Math 330 Differential Equations Fall 15 Midterm 2b

Name: Solutions

- I will count your best 8 of the following 10 questions.
- You may use yor textbooks and notes, but no electronic devices.

1	10	
2	10	
3	10	
4	10	
5	10	
6	10	
7	10	
8	10	
9	10	
10	10	
	80	

Midterm 1	
Overall	

(1) (10 points) Find all solutions to the following system of linear equations.

$$\begin{aligned} x_1 - x_2 + x_4 + 2x_5 &= 0 \\ 2x_1 - x_2 + x_3 + 3x_5 &= 0 \\ 3x_1 - 2x_2 + x_3 + x_4 + 6x_5 &= 0 \\ x_1 + x_3 - x_4 + 4x_5 &= 0 \end{aligned}$$

You may use the fact that

$$\begin{bmatrix} 1 & -1 & 0 & 1 & 2 \\ 2 & -1 & 1 & 0 & 3 \\ 3 & -2 & 1 & 1 & 6 \\ 1 & 0 & 1 & -1 & 4 \end{bmatrix} \text{ row reduces to } \begin{array}{c} \begin{array}{ccccc} x_1 & x_2 & x_3 & x_4 & x_5 \end{array} \\ \begin{bmatrix} \boxed{1} & 0 & 1 & -1 & 0 \\ 0 & \boxed{1} & 1 & -2 & 0 \\ 0 & 0 & 0 & 0 & \boxed{1} \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \end{array}.$$

free vars

$s \quad t$

$x_5 = 0$

$x_4 = t$

$x_3 = s$

$x_2 + s - 2t = 0 \qquad x_2 = -s + 2t$

$x_1 + s - t = 0 \qquad x_1 = -s + t$

$$\begin{bmatrix} -s+t \\ -s+2t \\ s \\ t \\ 0 \end{bmatrix} = \begin{bmatrix} -1 \\ -1 \\ 1 \\ 0 \\ 0 \end{bmatrix} s + \begin{bmatrix} 1 \\ 2 \\ 0 \\ 1 \\ 0 \end{bmatrix} t$$

(2) (10 points) Find an expression for a matrix (with respect to the standard basis) for the linear map from $\mathbb{R}^2 \to \mathbb{R}^2$ which expands by a factor of 2 in the direction $\underbrace{\begin{bmatrix} 1 \\ 2 \end{bmatrix}}_{v_1}$ and reverses the direction $\underbrace{\begin{bmatrix} 1 \\ 1 \end{bmatrix}}_{v_2}$.

$$\begin{tikzcd} \mathbb{R}^2_{e_1 e_2} \arrow[r, "A"] & \mathbb{R}^2_{e_1, e_2} \\ \mathbb{R}^2_{v_1 v_2} \arrow[u, "T"] \arrow[r, "D"', "\begin{bmatrix} 2 & 0 \\ 0 & -1 \end{bmatrix}"'] & \mathbb{R}^2_{v_1 v_2} \arrow[u, "T"] \end{tikzcd}$$

$$T = [v_1, v_2] = \begin{bmatrix} 1 & 1 \\ 2 & 1 \end{bmatrix}$$

$$A = T D T^{-1} = \begin{bmatrix} 1 & 1 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} 2 & 0 \\ 0 & -1 \end{bmatrix} \frac{1}{-1} \begin{bmatrix} 1 & -1 \\ -2 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 2 & -1 \\ 4 & -1 \end{bmatrix} \begin{bmatrix} -1 & +1 \\ +2 & -1 \end{bmatrix} = \begin{bmatrix} -4 & +3 \\ -6 & +5 \end{bmatrix}$$

(3) (10 points) Are the following two vector spaces the same?

$$V = \text{span}\{\underset{\underline{v_1}}{\begin{bmatrix} 1 \\ 1 \\ -1 \\ 2 \end{bmatrix}}, \underset{\underline{v_2}}{\begin{bmatrix} 1 \\ -1 \\ 1 \\ -1 \end{bmatrix}}\}, \quad W = \text{span}\{\underset{\underline{w_1}}{\begin{bmatrix} 2 \\ 0 \\ 0 \\ 1 \end{bmatrix}}, \underset{\underline{w_2}}{\begin{bmatrix} 0 \\ 2 \\ -2 \\ 1 \end{bmatrix}}\}.$$

You may use the fact that

$$\begin{bmatrix} 1 & 1 & 2 & 0 \\ 1 & -1 & 0 & 2 \\ -1 & 1 & 0 & -2 \\ 2 & -1 & 1 & 1 \end{bmatrix} \text{ row reduces to } \begin{bmatrix} \boxed{1} & 0 & 1 & 0 \\ 0 & \boxed{1} & 1 & 0 \\ 0 & 0 & 0 & \boxed{1} \\ 0 & 0 & 0 & 0 \end{bmatrix};$$

Explain your answer.

$\underline{w_1}$ is a linear combination of $\underline{v_1}, \underline{v_2}$ so $\underline{w_1} \in V$

$\underline{w_2}$ is $\underline{\text{not}}$ a linear combination of $\underline{v_1}, \underline{v_2}$ so $\underline{w_2} \notin V$

$\Rightarrow V \neq W$

(4) (10 points) Find the eigenvalues and eigenvectors of the matrix

$$A = \begin{bmatrix} -4 & 3 \\ -6 & 5 \end{bmatrix}.$$

eigenvalues: $\det(A-\lambda I)=0$ $\quad \begin{vmatrix} -4-\lambda & 3 \\ -6 & 5-\lambda \end{vmatrix} = (4+\lambda)(\lambda-5)+18$

$= \lambda^2-\lambda-2$

$\lambda = 2, -1$

$= (\lambda-2)(\lambda+1)$

eigenvectors: $\lambda=2$ $\quad \begin{bmatrix} -6 & 3 \\ -6 & 3 \end{bmatrix} \rightsquigarrow \begin{bmatrix} 2 & -1 \\ 0 & 0 \end{bmatrix} \quad \underline{v} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$

$\lambda=-1$ $\quad \begin{bmatrix} -3 & 3 \\ -6 & 6 \end{bmatrix} \rightsquigarrow \begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix} \quad \underline{v} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

(5) (10 points) Find the general solution to $X' = AX$, where

$$A = \begin{bmatrix} -4 & 3 \\ -6 & 5 \end{bmatrix}.$$

You may use your answer to the previous question.

$$X(t) = c_1 \begin{bmatrix} 1 \\ 2 \end{bmatrix} e^{2t} + c_2 \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{-t}$$

(6) (10 points) Solve the initial value problem $X' = AX$, where

$$A = \begin{bmatrix} -4 & 3 \\ -6 & 5 \end{bmatrix}, \text{ and } X(0) = \begin{bmatrix} 1 \\ 1 \end{bmatrix}.$$

You may use your answer to the previous question.

$$c_1 \begin{bmatrix} 1 \\ 2 \end{bmatrix} + c_2 \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \Rightarrow \quad c_2 = 1, c_1 = 0$$

$$\left[\begin{array}{cc|c} 1 & 1 & 1 \\ 2 & 1 & 1 \end{array}\right] \rightsquigarrow \left[\begin{array}{cc|c} 1 & 1 & 1 \\ 0 & -1 & -1 \end{array}\right]$$

$c_1 + 1 = 1 \Rightarrow c_1 = 0.$

$-c_2 = -1 \Rightarrow c_2 = 1$

so $X(t) = \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{-t}$

(7) (10 points) Find the general solution to the differential equation $X' = AX$, where
$$A = \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}.$$

eigenvalues : $\begin{vmatrix} 1-\lambda & 1 \\ -1 & 1-\lambda \end{vmatrix} = (1-\lambda)^2 + 1 = \lambda^2 - 2\lambda + 2 \qquad \lambda = \dfrac{2 \pm \sqrt{4-8}}{2}$

$\lambda = 1 \pm i$

eigenvectors: $\lambda = 1+i$
$\bar{\lambda} = 1-i$

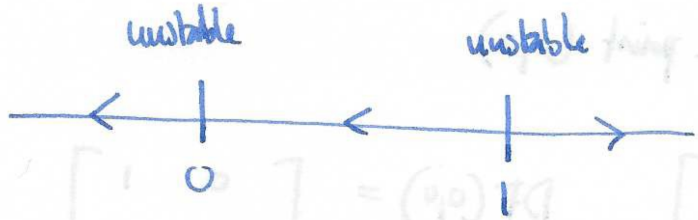
$\begin{bmatrix} -i & 1 \\ -1 & -i \end{bmatrix} \rightsquigarrow \begin{bmatrix} 1 & i \\ 0 & 0 \end{bmatrix} \qquad \underline{v} = \begin{bmatrix} i \\ -1 \end{bmatrix}$

$\underline{v} = \begin{bmatrix} 0 \\ -1 \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \end{bmatrix} i \qquad \bar{\underline{v}} = \begin{bmatrix} -i \\ -1 \end{bmatrix} \cancel{\begin{bmatrix} i \\ 1 \end{bmatrix}}$

general solution $X(t) = c_1 e^t \left(\begin{bmatrix} 0 \\ -1 \end{bmatrix} \cos t - \begin{bmatrix} 1 \\ 0 \end{bmatrix} \sin t \right)$
$+ c_2 e^t \left(\begin{bmatrix} 0 \\ -1 \end{bmatrix} \sin t + \begin{bmatrix} 1 \\ 0 \end{bmatrix} \cos t \right)$

(8) (10 points) Find the equilibrium solutions and draw the phase portrait for the differential equation $x' = x^3 - x^2$ and discuss their stability.

$x' = 0 = x^3 - x^2 = x^2(x-1)$ $\quad x = 0, 1$



x^2	+	+	+
$(x-1)$	−	−	+
x'	−	−	+

(9) (10 points) For what values of k are there stable equilibrium solutions for $x'' = -x + kx' - (x')^3$?

$x' = y$

$y' = -x + ky - (y)^3$

$X' = F(X)$

equilibrium solutions $F(X) = 0$ $\quad y = 0$

$-x + 0 = 0 \Rightarrow x = 0$

critical point $(0,0)$

linearize: $DF = \begin{bmatrix} 0 & 1 \\ -1 & k - 3y^2 \end{bmatrix}$ $\quad DF(0,0) = \begin{bmatrix} 0 & 1 \\ -1 & k \end{bmatrix}$

eigenvalues: $\begin{vmatrix} -\lambda & 1 \\ -1 & k-\lambda \end{vmatrix} = (\lambda - k)\lambda + 1 = \lambda^2 - k\lambda + 1 = 0$

$\lambda = \dfrac{k \pm \sqrt{k^2-4}}{2}$

note: $k > \sqrt{k^2-4}$ if $k^2-4>0$

~~$k > 0$ unstable~~
~~$k < 0$ sta~~

$k > 0$ unstable

$k \leq 0$ stable

real | complex | complex | real $\to k$ (marks at -2, 0, 2)

$k^2 > 4$, 2 real solutions, $\lambda_2 < \lambda_1 < 0$, ~~saddle~~

-2 real$(\lambda) < 0$ stable

0 real$(\lambda) > 0$ unstable

2 $k^2 > 4$, 2 real solution, $0 < \lambda_1 < \lambda_2$, unstable

stable sink

$k = 0$

$\lambda = \pm i$ stable elliptic

$k = -2$

$-1, -1$

stable

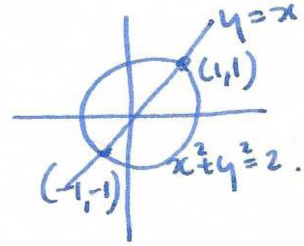
$k = 2 : 1, 1$

unstable

(10) (10 points) Find the equilibrium solutions for the following system of differential equations and decide whether or not they are stable.

$$x' = y - x$$
$$y' = x^2 + y^2 - 2$$

$\dot{X} = F(X)$



$F(X) = 0$ $\quad y = x$

$x^2+y^2=2 \Rightarrow 2x^2=2 \quad x=\pm 1 \quad (+1,1) \; (-1,-1)$

$DF = \begin{bmatrix} -1 & 1 \\ 2x & 2y \end{bmatrix}$ $\quad DF(1,1) = \begin{bmatrix} -1 & 1 \\ 2 & 2 \end{bmatrix}$ eigenvalues

$(-1-\lambda)(2-\lambda) - 2 = 0$

$\lambda^2 - \lambda - 4 = 0$

$\lambda = \dfrac{1 \pm \sqrt{1+16}}{2}$ two real, different signs saddle (unstable)

$DF(-1,-1) \quad \begin{bmatrix} -1 & 1 \\ -2 & -2 \end{bmatrix}$ eigenvalues

$(-1-\lambda)(-2-\lambda) + 2 = 0$

$\lambda^2 + 3\lambda + 4 = 0$

$\lambda = \dfrac{-3 \pm \sqrt{9-16}}{2}$ complex real$(\lambda) < 0$ stable sink