Math 330 Differential Equations Fall 15 Midterm 2a

Name: Solutions

- I will count your best 8 of the following 10 questions.
- You may use yor textbooks and notes, but no electronic devices.

1	10	
2	10	
3	10	
4	10	
5	10	
6	10	
7	10	
8	10	
9	10	
10	10	
	80	

Midterm 1	
Overall	

(1) (10 points) Find all solutions to the following system of linear equations.

$$x_1 - x_2 + x_3 - x_4 + x_5 = 0$$
$$x_1 - x_2 + 2x_3 - 3x_4 + x_5 = 0$$
$$2x_1 - 2x_2 + x_3 + 3x_5 = 0$$
$$2x_1 - 2x_2 + 2x_3 - 2x_4 + 4x_5 = 0$$

You may use the fact that

$$\begin{bmatrix} 1 & -1 & 1 & -1 & 1 \\ 1 & -1 & 2 & -3 & 1 \\ 2 & -2 & 1 & 0 & 3 \\ 2 & -2 & 2 & -2 & 4 \end{bmatrix} \text{ row reduces to } \begin{array}{c} \begin{matrix} x_1 & x_2 & x_3 & x_4 & x_5 \end{matrix} \\ \begin{bmatrix} \boxed{1} & -1 & 0 & 1 & 0 \\ 0 & 0 & \boxed{1} & -2 & 0 \\ 0 & 0 & 0 & 0 & \boxed{1} \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \\ \begin{matrix} & \uparrow \text{free var } s & & \uparrow \text{free var } t & \end{matrix} \end{array}.$$

$x_5 = 0$

$x_4 = t$

$x_3 - 2t = 0 \qquad x_3 = 2t$

$x_2 = s$

$x_1 - s + t = 0 \qquad x_1 = s - t$

$$\begin{bmatrix} s-t \\ s \\ 2t \\ t \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} s + \begin{bmatrix} -1 \\ 0 \\ 2 \\ 1 \\ 0 \end{bmatrix} t$$

(2) (10 points) Find an expression for a matrix (with respect to the standard basis) for the linear map from $\mathbb{R}^2 \to \mathbb{R}^2$ which expands by a factor of 2 in the direction $\underbrace{\begin{bmatrix}1\\1\end{bmatrix}}_{\underline{v_1}}$ and reverses the direction $\underbrace{\begin{bmatrix}1\\2\end{bmatrix}}_{\underline{v_2}}$.

$$\begin{tikzcd}
\mathbb{R}^2_{\underline{e_1},\underline{e_2}} \arrow[r, "A"] \arrow[d, leftrightarrow, "T"] & \mathbb{R}^2_{\underline{e_1},\underline{e_2}} \arrow[d, leftrightarrow, "T"] \\
\mathbb{R}^2_{\underline{v_1},\underline{v_2}} \arrow[r] & \mathbb{R}^2_{\underline{v_1},\underline{v_2}}
\end{tikzcd}$$

$$T = [\underline{v_1}\ \underline{v_2}] = \begin{bmatrix}1 & 1\\1 & 2\end{bmatrix}$$

$$\begin{bmatrix}2 & 0\\0 & -1\end{bmatrix} = D$$

$$A = TDT^{-1} = \begin{bmatrix}1 & 1\\1 & 2\end{bmatrix}\begin{bmatrix}2 & 0\\0 & -1\end{bmatrix}\begin{bmatrix}2 & -1\\-1 & 1\end{bmatrix}$$

$$= \begin{bmatrix}2 & -1\\2 & -2\end{bmatrix}\begin{bmatrix}2 & -1\\-1 & 1\end{bmatrix} = \begin{bmatrix}5 & -3\\6 & -4\end{bmatrix}$$

(3) (10 points) Are the following two vector spaces the same?

$$V = \text{span}\{\underset{\underline{v_1}}{\begin{bmatrix} 1 \\ -1 \\ 1 \\ 2 \end{bmatrix}}, \underset{\underline{v_2}}{\begin{bmatrix} 1 \\ 1 \\ -1 \\ -1 \end{bmatrix}}\}, \quad W = \text{span}\{\underset{\underline{w_1}}{\begin{bmatrix} 2 \\ 0 \\ 0 \\ 1 \end{bmatrix}}, \underset{\underline{w_2}}{\begin{bmatrix} 0 \\ 2 \\ -2 \\ 1 \end{bmatrix}}\}.$$

You may use the fact that

$$\begin{bmatrix} 1 & 1 & 2 & 0 \\ -1 & 1 & 0 & 2 \\ 1 & -1 & 0 & -2 \\ 2 & -1 & 1 & 1 \end{bmatrix} \text{ row reduces to } \begin{bmatrix} \boxed{1} & 0 & 1 & 0 \\ 0 & \boxed{1} & 1 & 0 \\ 0 & 0 & 0 & \boxed{1} \\ 0 & 0 & 0 & 0 \end{bmatrix},$$

Explain your answer.

no pivot in col 3 $\Rightarrow$ $\underline{w_1}$ is a linear combination of $\underline{v_1}, \underline{v_2}$ $\Rightarrow$ $\underline{w_1} \in V$

pivot in col 4 $\Rightarrow$ $\underline{w_2}$ is $\underline{\text{not}}$ a linear combination of $\underline{v_1}, \underline{v_2}$ $\Rightarrow$ $w_2 \notin V$

$\Rightarrow$ $V \neq W$.

(4) (10 points) Find the eigenvalues and eigenvectors of the matrix

$$A = \begin{bmatrix} 5 & -3 \\ 6 & -4 \end{bmatrix}.$$

$$\begin{vmatrix} 5-\lambda & -3 \\ 6 & -4-\lambda \end{vmatrix} = -(5-\lambda)(4+\lambda) + 18$$

$$= \lambda^2 - \lambda \cancel{+12} - 2 = \cancel{(\lambda+4)(\lambda+3)}$$

$$= (\lambda-2)(\lambda+1)$$

$$\lambda = 2: \quad \begin{bmatrix} 3 & -3 \\ 6 & -6 \end{bmatrix} \sim \begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix} \quad \underline{v} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$\lambda = -1: \quad \begin{bmatrix} 6 & -3 \\ 6 & -3 \end{bmatrix} \sim \begin{bmatrix} 2 & -1 \\ 0 & 0 \end{bmatrix} \quad \underline{v} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

(5) (10 points) Find the general solution to $X' = AX$, where

$$A = \begin{bmatrix} 5 & -3 \\ 6 & -4 \end{bmatrix}.$$

You may use your answer to the previous question.

$$X(t) = c_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{2t} + c_2 \begin{bmatrix} 1 \\ 2 \end{bmatrix} e^{-t}$$

(6) (10 points) Solve the initial value problem $X' = AX$, where

$$A = \begin{bmatrix} 5 & -3 \\ 6 & -4 \end{bmatrix}, \text{ and } X(0) = \begin{bmatrix} 1 \\ 1 \end{bmatrix}.$$

You may use your answer to the previous question.

$$c_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} + c_2 \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \Rightarrow \begin{array}{l} c_1 = 1 \\ c_2 = 0 \end{array}$$

$$\left(\left[\begin{array}{cc|c} 1 & 1 & 1 \\ 1 & 2 & 1 \end{array}\right] \sim\!> \left[\begin{array}{cc|c} 1 & 1 & 1 \\ 0 & 1 & 0 \end{array}\right] \Rightarrow c_2 = 0 \quad c_1 + 0 = 1 \Rightarrow c_1 = 1.$$

$$X(t) = \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{2t}$$

(7) (10 points) Find the general solution to the differential equation $X' = AX$, where

$$A = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}.$$

eigenvalues: $\det(A - \lambda I) = 0$ $\quad \begin{vmatrix} 1-\lambda & -1 \\ 1 & 1-\lambda \end{vmatrix} = (1-\lambda)^2 + 1 = \lambda^2 - 2\lambda + 2$

$\lambda = \frac{2 \pm \sqrt{4-8}}{2} = 1 \pm i$

v: $(A - \lambda I)\underline{v} = 0$ $\quad \begin{bmatrix} -i & -1 \\ 1 & -i \end{bmatrix} \rightsquigarrow \begin{bmatrix} 1 & -i \\ 0 & 0 \end{bmatrix}$ $\quad \underline{v} = \begin{bmatrix} i \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \end{bmatrix} i$

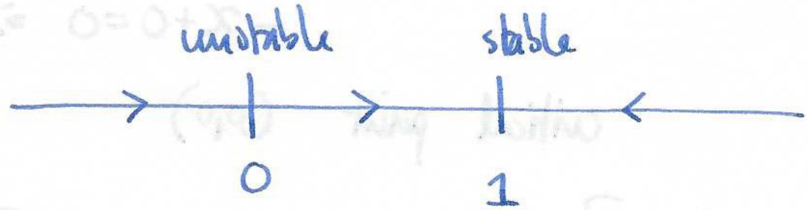
$\lambda = 1 + i$

$\bar{\lambda} = 1 - i$ $\quad \bar{\underline{v}} = \begin{bmatrix} -i \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix} - \begin{bmatrix} 1 \\ 0 \end{bmatrix} i$

$$X(t) = c_1 e^t \left(\begin{bmatrix} 0 \\ 1 \end{bmatrix} \cos t - \begin{bmatrix} 1 \\ 0 \end{bmatrix} \sin t \right) + c_2 e^t \left(\begin{bmatrix} 0 \\ 1 \end{bmatrix} \sin t + \begin{bmatrix} 1 \\ 0 \end{bmatrix} \cos t \right)$$

(8) (10 points) Find the equilibrium solutions and draw the phase portrait for the differential equation $x' = x^2 - x^3$ and discuss their stability.

$$x' = 0 = x^2 - x^3 = x^2(1-x) \qquad x = 0, 1$$



x^2	+	+	+
$(1-x)$	+	+	−
x'	+	+	−

(9) (10 points) For what values of k are there stable equilibrium solutions for $x'' = -x + kx' - (x')^3$?

$x' = y$

$y' = -x + ky - (y)^3$

$X' = F(X)$

equilibrium solutions $F(X) = 0$ $\quad y = 0$

$-x + 0 = 0 \Rightarrow x = 0$

critical point $(0,0)$

linearize: $DF = \begin{bmatrix} 0 & 1 \\ -1 & k-3y^2 \end{bmatrix}$ $\quad DF(0,0) = \begin{bmatrix} 0 & 1 \\ -1 & k \end{bmatrix}$

eigenvalues: $\begin{vmatrix} -\lambda & 1 \\ -1 & k-\lambda \end{vmatrix} = (\lambda - k)\lambda + 1 = \lambda^2 - k\lambda + 1 = 0$

$\lambda = \dfrac{k \pm \sqrt{k^2-4}}{2}$

note

$k > \sqrt{k^2-4}$ if $k^2-4>0$

~~k > 0 unstable~~

~~k < 0 sta~~

$k > 0$ unstable

$k \leq 0$ stable

real $\quad$ complex $\quad$ complex $\quad$ real

$-2 \quad 0 \quad 2 \quad k$

$k^2 > 4$

2 real solutions

$\lambda_2 < \lambda_1 < 0$

~~saddle~~

stable sink

real(λ) < 0

stable

real(λ) > 0

unstable

$k^2 > 4$

2 real solution

$0 < \lambda_1 < \lambda_2$

unstable

$k = 0$

$\lambda = \pm i$ stable elliptic

$k = -2$

$-1, -1$

stable

$k = 2 : 1, 1$

unstable

(10) (10 points) Find the equilibrium solutions for the following system of differential equations and decide whether or not they are stable.

$$x' = x - y$$
$$y' = x^2 + y^2 - 2$$

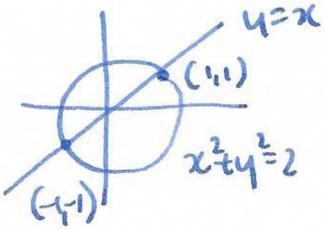
$\mathbf{x}' = F(\mathbf{x})$

solve $F(\mathbf{x}) = 0$

$x - y = 0 \Rightarrow x = y$

$x^2 + y^2 = 2 \Rightarrow 2x^2 = 2$

$x = \pm 1$



critical points $(1,1), (-1,-1)$

linearize: $DF = \begin{bmatrix} 1 & -1 \\ 2x & 2y \end{bmatrix}$

$DF(1,1) = \begin{bmatrix} 1 & -1 \\ 2 & 2 \end{bmatrix}$ eigenvalues $(1-\lambda)(2-\lambda) + 2 = 0$

$\lambda^2 - 3\lambda + 4 = (\lambda - 2)(\lambda - 1)$

$\lambda = \frac{3 \pm \sqrt{9-16}}{2}$ ~~$\lambda = 2, 1$~~ real part > 0 unstable

$DF(-1,-1) = \begin{bmatrix} 1 & -1 \\ -2 & -2 \end{bmatrix}$ eigenvalues $(1-\lambda)(-2-\lambda) - 2 = 0$

$\lambda^2 + \lambda - 4 = 0$

$\lambda = \frac{-1 \pm \sqrt{1+16}}{2}$ saddle (unstable).