Math 330 Differential Equations Fall 15 Midterm 1a

Name: Solutions

- I will count your best 8 of the following 10 questions.
- You may use a calculator, and a US Letter page of notes; you may write on both sides. No cell phones.

1	10	
2	10	
3	10	
4	10	
5	10	
6	10	
7	10	
8	10	
9	10	
10	10	
	80	

Midterm 1	
Overall	

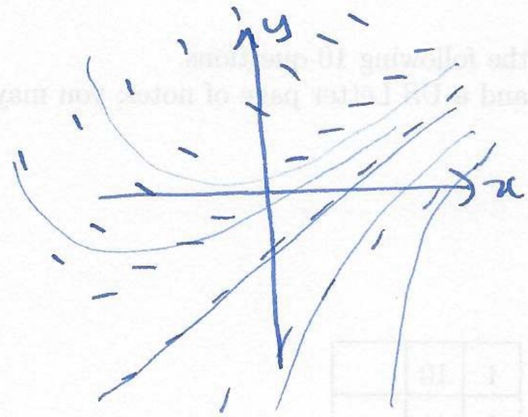
(1) (10 points) Sketch the flow vectors for the differential equation

$$\frac{dy}{dx} = x - y.$$

Find and sketch the family of solutions determined by this equation.

$x - y = c$

$y = x - c$



$y' + y = x$

solve $y' + y = 0$: try $\left.\begin{array}{l} y = e^{\lambda x} \\ y' = \lambda e^{\lambda x} \end{array}\right\}$ $e^{\lambda x}(\lambda + 1) = 0$ $y = c_1 e^{-x}$

particular solution: try $\left.\begin{array}{l} y = Ax + B \\ y' = A \end{array}\right\}$ $A + Ax + B = x$ $\left.\begin{array}{l} A = 1 \\ A + B = 0 \end{array}\right\}$ $x - 1$

general solution $y = c_1 e^{-x} + x - 1$

(2) (10 points) Find the general solution to

$$y^3y' - \tan x = 1.$$

$y^3 y' = 1 + \tan x$

$\int y^3 dy = \int 1 + \tan x \, dx$

$\frac{1}{4} y^4 = x + - \ln |\cos x| + c$

$y = (4(x - \ln|\cos x| + c))^{1/4}$

(3) (10 points) Find the general solution to

$$e^x y' + ye^x - 1 = 0.$$

check exact: $\frac{\partial}{\partial x}(e^x) = e^x$ $\quad \frac{\partial}{\partial y}(ye^x - 1) = e^x$ $\quad$ yes.

$\int e^x dy = ye^x$ $\quad \int ye^x - 1 \, dx = ye^x - x$

so $\quad (ye^x - x)' = 0$

$ye^x - x = c$

$y = (c+x)e^{-x}$

(4) (10 points) Find the general solution to

$$y' - 2xy = 0.$$

integrating factor $e^{\int -2x\,dx} = e^{-x^2}$

$e^{-x^2} y' - 2xye^{-x^2} = 0$

$(ye^{-x^2})' = 0$

$ye^{-x^2} = c$

$y = ce^{+x^2}$

(5) (10 points) Find the general solution to

$$y'' - 4y' + 4y = e^x.$$

Solve $y''-4y'+4y=0$ try $y=e^{\lambda x}$ $\quad e^{\lambda x}(\lambda^2-4\lambda+4)=0$

$(\lambda-2)^2$

general solution $\quad y = c_1e^{2x} + c_2xe^{2x}$

particular solution: try $\left.\begin{aligned} y &= Ae^x \\ y' &= Ae^x \\ y'' &= Ae^x \end{aligned}\right\}$ $\quad Ae^x(1-4+4) = e^x \Rightarrow A = 1.$

general solution: $\quad y = c_1e^{2x} + c_2xe^{2x} + e^x$

(6) (10 points) Find the solution to

$$y'' - 9y = e^{-3x}.$$

which satisfies $y(0) = 1$ and stays bounded as $x \to \infty$

solve $y'' - 9y = 0$ try $y = e^{\lambda x}$ $\quad e^{\lambda x}(\lambda^2 - 9) = 0$

$(\lambda - 3)(\lambda + 3)$

general solution $y = c_1 e^{3x} + c_2 e^{-3x}$

particular solution: try $y = Axe^{-3x}$

$y' = Ae^{-3x} - 3Axe^{-3x}$

$y'' = -3Ae^{-3x} - 3Ae^{-3x} + 9Axe^{-3x}$

$e^{-3x}(-6A + 9xA - 9Ax) = e^{-3x}$

$A = -\frac{1}{6}$

general solution : $\quad y = c_1 e^{3x} + c_2 e^{-3x} - \frac{1}{6} x e^{-3x}$

stays bounded $\Rightarrow c_1 = 0$

$y(0) = 1 = c_2 \cancel{\frac{4}{6}} \Rightarrow c_2 = \cancel{\frac{7}{6}}\, 1$

so $\quad y = \cancel{\frac{7}{6}}\, e^{-3x} - \frac{1}{6} x e^{-3x}$

(7) (10 points)

(a) Find the general solution to
$$xy' - y = 0.$$

(b) Find a particular solution to $xy' - y = x^\alpha$, $\alpha \neq 1$, by looking for a solution of the form $y = Ax^\alpha$.

a) try $\left.\begin{array}{l} y = x^\lambda \\ y' = \lambda x^{\lambda-1} \end{array}\right\}$ $\quad x^\lambda(\lambda - 1) = 0 \quad \lambda = 1$ solution: $y = c_1 x$

b) $\left.\begin{array}{l} y = Ax^\alpha \\ y' = A\alpha x^{\alpha-1} \end{array}\right\}$ $\quad A\alpha x^\alpha - Ax^\alpha = x^\alpha$

$x^\alpha(A\alpha - A) = x^\alpha$

$A(\alpha - 1) = 1 \qquad A = \frac{1}{\alpha - 1}$

so $\quad y = c_1 x + \frac{1}{\alpha-1} x^\alpha$

(8) (10 points)

(a) Use your solution to the previous problem to find the solution to the initial value problem

$$xy' - y = x^\alpha, \quad y(1) = 0, \quad \alpha \neq 1.$$

(b) Solve

$$xy' - y = x, \quad y(1) = 0$$

by taking the limit as $\alpha \to 1$ of your answer to (a).

a) $y = c_1 x + \frac{1}{\alpha - 1} x^\alpha$ $\quad y(1) = 0 = c_1 + \frac{1}{\alpha-1} \Rightarrow c_1 = \frac{-1}{\alpha - 1}$

so $\quad y = \dfrac{x^\alpha - x}{\alpha - 1}$

b) $\lim\limits_{\alpha \to 1} \dfrac{x^\alpha - x}{\alpha - 1} = \lim\limits_{\alpha \to 1} \dfrac{e^{\alpha \ln(x)} - x}{\alpha - 1} \underset{\text{l'H}}{=} \lim\limits_{\alpha \to 1} \dfrac{\ln(x)\, e^{\alpha \ln(x)}}{1} = x \ln(x).$

(9) (10 points). You jump out of an aeroplane and fall with constant gravitational acceleration g. Suppose you have mass m and the force of air resistance is equal to your surface area A times your speed. If $y(t)$ is your height above the ground, show that your equation of motion is

$$y'' = -g - \frac{A}{m}y'.$$

If you start at height $y(0) = 0$ with velocity $y'(0) = 0$, find $y(t)$, and use this to show that your terminal velocity is $-mg/A$.

F=ma: $my'' = -mg - Ay'$

$$y'' = -g - \frac{A}{m}y' \iff y'' + \frac{A}{m}y' = -g$$

solve: $y'' + \frac{A}{m}y' = 0$ try $y = e^{\lambda t}$: $e^{\lambda x}\left(\lambda^2 + \frac{A}{m}\lambda\right) = 0$

$\lambda\left(\lambda + \frac{A}{m}\right)$ $\lambda = 0, -\frac{A}{m}$.

~~gen~~ so $y = c_1 + c_2 e^{-\frac{A}{m}t}$

particular solution: $\left.\begin{array}{l} y = Bx \\ y' = B \\ y'' = 0 \end{array}\right\}$ $\frac{AB}{m} = -g$ $B = -\frac{gm}{A}$

general solution $y(t) = c_1 + c_2 e^{-\frac{A}{m}t} \cancel{+} - \frac{gmt}{A}$ $y'(t) = -\frac{Ac_2}{m}e^{-\frac{A}{m}t} \cancel{+} - \frac{gm}{A}$

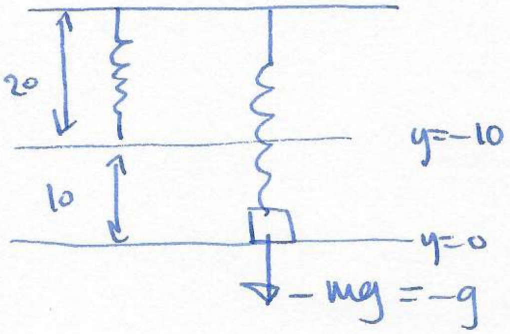
$y(0) = 0 = c_1 + c_2 \Rightarrow \cancel{c_2}$

$y'(0) = 0 = -\frac{A}{m}c_2 \cancel{+} - \frac{gm}{A}$ $c_2 = -\frac{gm^2}{A^2}$ $c_1 = +\frac{gm^2}{A^2}$.

$$y(t) = +\frac{gm^2}{A^2} \cancel{+} - \frac{gm^2}{A^2}e^{-\frac{A}{m}t} \cancel{+} - \frac{gmt}{A}.$$

$$y'(t) = +\frac{gm}{A}e^{-\frac{A}{m}t} \cancel{+} - \frac{gm}{A} \qquad \lim_{t \to \infty} y'(t) = -\frac{gm}{A}$$

(10) An unloaded spring has length 20cm, and is in equilibrium at length 30cm with a 1 kg mass attached. Suppose there is a damping force equal to k times the velocity. Derive the equation of motion for the spring and show that the long time behaviour of the spring always tends to zero velocity.



Newton: $F = ma$.

Hooke: $F = \cancel{k}\, c\, e$ e extension.

in equilibrium: $-mg = c(\cancel{y}+10) \Rightarrow -mg = 10c$

in general $my'' = -mg - c(y+10) - ky'$

$= \underbrace{-mg}_{10c} - cy - 10c - ky'$.

$my'' = -cy - ky'$.

so $y'' + \frac{k}{m}y' + \frac{c}{m}y = 0$

try $y = e^{\lambda x}$: $e^{\lambda x}\left(\lambda^2 + \frac{k}{m}\lambda + \frac{c}{m}\right) = 0$

$$\lambda = \frac{-\frac{k}{m} \pm \sqrt{\frac{k^2}{m^2} - \frac{4c}{m}}}{2}$$

$k, m, c > 0$

so $\sqrt{\frac{k^2}{m^2} - \frac{4c}{m}} < \frac{k}{m}$

$\Rightarrow$ if both roots real, both negative: $c_1 e^{-\lambda_1 t} + c_2 e^{-\lambda_2 t}$ $\lambda_1, \lambda_2 > 0$

if complex roots, real part negative: $e^{-k/2m\, t}(A\cos\omega t + B\sin\omega t)$.

in either case, $\lim_{t\to\infty} y(t) = 0$ as multiplied by negative exponential.