Solutions MTH 232 Sample final

Q1 a) $\int_0^\infty xe^{-2x^2}\,dx$ $\quad u=-2x^2$, $\frac{du}{dx}=-4x$ $\quad \int_0^{-\infty} x e^u \frac{dx}{du}\,du$

$= \int_0^{-\infty} -\frac{1}{4}e^u\,du = \lim_{R\to\infty}\left[-\frac{1}{4}e^R+\frac{1}{4}\right] = \frac{1}{4}.$

b) $\int_0^\infty xe^{-2x}\,dx = \lim_{R\to\infty}\int_0^R xe^{-2x}\,dx$ $\quad \int uv'\,dx = uv-\int u'v\,dx$

$u=x \quad u'=1$

$v'=e^{-2x} \quad v=-\frac{1}{2}e^{-2x}$

$= \lim_{R\to\infty}\left[-\frac{1}{2}xe^{-2x}\right]_0^R + \int_0^R \frac{1}{2}e^{-2x}\,dx$

$= \lim_{R\to\infty} -\frac{1}{2}Re^{-2R} + \left[-\frac{1}{4}e^{-2x}\right]_0^R dx = \lim_{R\to\infty} \underbrace{-\frac{1}{2}Re^{-2R}}_{\to 0} \underbrace{-\frac{1}{4}e^{-2R}}_{\to 0} + \frac{1}{4} = \frac{1}{4}.$

c) $\int \sin^2 x\cos^3 x\,dx$ $\quad u=\sin x$, $\frac{du}{dx}=\cos x$ $\quad \int u^2\cos^3 x \cdot \frac{dx}{du}\,du = \int u^2\cos^2 x\,du$

$= \int u^2(1-\sin^2 x)\,du = \int u^2(1-u^2)\,du = \int u^2-u^4\,du = \frac{1}{3}u^3-\frac{1}{5}u^5+c$

$= \frac{1}{3}\sin^3 x-\frac{1}{5}\sin^5 x+c$

d) $\int \sin 2x\cos 3x\,dx$

$\sin(A+B) = \sin A\cos B+\cos A\sin B$

$\sin(A-B) = \sin A\cos B-\cos A\sin B$

$\sin(A+B)+\sin(A-B) = 2\sin A\cos B$

$= \frac{1}{2}\int \sin 5x+\sin(-x)\,dx = -\frac{1}{10}\cos 5x+\frac{1}{2}\cos x+c$

②

Q2 $f(x) = \ln(1+x^2)$ $\quad f(1) = \ln(2)$

$f'(x) = \frac{1}{1+x^2}\cdot 2x$ $\quad f'(1) = 1$

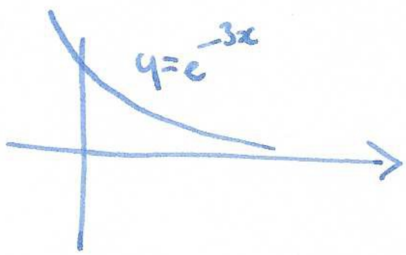
$f''(x) = \frac{(1+x^2).2 - 4x^2}{(1+x^2)^2} = \frac{2-2x^2}{(1+x^2)^2}$ $\quad f''(1) = 0$

$f^{(3)}(x) = \frac{(1+x^2)^2.-4x - 2(1+x^2).2x(2-2x^2)}{(1+x^2)^4}$ $\quad f^{(3)}(1) = \frac{-16}{16} = -1$

$$T_3 = f(1) + f'(1)(x-1) + \frac{f''(1)}{2!}(x-1)^2 + \frac{f^{(3)}(1)}{3!}(x-1)^3$$

$$= \ln(2) + (x-1) - \frac{1}{6}(x-1)^3$$

Q3



$$V = \int_0^\infty \pi(e^{-3x})^2\,dx$$

$$= \lim_{R\to\infty}\int_0^R \pi e^{-6x}\,dx = \lim_{R\to\infty}\left[\frac{\pi e^{-6x}}{-6}\right]_0^R$$

$$= \lim_{R\to\infty} \underbrace{\frac{-\pi}{6}e^{-6R}}_{\to 0} + \frac{\pi}{6} = \frac{\pi}{6}$$

Q4 $e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots + \frac{x^n}{n!} + \cdots$

$e^{-x^2} = 1 - x^2 + \frac{x^4}{2!} - \frac{x^6}{3!} + \cdots + \frac{x^{2n}}{n!} + \cdots$

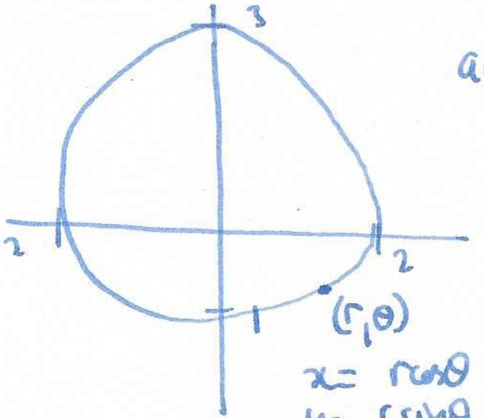
$x^2e^{-x^2} = x^2 - x^4 + \frac{x^6}{2!} - \frac{x^8}{3!} + \cdots + \frac{x^{2n+2}}{n!} + \cdots$

ratio test $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{x^{2n+4}}{(n+1)!}\cdot\frac{n!}{x^{2n+2}}\right| = \lim_{n\to\infty}\left|\frac{x^2}{n+1}\right| = 0$

$R = \infty$.

③

Q5



$x = r\cos\theta$
$y = r\sin\theta$

$$\text{area} = \int_0^{2\pi} \frac{1}{2}(2+\sin\theta)^2 \, d\theta$$

$r = 2 + \sin\theta$

$x = (2+\sin\theta)\cos\theta$
$y = (2+\sin\theta)\sin\theta$

$x = 2\cos\theta + \underbrace{\sin\theta\cos\theta}_{\frac{1}{2}\sin 2\theta}$
$y = 2\sin\theta + \sin^2\theta$

$$\frac{dx}{d\theta} = -2\sin\theta + \cos 2\theta$$

$$\frac{dy}{d\theta} = 2\cos\theta + 2\sin\theta\cos\theta$$

$$\frac{dy}{dx} = \frac{dy}{d\theta} \Big/ \frac{dx}{d\theta} = \frac{2\cos\theta + 2\sin\theta\cos\theta}{-2\sin\theta + \cos 2\theta}$$

$$\frac{dy}{dx}\left(\theta = \frac{-\pi}{4}\right) = \frac{\sqrt{2} + -2/2}{-2/\sqrt{2} + 0} = \frac{\sqrt{2}-1}{-\sqrt{2}}.$$

Q6 a) $$\int x + \frac{2}{x} \, dx = \frac{1}{2}x^2 + 2\ln|x| + c.$$

b)
$$\begin{array}{r} x - 2 \\ x+2 \overline{\smash{)}\, x^2 \qquad + 1} \\ \underline{x^2 + 2x \quad} \\ -2x + 1 \\ \underline{-2x - 4} \\ 5 \end{array}$$

$$\int x - 2 + \frac{5}{x+2} \, dx = \frac{1}{2}x^2 - 2x + 5\ln|x+2| + c$$

c) $$\int \frac{x}{2x^2+3} \, dx = \cancel{\frac{1}{2}\int \frac{x}{x^2+3/2} \, dx = \frac{1}{3}\int \frac{x}{\frac{2}{3}x^2+1} \, dx} \quad \cancel{u = \sqrt{\tfrac{2}{3}}}$$

~~$\sin^2x + \cos^2x = 1$~~ $\quad u = 2x^2 + 3$

$$\frac{du}{dx} = 6x \qquad \int \frac{x}{u} \frac{dx}{du} \, du = \int \frac{x}{u} \cdot \frac{1}{6x} \, du$$

$$= \frac{1}{6}\int \frac{1}{u} \, du = \frac{1}{6}\ln|u| + c = \frac{1}{6}\ln|2x^2+3| + c.$$

d) $\int \frac{1}{1+2x^2}\,dx$ $\sin^2 x+\cos^2 x=1$ $\tan^2 x+1=\sec^2 x$

$\sqrt{2}x=\tan u$

$x=\frac{1}{\sqrt{2}}\tan u$

$\frac{dx}{du}=\frac{1}{\sqrt{2}}\sec^2 u.$

$$\int \frac{1}{1+\tan^2 u}\frac{dx}{du}\,du=\int \frac{1}{\sec^2 u}\frac{1}{\sqrt{2}}\sec^2 u\,du=\frac{1}{\sqrt{2}}\int du=\frac{1}{\sqrt{2}}u+C$$

$$=\frac{1}{\sqrt{2}}\tan^{-1}(\sqrt{2}x)+C.$$

Q7 a) $\sum_{n=0}^{\infty}\left(\frac{-\sqrt{2}}{e}\right)^n$ geometric series $a+ar+ar^2+\dots=\frac{a}{1-r}$ if $|r|<1$

$a=1$ $r=\frac{-\sqrt{2}}{e}<1$. so converges to $\frac{1}{1+\sqrt{2}/e}$.

b) $\sum_{n=0}^{\infty}\frac{1}{2+n^2}$ comparison test: $\frac{1}{2+n^2}<\frac{1}{n^2}$. $\sum_{n=1}^{\infty}\frac{1}{n^2}$ converges (p-series or integral test). $p>1$

$\Rightarrow \sum_{n=0}^{\infty}\frac{1}{2+n^2}$ converges.

c) $\sum_{n=0}^{\infty}\frac{(-1)^n}{2+n^2}$ alternating series test: $a_n=\frac{1}{2+n^2}$ decreasing, $a_n\to 0$ $\Rightarrow$ converges.

d) $\sum_{n=0}^{\infty}\frac{10^n}{n!}$ ratio test: $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right|=\frac{10^{n+1}}{(n+1)!}\frac{n!}{10^n}=\frac{10}{n+1}\to 0<1$

$\Rightarrow$ converges

Q8 a) $\lim_{n\to\infty} 2-\frac{1}{n+1}=2$ converges

b) $\sum_{n=0}^{\infty}2-\frac{1}{n+1}$ does not converge as $a_n\not\to 0$.