Sample midterm 2 Solutions

Q1 a) $\det(A-\lambda I) = \begin{vmatrix} 1-\lambda & 1 & 0 \\ -1 & 1-\lambda & 0 \\ 0 & 2 & 1-\lambda \end{vmatrix} = (1-\lambda)\begin{vmatrix} 1-\lambda & 0 \\ 2 & 1-\lambda \end{vmatrix} - \begin{vmatrix} -1 & 0 \\ 0 & 1-\lambda \end{vmatrix}.$

$= (1-\lambda)\left((1-\lambda)^2 + \overset{1}{\cancel{(1-\lambda)}} \right).$

$= (1-\lambda)(\lambda^2 - 2\lambda + 2)$

$\lambda = 1, \quad \dfrac{2 \pm \sqrt{4-8}}{2} = 1 \pm i$

b) $\lambda = 1$: $\begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 2 & 0 \end{bmatrix} \rightsquigarrow \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ $\quad \underline{v}_1 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$

$\lambda = 1+i$ $\quad \begin{bmatrix} -i & 1 & 0 \\ -1 & -i & 0 \\ 0 & 2 & -i \end{bmatrix} \rightsquigarrow \begin{bmatrix} 1 & i & 0 \\ 0 & 2 & -i \\ 0 & 0 & 0 \end{bmatrix}$

$x_3 = t$

$2x_2 - it = 0 \quad x_2 = \frac{1}{2} i t$

$x_1 + -\frac{1}{2}t = 0 \quad x_1 = \frac{1}{2}t.$

$\underline{v} = \begin{bmatrix} 1/2 \\ 1/2\, i \\ 1 \end{bmatrix}$ so $\bar{\lambda} = 1 - i$ has eigenvalue $\bar{\underline{v}} = \begin{bmatrix} 1/2 \\ -1/2\, i \\ 1 \end{bmatrix}$

Q2

a) $[\underline{e}_1\ \underline{e}_2\ \underline{e}_3] = \begin{bmatrix} 0 & -2 & 3 \\ -1 & -2 & -6 \\ 3 & 1 & 12 \end{bmatrix} \rightsquigarrow \begin{bmatrix} 1 & 2 & 6 \\ 3 & 1 & 12 \\ 0 & -2 & 3 \end{bmatrix} \rightsquigarrow \begin{bmatrix} 1 & 2 & 6 \\ 0 & -5 & -6 \\ 0 & -2 & 3 \end{bmatrix}$

$\rightsquigarrow \begin{bmatrix} 1 & 2 & 6 \\ 0 & 2 & -3 \\ 0 & 10 & 12 \end{bmatrix} \rightsquigarrow \begin{bmatrix} \boxed{1} & 2 & 6 \\ 0 & \boxed{2} & -3 \\ 0 & 0 & \boxed{27} \end{bmatrix}$ $\Rightarrow$ linearly dependent $\overset{\text{3 vectors in } \mathbb{R}^3}{\Rightarrow}$ basis.

b) $\left[\begin{array}{ccc|c} 0 & -2 & 3 & 13 \\ -1 & -2 & -6 & -12 \\ 3 & 1 & 12 & 28 \end{array}\right] \rightsquigarrow \left[\begin{array}{ccc|c} 1 & 2 & 6 & 12 \\ 3 & 1 & 12 & 28 \\ 0 & -2 & 3 & 13 \end{array}\right] \rightsquigarrow \left[\begin{array}{ccc|c} 1 & 2 & 6 & 12 \\ 0 & -5 & -6 & -8 \\ 0 & -2 & 3 & 13 \end{array}\right]$

$\rightsquigarrow \left[\begin{array}{ccc|c} 1 & 2 & 6 & 12 \\ 0 & -2 & 3 & 13 \\ 0 & -1 & -12 & -34 \end{array}\right] \rightsquigarrow \left[\begin{array}{ccc|c} 1 & 2 & 6 & 12 \\ 0 & 1 & 12 & 34 \\ 0 & 0 & 27 & 81 \end{array}\right]$ $\quad x_3 = 3 \quad x_2 = -2 \quad x_1 = -2$

c) $\underline{e}_1 \cdot \underline{e}_3 = \|\underline{e}_1\| \|\underline{e}_3\| \cos\theta = \sqrt{10}\sqrt{9+36+144}\cos\theta = 6+36$

$\cos\theta = \dfrac{42}{\sqrt{10}\sqrt{189}} \qquad \theta = \cos^{-1}\left(\dfrac{42}{\sqrt{10}\sqrt{189}}\right).$

volume of parallelepiped $= \underline{e}_1 \cdot \underline{e}_2 \times \underline{e}_3 = \begin{vmatrix} 0 & -1 & 3 \\ -2 & -2 & 1 \\ 3 & -6 & 12 \end{vmatrix} = \begin{vmatrix} -2 & 1 \\ 3 & 12 \end{vmatrix} + 3\begin{vmatrix} -2 & -2 \\ 3 & -6 \end{vmatrix}$

$= -24 + 3 + 3(12+6) = -21 + 54 = 33.$

<u>Q3</u> $X' = AX \quad A = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$ find eigenvalues $\det(A - \lambda I) = \begin{vmatrix} 1-\lambda & 0 & 1 \\ 0 & 1-\lambda & 0 \\ 1 & 0 & 1-\lambda \end{vmatrix}$

$= (1-\lambda)\begin{vmatrix} 1-\lambda & 0 \\ 0 & 1-\lambda \end{vmatrix} + \begin{vmatrix} 0 & 1-\lambda \\ 1 & 0 \end{vmatrix} = (1-\lambda)\left[(1-\lambda)^2 - 1\right] = \lambda(1-\lambda)(\lambda-2)$

$\lambda = 0, 1, 2$ find eigenvalues $\lambda = 0: \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} \rightsquigarrow \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad \underline{v}_0 = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$

$\lambda = 1: \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix} \rightsquigarrow \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \quad \underline{v}_1 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$

$\lambda_2: \begin{bmatrix} -1 & 0 & 1 \\ 0 & -1 & 0 \\ 1 & 0 & -1 \end{bmatrix} \rightsquigarrow \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad \underline{v}_2 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$

general solution: $\underline{X}(t) = c_1\begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} + c_2\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}e^{t} + c_3\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}e^{2t}$

$\underline{X}(0) = \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix}$: $\left[\begin{array}{ccc|c} -1 & 0 & 1 & 1 \\ 0 & 1 & 0 & -1 \\ 1 & 0 & 1 & 2 \end{array}\right] \rightsquigarrow \left[\begin{array}{ccc|c} -1 & 0 & 1 & 1 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 2 & 3 \end{array}\right] \quad 2c_3 = 3$

$c_3 = 3/2 \quad c_2 = -1 \quad -c_1 + \frac{3}{2} = 1 \quad c_1 = +1/2$

particular solution: $X(t) = \frac{1}{2}\begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} - \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}e^{t} + \frac{3}{2}\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}e^{2t}.$

Q4 $X' = AX$ $\quad A = \begin{bmatrix}-1 & 2\\ -1 & -3\end{bmatrix}$ find eigenvalues $\det(A-\lambda I) = \begin{vmatrix}-1-\lambda & 2\\ -1 & -3-\lambda\end{vmatrix}$

$= +(1+\lambda)(3+\lambda)+2 = \lambda^2+4\lambda+5$ $\quad \lambda = \frac{-4\pm\sqrt{16-20}}{2} = -2\pm i$

$\lambda = 1+i$

find eigenvectors: $(A-\lambda I)\underline{v} = 0$: $\begin{bmatrix}1-i & 2\\ -1 & -1-i\end{bmatrix} \rightsquigarrow \begin{bmatrix}1 & 1+i\\ 0 & 0\end{bmatrix}$ $\quad \underline{v} = \begin{bmatrix}1+i\\ -1\end{bmatrix}$

$\underline{v} = \underline{a}+\underline{b}i = \begin{bmatrix}1\\ -1\end{bmatrix} + \begin{bmatrix}1\\ 0\end{bmatrix} i$

$\lambda = \alpha+\beta t$

general solution: $c_1 e^{\alpha t}(\underline{a}\cos\beta t - \underline{b}\sin\beta t) + c_2 e^{\alpha t}(\underline{a}\sin\beta t + \underline{b}\cos\beta t)$.

$\underline{X}(t) = c_1 e^{t}\left(\begin{bmatrix}1\\ -1\end{bmatrix}\cos t - \begin{bmatrix}1\\ 0\end{bmatrix}\sin t\right) + c_2 e^{t}\left(\begin{bmatrix}1\\ -1\end{bmatrix}\sin t + \begin{bmatrix}1\\ 0\end{bmatrix}\cos t\right)$.

$\underline{X}(0) = \begin{bmatrix}2\\ -1\end{bmatrix} = c_1\begin{bmatrix}1\\ -1\end{bmatrix} + c_2\begin{bmatrix}1\\ 0\end{bmatrix}$ $\qquad \left[\begin{array}{cc|c}1 & 1 & 2\\ -1 & 0 & -1\end{array}\right] \rightsquigarrow$ $\quad c_1 = 1$, $c_2 = 1$

particular solution $\underline{X}(t) = e^{t}\left(\begin{bmatrix}2\\ -1\end{bmatrix}\cos t + \begin{bmatrix}0\\ -1\end{bmatrix}\sin t\right)$.

Q5 $\left[\begin{array}{ccc|c}1 & -3 & 1 & 4\\ 2 & -8 & 8 & -2\\ 6 & -3 & 15 & -9\end{array}\right] \rightsquigarrow \left[\begin{array}{ccc|c}1 & -3 & 1 & 4\\ 0 & -2 & 6 & -10\\ 0 & 15 & 9 & -33\end{array}\right] \rightsquigarrow \left[\begin{array}{ccc|c}1 & -3 & 1 & 4\\ 0 & 1 & -3 & 5\\ 0 & 5 & 3 & -11\end{array}\right]$

$\rightsquigarrow \left[\begin{array}{ccc|c}1 & -3 & 1 & 4\\ 0 & 1 & -3 & 5\\ 0 & 0 & 18 & -36\end{array}\right]$ $\quad x_3 = -2$ $\quad x_2 = -1$ $\quad x_1 = 4-3+2 = 3$.

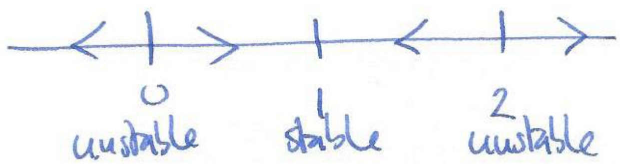
Q6 $\begin{bmatrix}1 & 2 & 3 & 4 & 3\\ 2 & 4 & 6 & 2 & 6\\ 3 & 6 & 18 & 9 & 9\\ 4 & 8 & 12 & 10 & 12\\ 5 & 10 & 24 & 11 & 15\end{bmatrix} \rightsquigarrow \begin{bmatrix}1 & 2 & 3 & 4 & 3\\ 0 & 0 & 0 & -6 & 0\\ 0 & 0 & 9 & -3 & 0\\ 0 & 0 & 0 & -6 & 0\\ 0 & 0 & 9 & -9 & 0\end{bmatrix} \rightsquigarrow \begin{bmatrix}\boxed{1} & 2 & 3 & 4 & 3\\ 0 & 0 & \boxed{1} & -1 & 0\\ 0 & 0 & 0 & \boxed{1} & 0\\ 0 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 0\end{bmatrix}$

$x_5 = t$ $\quad x_4 = 0$ $\quad x_3 - 0 = 0$ $\quad x_3 = 0$ $\quad x_2 = s$ $\quad x_1 + 2s + 3t = 0$

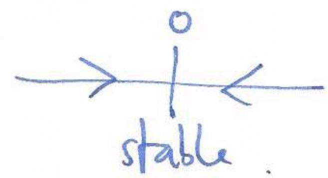
$\begin{bmatrix}-2s-3t\\ s\\ 0\\ 0\\ t\end{bmatrix}$.

Solutions: $\begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} s + \begin{bmatrix} -3 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} t$.

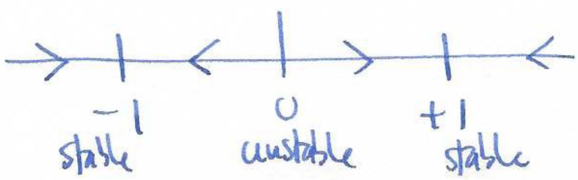
Q7 a) $y' = y(y-1)(y-2)$ equilibrium solutions: $y=0$, $y=1$, $y=2$.



b) $y' = -2\tan^{-1}\left(\frac{y}{1+y^2}\right)$ equilibrium solution: $y=0$



c) $y' = y(1-y^2)$ equilibrium solutions $y = 0, \pm 1$



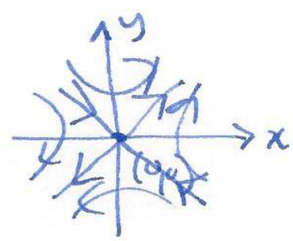
Q8 a) $x'' - x = 0$ $\quad$ $x' = y$, $y' = x$ $\quad$ critical points: $y=0$, $x=0$ $(0,0)$

linearize: $\underline{F}(\underline{x}) = \begin{bmatrix} y \\ x \end{bmatrix}$ $\quad DF = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ $\quad$ eigenvalues: $\begin{vmatrix} -\lambda & 1 \\ 1 & -\lambda \end{vmatrix} = \lambda^2 - 1$

$\lambda = \pm 1$ unstable (saddle).

eigenvectors: $\underline{v}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ $\quad \underline{v}_{-1} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$

phase portrait:



b) $x'' - x + x^3 = 0$ $\quad$ $x' = y$, $\quad y' = x - x^3$ $\quad$ $\underline{F}(\underline{x}) = \begin{bmatrix} y \\ x - x^3 \end{bmatrix}$ $\quad$ critical points (5)

$y = 0$

$x - x^3 = 0 \quad x(1-x^2) = 0$

$x = 0, \pm 1.$

critical points $(0,0)$ $(-1,0)$ $(1,0)$

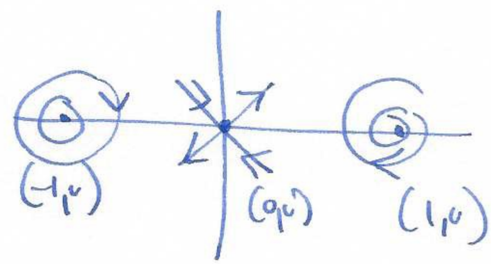
$DF = \begin{bmatrix} 0 & 1 \\ 1-3x^2 & 0 \end{bmatrix}$ $\quad$ $DF\left(\begin{bmatrix} 0 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ eigenvalues: $\lambda^2 - 1 = 0$

$\lambda = \pm 1$ unstable saddle

$DF\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} 0 & 1 \\ -2 & 0 \end{bmatrix}$ eigenvalues $\lambda^2 + 2 = 0$ $\quad \lambda = \pm\sqrt{2}\, i$ stable elliptic

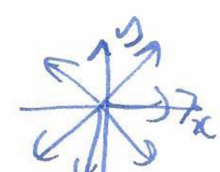
$DF\left(\begin{bmatrix} -1 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} 0 & 1 \\ -2 & 0 \end{bmatrix}$ " " " "

phase portrait



<u>Q9</u> a) $x' = x + x^2 + xy^2$, $\quad y' = y + y^{3/2}$ $\quad$ $DF = \begin{bmatrix} 1 + 2x + y^2 & 2xy \\ 0 & 1 + \frac{3}{2}y^{1/2} \end{bmatrix}$

$DF\left(\begin{bmatrix} 0 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ eigenvalues $1, 1$ eigenvectors $\begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix}$



unstable source.

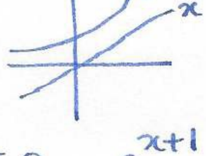
b) $x' = x^2 e^y$, $\quad y' = ye^x - y$ $\quad$ $DF = \begin{bmatrix} 2xe^y & x^2 e^y \\ ye^x & e^x - 1 \end{bmatrix}$ $\quad$ $DF\left(\begin{bmatrix} 0 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$

no information.

<u>Q10</u> $x' = e^{x+y} - y$, $\quad y' = -x + xy$ $\quad$ find critical points: $e^{x+y} - y = 0$

$-x + xy = 0 \Rightarrow x(y-1) = 0$

$x = 0 \Rightarrow e^y = y$

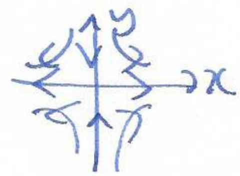


no solutions

~~or~~ $y = 1$ $\quad e^{x+1} - 1 = 0 \quad e^{x+1} = 1 \Rightarrow x = -1$ $\quad$ unique critical point $(-1, 1)$

linearization: $DF = \begin{bmatrix} e^{x+y} & e^{x+y}-1 \\ -1+y & x \end{bmatrix}$

$DF\left(\begin{bmatrix}-1\\1\end{bmatrix}\right) = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$ eigenvalues $\lambda = 1, -1$ unstable saddle



<u>Q11</u> $I' = I - \alpha C$ $\alpha > 1$, $\beta \geq 1$

$C' = \beta(I - C - G)$

a) find equilibrium solution: $I - \alpha C = 0$ $\quad I = \alpha C$

$\beta(I - C - G_0) = 0 \quad \beta(\alpha C - C - G_0) = 0$

$\beta(C(\alpha-1) - G_0) = 0$

critical point $\cancel{(C, I) = \left(\frac{G_0}{\alpha-1}, \frac{\alpha}{\alpha-1}G_0\right)}$ $\quad C = \frac{G_0}{\alpha-1} \quad I = \frac{\alpha}{\alpha-1}G_0$.

$(I, C) = \left(\frac{\alpha}{\alpha-1}G_0, \frac{G_0}{\alpha-1}\right)$.

$DF = \begin{bmatrix} 1 & -\alpha \\ \beta & -\beta \end{bmatrix}$ eigenvalues if $\beta = 1$: $\begin{bmatrix} 1 & -\alpha \\ 1 & -1 \end{bmatrix}$ eigenvalues $\begin{vmatrix} 1-\lambda & -\alpha \\ 1 & -1-\lambda \end{vmatrix}$

$\boxed{\text{trace} = 1 - \beta = 0 \Rightarrow \text{elliptic if complex}}$

$= -(1-\lambda)(\lambda+1) + \alpha = \lambda^2 + \alpha - 1 = 0 \quad \lambda = \pm\sqrt{\alpha-1}\, i$ as $\alpha > 1$. $\Rightarrow$ oscillates (elliptic).

b) equilibrium solution: $I - \alpha C = 0$ $\quad I = \alpha C$

$\beta(I - C - G_0 - kI) = 0 \quad \alpha C - C - G_0 - k\alpha C = 0$

$C(\underbrace{\alpha - 1 - k\alpha}_{\text{need this} > 0}) = G_0$

$\alpha(1-k) - 1 > 0 \quad \alpha(1-k) > 1 \quad 1 - k > \frac{1}{\alpha} \quad k < \frac{\alpha-1}{\alpha}$.

if $k > \frac{\alpha-1}{\alpha}$ C' always < 0 $\quad C \to 0 \quad I \to \infty$

c) $I' = I - \alpha C$

$C' = \beta(I - C - G_0 - kI^2)$

equilibrium solution: $I - \alpha C = 0 \Rightarrow I = \alpha C$

$I - C - G_0 - kI^2 = 0$ $\quad \alpha < 1$

$\Rightarrow I - \frac{1}{\alpha} I - G_0 - kI^2 = 0$

$kI^2 + I\left(\frac{1}{\alpha} - 1\right) + G_0 = 0 \qquad I = \frac{-k \pm \sqrt{\left(\frac{1}{\alpha}-1\right)^2 - 4kG_0}}{2}$.

assume $\left(\frac{1}{\alpha}-1\right)^2 > 4kG_0 \Rightarrow$ equilibrium solution

$DF = \begin{bmatrix} 1 & -\alpha \\ \beta - 2kI & -\beta \end{bmatrix}$

trace $= 1 - \beta < 0 \Rightarrow$ stable. if complex

det $= -\beta + \alpha(\beta - 2kI)$

$= \beta(1-\alpha) - 2k\alpha I < 0 \Rightarrow$ saddle unstable.