Math 330 Differential Equations Fall 15 Midterm 1b

Name: Solutions

- I will count your best 8 of the following 10 questions.
- You may use a calculator, and a US Letter page of notes; you may write on both sides. No cell phones.

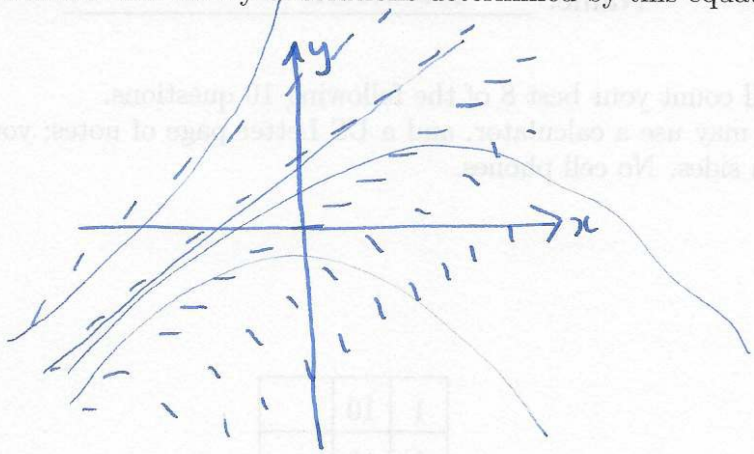
1	10	
2	10	
3	10	
4	10	
5	10	
6	10	
7	10	
8	10	
9	10	
10	10	
	80	

Midterm 1	
Overall	

(1) (10 points) Sketch the flow vectors for the differential equation

$$\frac{dy}{dx} = y - x.$$

Find and sketch the family of solutions determined by this equation.



$y - x = c$

$y = c + x$

$y' - y = -x$

solve $y' - y = 0$ try $y = e^{\lambda x}$: $e^{\lambda x}(\lambda - 1) = 0$ $\quad y = c_1 e^{x}$

particular solution try $\left.\begin{array}{l} y = Ax + B \\ y' = A \end{array}\right\}$ $\quad A - Ax - B = -x \quad A = +1$

$A - B = 0 \quad B = +1$

general solution $y = c_1 e^{x} + 1 + x$

(2) (10 points) Find the general solution to

$$y^2y' + \tan x = 1.$$

$y^2 \frac{dy}{dx} = 1 - \tan x$

$\int y^2 dy = \int 1 - \tan x \, dx$

$\frac{1}{3} y^3 = x + \ln|\cos x| + c$

$y = (3(x + \ln|\cos x| + c))^{1/3}$

(3) (10 points) Find the general solution to

$$e^x y' + ye^x + 1 = 0.$$

$\frac{\partial}{\partial x}(e^x) = e^x \qquad \frac{\partial}{\partial y}(ye^x + 1) = e^x \Rightarrow$ exact

$\int e^x dy = ye^x \qquad \int ye^x + 1 \, dx = ye^x + x$

$(ye^x + x)' = 0$

$ye^x + x = c$

$y = \frac{c - x}{e^x}$

(4) (10 points) Find the general solution to

$$y' + 2xy = 0.$$

integrating factor $e^{\int 2x\,dx} = e^{x^2}$

$e^{x^2} y' + 2xe^{x^2} y = 0$

$(e^{x^2} y)' = 0$

$e^{x^2} y = c$

$y = ce^{-x^2}$

(5) (10 points) Find the general solution to

$$y'' - 6y' + 9y = e^x.$$

solve $y'' - 6y' + 9y = 0$ try $y = e^{\lambda x}$ $e^{\lambda x}(\lambda^2 - 6\lambda + 9) = 0$

$(\lambda - 3)^2$

sol$\underline{^n}$: $y = c_1 e^{+3x} + c_2 x e^{3x}$

particular solution: try $\left.\begin{array}{l} y = Ae^x \\ y' = Ae^x \\ y'' = Ae^x \end{array}\right\}$ $e^x(A - 6A + 9A) = e^x$

$4A = 1$ $A = 1/4$

general solution:

$c_1 e^{3x} + c_2 x e^{3x} + \frac{1}{9} e^x$

(6) (10 points) Find the solution to

$$y'' - 4y = e^{-2x}.$$

which satisfies $y(0) = 1$ and stays bounded as $x \to \infty$

solve $y''-4y=0$ try $y=e^{\lambda x}$ $e^{\lambda x}(\lambda^2-4)=0$

$(\lambda-2)(\lambda+2)$

solution $c_1e^{2x}+c_2e^{-2x}$

particular solution : try $y = Axe^{-2x}$

$y' = Ae^{2x} - 2Axe^{-2x}$

$y'' = -2Ae^{-2x} - 2Ae^{-2x} + 4Axe^{-2x}$

$e^{-2x}\left[-4A + 4Ax - 4Ax\right] = e^{-2x}$ $\Rightarrow$ $-4A=1$ $A=-1/4$

general solution : $y = c_1e^{2x} + c_2e^{-2x} - \frac{1}{4}xe^{-2x}$

stays bounded $\Rightarrow$ $c_1=0$

$y(0)=1 = c_2$ $\Rightarrow c_2=1$ $y = e^{-2x} - \frac{1}{4}xe^{-2x}$.

(7) (10 points)

(a) Find the general solution to
$$xy' - y = 0.$$

(b) Find a particular solution to $xy' - y = x^\alpha$, $\alpha \neq 1$, by looking for a solution of the form $y = Ax^\alpha$.

a) try $\left.\begin{array}{l} y = x^\lambda \\ y' = \lambda x^{\lambda-1} \end{array}\right\}$ $\quad x^\lambda(\lambda - 1) = 0 \quad \lambda = 1 \quad$ solution: $y = c_1 x$

b) $\left.\begin{array}{l} y = Ax^\alpha \\ y' = A\alpha x^{\alpha-1} \end{array}\right\}$

$$A\alpha x^\alpha - Ax^\alpha = x^\alpha$$
$$x^\alpha(A\alpha - A) = x^\alpha$$
$$A(\alpha - 1) = 1 \qquad A = \frac{1}{\alpha - 1}$$

so $\quad y = c_1 x + \frac{1}{\alpha - 1} x^\alpha$

(8) (10 points)

(a) Use your solution to the previous problem to find the solution to the initial value problem

$$xy' - y = x^{\alpha}, \quad y(1) = 0, \quad \alpha \neq 1.$$

(b) Solve

$$xy' - y = x, \quad y(1) = 0$$

by taking the limit as $\alpha \to 1$ of your answer to (a).

a) $y = c_1 x + \frac{1}{\alpha - 1} x^{\alpha}$ $\quad y(1) = 0 = c_1 + \frac{1}{\alpha-1} \Rightarrow c_1 = \frac{-1}{\alpha - 1}$

so $y = \dfrac{x^{\alpha} - x}{\alpha - 1}$

b) $\lim\limits_{\alpha \to 1} \dfrac{x^{\alpha} - x}{\alpha - 1} = \lim\limits_{\alpha \to 1} \dfrac{e^{\alpha \ln(x)} - x}{\alpha - 1} \underset{\text{l'H}}{=} \lim\limits_{\alpha \to 1} \dfrac{\ln(x)\, e^{\alpha \ln(x)}}{1} = x \ln(x).$

(9) (10 points). You jump out of an aeroplane and fall with constant gravitational acceleration g. Suppose you have mass m and the force of air resistance is equal to your surface area A times your speed. If $y(t)$ is your height above the ground, show that your equation of motion is

$$y'' = -g - \frac{A}{m}y'.$$

If you start at height $y(0) = 0$ with velocity $y'(0) = 0$, find $y(t)$, and use this to show that your terminal velocity is $-mg/A$.

F=ma : $my'' = -mg - Ay'$

$y'' = -g - \frac{A}{m}y' \iff y'' + \frac{A}{m}y' = -g$

Solve : $y'' + \frac{A}{m}y' = 0$ ~~try~~ try $y = e^{\lambda t}$: $e^{\lambda x}\left(\lambda^2 + \frac{A}{m}\lambda\right) = 0$

$\lambda\left(\lambda + \frac{A}{m}\right)$ $\lambda = 0, -\frac{A}{m}$.

~~gen~~ so $y = c_1 + c_2 e^{-\frac{A}{m}t}$

particular solution : $\left.\begin{array}{l} y = Bx \\ y' = B \\ y'' = 0 \end{array}\right\}$ $\frac{AB}{m} = -g$ $B = -\frac{gm}{A}$

general solution $y(t) = c_1 + c_2 e^{-\frac{A}{m}t} - \frac{gmt}{A}$ $y'(t) = -\frac{Ac_2}{m}e^{-\frac{A}{m}t} - \frac{gm}{A}$

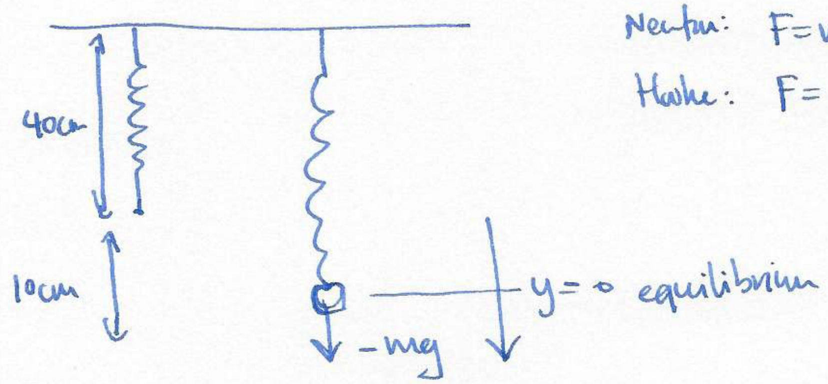
$y(0) = 0 = c_1 + c_2 \Rightarrow$ ~~c_2~~

$y'(0) = 0 = -\frac{A}{m}c_2 - \frac{gm}{A}$ $c_2 = -\frac{gm^2}{A^2}$ $c_1 = +\frac{gm^2}{A^2}$.

$y(t) = +\frac{gm^2}{A^2} - \frac{gm^2}{A^2}e^{-\frac{A}{m}t} - \frac{gmt}{A}$.

$y'(t) = +\frac{gm}{A}e^{-\frac{A}{m}t} - \frac{gm}{A}$ $\lim_{t\to\infty} y'(t) = -\frac{gm}{A}$

(10) An unloaded spring has length 40cm, and is in equilibrium at length 50cm with a 1 kg mass attached. Suppose there is a damping force equal to k times the velocity. Derive the equation of motion for the spring and show that the long time behaviour of the spring always tends to zero velocity.



Newton: $F=ma$

Hooke: $F = ce$ e=extension c = ~~hook~~ spring constant

$$-mg = c(0+10)$$

in general: $my'' = -mg - c(y+10) - ky'$

$$y'' = -\cancel{g} - \frac{cy}{m} - \cancel{\frac{10c}{m}} - \frac{ky'}{m}$$

$$y'' + \frac{k}{m}y' + \frac{c}{m}y = 0 \qquad k,c,m > 0$$

try $y=e^{\lambda x}$ $\quad e^{\lambda x}\left(\lambda^2 + \frac{k}{m}\lambda + \frac{c}{m}\right) = 0$

$$\lambda = \frac{-\frac{k}{m} \pm \sqrt{\frac{k^2}{m^2} - \frac{4c}{m}}}{2} \qquad \sqrt{\frac{k^2}{m^2} - \frac{4c}{m}} < \frac{k}{m} \text{ so if}$$

two real roots: both negative

complex roots: $\underbrace{-\frac{k}{2m}}_{\text{negative}} \pm i\beta$

⇒ all solutions product with negative exponential $\to 0$ as $t \to \infty$.