}{2}\sin^2 2\theta \, d\theta$$

$$\cos 2\theta = \cos^2\theta - \sin^2\theta$$
$$= 1 - 2\sin^2\theta$$

$$\sin^2\theta = \frac{1}{2} - \frac{1}{2}\cos 2\theta.$$

$$2\int_0^{\pi/2} \frac{1}{2} - \frac{1}{2}\cos 4\theta \, d\theta$$

$$\left[\theta - \frac{1}{4}\sin 4\theta\right]_0^{\pi/2} = \pi/2.$$

(5) Find the degree three Taylor polynomial for $e^{\sqrt{x}}$.

at $x = 1$

$f(x) = e^{\sqrt{x}} \qquad f(1) = e$

$f'(x) = e^{\sqrt{x}} \cdot \frac{1}{2}x^{-1/2} \qquad f'(1) = \frac{e}{2}$

$f''(x) = e^{\sqrt{x}} \cdot \frac{1}{2}x^{-1/2} \cdot \frac{1}{2}x^{-1/2} + e^{\sqrt{x}} \cdot -\frac{1}{4}x^{-3/2} \qquad f''(1) = \frac{e}{4} - \frac{e}{4} = 0$

$\cancel{f^{(3)}(x)} = e^{\sqrt{x}} \cdot \frac{1}{4x} \; \cancel{+} - \frac{e^{\sqrt{x}}}{4} x^{-3/2}$

$f^{(3)}(x) = e^{\sqrt{x}} \frac{1}{2}x^{-1/2} \cdot \frac{1}{4x} + e^{\sqrt{x}} \cdot \frac{1}{4} - x^{-2} - e^{\sqrt{x}} \cdot \frac{1}{2}x^{-1/2} \cdot \frac{x^{-3/2}}{4} \; \cancel{+} - \frac{e^{\sqrt{x}}}{4} \cdot -\frac{3}{2}x^{-5/2}.$

$f^{(3)}(1) = e\left(\frac{1}{8} - \frac{1}{4} - \frac{1}{8} + \frac{3}{8}\right) = \frac{e}{8}.$

$T_3 = e + \frac{e}{2}(x-1) + \frac{e}{8}\frac{(x-1)^3}{3!}$

(6) Find $\int \frac{x+4}{x^2-4}\,dx$.

$$\frac{x+4}{x^2-4} = \frac{A}{x+2} + \frac{B}{x-2} = \frac{A(x-2)+B(x+2)}{(x+2)(x-2)}$$

$x=2$: $6 = 4B$ $B = 3/2$

$x=-2$: $2 = -4A$ $A = -1/2$.

$$\int \frac{-1/2}{x+2} + \frac{3/2}{x-2}\,dx$$

$$= -\frac{1}{2}\ln|x+2| + \frac{3}{2}\ln|x-2| + C.$$

(7) Find $\displaystyle\int_0^\infty \frac{1}{x^2-2x+2}\,dx$.

$$\lim_{R\to\infty} \int_0^R \frac{1}{(x-1)^2+1}\,dx \qquad u = x-1 \qquad \frac{du}{dx} = 1.$$

$$\lim_{R\to\infty} \int_{-1}^{R} \frac{1}{u^2+1}\,du = \lim_{R\to\infty} \left[\tan^{-1}(u)\right]_{-1}^{R}$$

$$\cancel{= \lim_{R\to\infty} \underbrace{-\tan^{-1}(-1)}_{-\frac{\pi}{4}} + \underbrace{\tan^{-1}(R)}_{\to \frac{\pi}{2}}}$$

$$= \lim_{R\to\infty} \tan^{-1}(R) - \tan^{-1}(-1)$$

$$= \frac{\pi}{2} + \frac{\pi}{4} = \frac{3\pi}{4}$$

(8) Explain whether the following series converge or diverge, indicating clearly which tests you use.

(a) $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt{n}}$.

alternating series test $\sum (-1)^n a_n$ $a_n = \frac{1}{\sqrt{n}}$.

$\frac{1}{\sqrt{a_n}}$ positive, decreasing $\to 0$.

so converges.

(b) $\displaystyle\sum_{n=2}^{\infty} \frac{1}{\sqrt{n}-1}$.

comparison test with $\frac{1}{\sqrt{n}}$ ← diverges by p-series.

$\frac{1}{\sqrt{n}} < \frac{1}{\sqrt{n}-1}$

so $\sum \frac{1}{\sqrt{n}}$ diverges $\Rightarrow$ $\sum \frac{1}{\sqrt{n}-1}$ diverges.

(9) Explain whether the following series converges or diverges, indicating clearly which tests you use.

$$\sum_{n=2}^{\infty} \frac{1}{n\ln(n)}.$$

Q1b

integral test $\lim_{R\to\infty} \int_2^R \frac{1}{x\ln(x)}\,dx = \lim_{R\to\infty} \left[\ln(\ln(x))\right]_2^R$

diverges as $R \to \infty$

$\Rightarrow \sum \frac{1}{n\ln(n)}$ diverges.

(10) Find the power series for $\dfrac{\sin(x)}{x}$. Use this to find a power series for $\displaystyle\int \frac{\sin(x)}{x}\,dx$. What is the radius of convergence for this power series?

$$\sin(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} \cdots + \frac{x^{2n+1}}{(2n+1)!} + \cdots$$

$$\frac{\sin(x)}{x} = 1 - \frac{x^2}{3!} + \frac{x^4}{5!} \cdots + \frac{x^{2n}}{(2n+1)!} + \cdots$$

$$\int \frac{\sin(x)}{x}\,dx = c + x - \frac{x^3}{3.3!} + \frac{x^5}{5.5!} \cdots + \frac{x^{2n+1}}{(2n+1)(2n+1)!} + \cdots$$

radius of convergence

ratio test:

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{x^{2n+3}}{(2n+3)(2n+3)!}\,\frac{(2n+1)(2n+1)!}{x^{2n+1}}\right|.$$

$$= \lim_{n\to\infty} |x^2|\,\frac{2n+1}{2n+3}\,\frac{\quad}{(2n+2)(2n+3)} \to 0 \text{ as } n\to\infty$$

converges for all x

$R = \infty$.