MTH 330 SMT1 Solutions

Q1 solve $y'-y=0$

$$\left.\begin{array}{l} \text{try } y=e^{\lambda x} \\ y'=\lambda e^{\lambda x} \end{array}\right\} \quad e^{\lambda x}(\lambda-1)=0 \Rightarrow \lambda=0$$

general solution $y=c_1e^{x}$

find particular solution for $y'-y=e^{\mu x}$. $\left.\begin{array}{l} \text{try } y=Ae^{\mu x} \\ y'=\mu Ae^{\mu x} \end{array}\right\}$ gives:

$$e^{\mu x}(A\mu-A)=e^{\mu x} \Rightarrow A(\mu-1)=1 \Rightarrow A=\frac{1}{\mu-1}$$

so general solution is $c_1e^{x}+\frac{1}{\mu-1}e^{\mu x}=\underbrace{\left(c_1+\frac{1}{\mu-1}\right)}_{A}e^{x}+\frac{e^{\mu x}-e^{x}}{\mu-1}$

$$\lim_{\mu\to 1}\frac{e^{\mu x}-e^{x}}{\mu-1}\underset{\text{l'H}}{=}\lim_{\mu\to 1}\frac{xe^{\mu x}}{1}=xe^{x}$$

general solution to $y'-y=e^{x}$ is $Ae^{x}+xe^{x}$

Q2 a) ① $y'-y=0$ $\left.\begin{array}{l} \text{try } y=e^{\lambda x} \\ y'=\lambda e^{\lambda x} \end{array}\right\}$ $e^{\lambda x}(\lambda-1)=0$, $\lambda=1$ $\quad y=c_1e^{x}$

② $y'-y=2e^{x}$ $\left.\begin{array}{l} \text{try } y=Axe^{x} \\ y'=Ae^{x}+Axe^{x} \end{array}\right\}$ $y'-y=2e^{x}$: $Ae^{x}+Axe^{x}-Axe^{x}=2e^{x}\Rightarrow A=2$

general solution: $y=c_1e^{x}+2xe^{x}$

b) ① $y'+2y=0$ $\left.\begin{array}{l} \text{try } y=e^{\lambda x} \\ y'=\lambda e^{\lambda x} \end{array}\right\}$ $e^{\lambda x}(\lambda+2)=0$, $\lambda=-2$ $\quad$ homogeneous solution $y=c_1e^{-2x}$

② try $y=(Ax^2+Bx+C)e^{-2x}$

$y'=(2Ax+B)e^{-2x}-2(Ax^2+Bx+C)e^{-2x}$

② try $y = Ax^2e^{-2x}$, $y' = 2Axe^{-2x} - 2Ax^2e^{-2x}$ $\Big\}$ $y' + 2y = xe^{-2x}$

$$xe^{-2x}(2A) + x^2e^{-2x}\underbrace{(-2A+2A)}_{0} = xe^{-2x}$$

$\Rightarrow 2A = 1,\ A = \frac{1}{2}$ $\quad y = \frac{1}{2}x^2e^{-2x}$

general solution: $\quad y = c_1e^{-2x} + \frac{1}{2}x^2e^{-2x}$

c) $y' = \dfrac{1}{e^y - x}$ $\quad$ try sub $y(x) = \ln|z(x)|$

$y' = \frac{1}{z}\cdot z'$ $\quad$ gives $\frac{1}{z}z' = \dfrac{1}{z-x}$

$z' = \dfrac{z}{z-x}$ ← homogeneous $f(\frac{x}{z})$ so sub $z(x) = u(x)x$

$z' = u'x + u$

$$u'x + u = \frac{ux}{ux - x} = \frac{u}{u-1}, \qquad u'x = \frac{u}{u-1} - u = \frac{2u - u^2}{u-1} = \frac{u(2-u)}{u-1}$$

$u' = \dfrac{1}{x}\dfrac{u(2-u)}{u-1}$ separable: $\displaystyle\int \frac{u-1}{u(2-u)}\,du = \int \frac{1}{x}\,dx = \ln|x| + c$

$$\frac{u-1}{u(2-u)} = \frac{A}{u} + \frac{B}{2-u} = \frac{A(2-u) + Bu}{u(2-u)}$$

$u = 0:\ -1 = 2A \quad A = -1/2$

$u = 2:\ 1 = 2B \quad B = 1/2$

$$\int \frac{-1/2}{u} + \frac{1/2}{2-u}\,du = -\frac{1}{2}\ln|u| - \frac{1}{2}\ln|2-u| = \ln|x| + c$$

$$\ln|u| + \ln|2-u| = -2\ln|x| + c$$

$$u(2-u) = Ax^{-2}$$

$u^2 - 2u + Ax^{-2} = 0 \qquad u = \dfrac{2 \pm \sqrt{4 - 4Ax^{-2}}}{2} = 1 \pm \sqrt{1 - Ax^{-2}}$

$z = ux \Rightarrow \qquad z = x \pm \sqrt{x^2 - A}$

$y = \ln|z| \Rightarrow \qquad y = \ln\left|x \pm \sqrt{x^2 - A}\right|.$

d) $y' \tan x + y = 1 \Rightarrow y' \sin x + y \cos x = \cos x$

$$(y \sin x)' = \cos x$$

$$y \sin x = \sin x + c$$

$$y = 1 + \frac{c}{\sin x}.$$

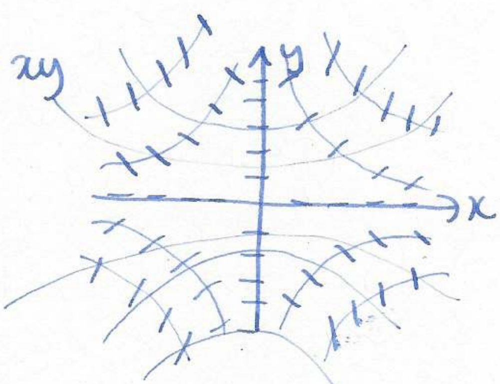
e) $y' x \sin x + (\sin x + x \cos x) y = x e^x$

$$(y x \sin x)' = x e^x$$

$$y x \sin x = \int x e^x dx = x e^x - \int e^x dx = x e^x - e^x + c.$$

$$y = \frac{x e^x - e^x + c}{x \sin x}$$

Q3 $\frac{dy}{dx} = xy$



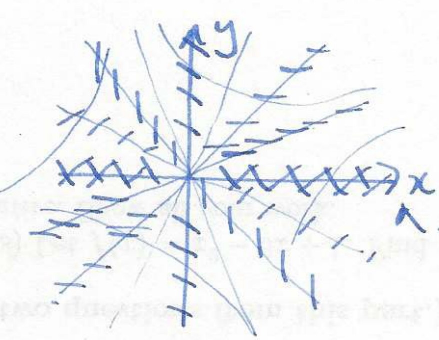
$\frac{dy}{dx} = c \quad xy = c \quad y = \frac{c}{x}$

$$\int \frac{dy}{y} = \int x dx$$

$$\ln|y| = \frac{1}{2}x^2 + c$$

$$y = Ae^{\frac{1}{2}x^2}$$

Q4 $\frac{dy}{dx} = \frac{x-y}{x+y}$



^ +1 along x-axis.

$\frac{x-y}{x+y} = c \Leftrightarrow y = \frac{1-c}{1+c} x$

$y = ux$

$y' = u'x + u \qquad \frac{dy}{dx} = u'x + u = \frac{x - ux}{x + ux} = \frac{1-u}{1+u}$

$$u'x = \frac{1-u}{1+u} - u = \frac{1-u-u-u^2}{1+u} = \frac{1-2u-u^2}{1+u}$$

$$\int \frac{1+u}{1-2u-u^2}\,du = \int \frac{1}{x}\,dx$$

$$-\frac{1}{2}\ln\left|1-2u-u^2\right| = \ln|x| + C$$

$$1-2u-u^2 = Ax^{-2}$$

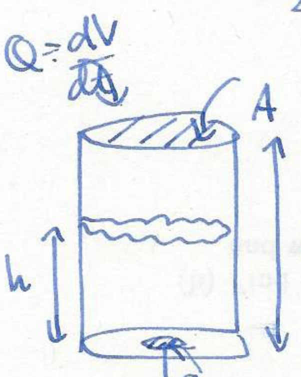
$$u^2+2u+Ax^{-2}-1=0$$

$$u = \frac{-2 \pm \sqrt{4-4(Ax^{-2}-1)}}{2}$$

$$u = \frac{-1 \pm \sqrt{4-4Ax^{-2}+4}}{2}$$

$$y = -x \pm \sqrt{2x^2 - A}$$

Q5



$$\downarrow \frac{dV}{dt} = Q$$

$$\downarrow \frac{dV}{dt} = a\sqrt{2gh}$$

$$\frac{dV}{dt} = Q - a\sqrt{2gh}$$

$$V = Ah \qquad V' = Ah'$$

$$\frac{dh}{dt} = \frac{1}{A}\left(Q - a\sqrt{2gh}\right)$$

equilibrium:

$$\frac{dh}{dt} = 0 \Rightarrow Q = a\sqrt{2gh} \qquad \frac{Q}{a} = \sqrt{2gh} \qquad \frac{Q^2}{a^2} = 2gh \qquad h = \frac{Q^2}{2a^2 g}$$

water stops going in: $\frac{dh}{dt} = -\frac{a}{A}\sqrt{2gh}$

$$\int \frac{dh}{\sqrt{h}} = \int -\frac{a\sqrt{2g}}{A}\,dt$$

$$2\sqrt{h} = -\frac{a\sqrt{2g}}{A}t + c$$

$$h = \left(-\frac{a\sqrt{2g}\,t}{2A} + c\right)^2$$

suppose $h(0) = H$

$$H = \cancel{\left(-\frac{a\sqrt{2g}}{2A} + c\right)^2}\ c^2$$

$$\cancel{\sqrt{H} + \frac{a\sqrt{2g}}{2A} = c}$$

find t s.t. $h(t) = 0$:

$$\left(-\frac{a\sqrt{2g}}{A}t + \sqrt{H}\right)^2 = 0$$

$$t = \frac{A\sqrt{H}}{9\sqrt{2g}}$$

Q6 a) $\underbrace{y'' + 5y' + 6y}_{L(y)} = 3e^{-2x} + e^{3x}$

homogeneous: try $y = e^{\lambda x}$: $e^{\lambda x}(\lambda^2 + 5\lambda + 6) = 0$

$(\lambda+3)(\lambda+2)$ $\quad y_h = c_1 e^{-3x} + c_2 e^{-2x}$

solve $Ly = 3e^{-2x}$ try $y = Axe^{-2x}$

$y' = Ae^{-2x} - 2Axe^{-2x}$

$y'' = -2Ae^{-2x} - 2Ae^{-2x} + 4Axe^{-2x}$

	xe^{-2x}	e^{-2x}
y''	$4A$	$-4A$
$5y'$	$-10A$	$5A$
$6y$	$6A$	0
	0 ✓	3

$A = 3$. so $y = 3xe^{-2x}$

solve $Ly = e^{3x}$ try $y = Ae^{3x}$

$y' = 3Ae^{3x}$ $\quad y'' + 5y' + 6y = e^{3x}$

$y'' = 9Ae^{3x}$ $\quad e^{3x}(9A + 15A + 6A) = e^{3x}$

$A = \frac{1}{30}$ $\quad y = \frac{1}{30}e^{3x}$

general solution: $y = c_1 e^{-3x} + c_2 e^{-2x} + 3xe^{-2x} + \frac{1}{30}e^{3x}$.

b) $y'' - 2y' + y = (x-1)e^x = xe^x - e^x$

homogeneous: try $y = e^{\lambda x}$: $e^{\lambda x}(\lambda^2 - 2\lambda + 1) = 0$

$(\lambda - 1)^2 = 0$ gen sol^n $y = c_1 e^x + c_2 x e^x$

particular solution: try $y = Ax^3e^x + Bx^2e^x = (Ax^3 + Bx^2)e^x$

$y' = (3Ax^2 + 2Bx)e^x + (Ax^3 + Bx^2)e^x$

$y'' = (6Ax + 2B)e^x + 2(3Ax^2 + 2Bx)e^x + (Ax^3 + Bx^2)e^x$

	x^3	x^2	x	1
y''	A	$B+6A$	$4B+6A$	$2B$
$-2y'$	$-2A$	$-6A\overset{-2B}{\cancel{+4B}}$	$-4B$	
y	A	B		
			$\underbrace{1}$	$\underbrace{-1}$
			$A=\frac{1}{6}$	$B=-\frac{1}{2}$.

$(\ \ )e^x$

general solution

$$y = c_1e^x + c_2xe^x + \frac{1}{6}x^3e^x - \frac{1}{2}x^2e^x$$

Q7 a) $x(x+1)y'' + (x-1)y' - y = 0$ has solution $u = \frac{1}{x+1}$, $u' = -(x+1)^{-2}$

look for solution
$$\left.\begin{aligned} y &= uv \\ y' &= u'v + uv' \\ y'' &= u''v + 2u'v' + uv'' \end{aligned}\right\}\ \text{plug in:}$$

$$v\underbrace{\left[x(x+1)u'' + (x-1)u' - u\right]}_{0 \text{ as } u \text{ is sol}^{\underline{n}}.} + x(x+1)(2u'v' + uv'') + (x-1)uv' = 0$$

$$v''\left[x(x+1)u\right] + v'\left[x(x+1)2u' + (x-1)u\right] = 0$$

$$xv'' + v'\left[-\frac{2x(x+1)}{(x+1)^2} + \frac{(x-1)}{x+1}\right] = 0 \qquad \frac{x-1-2x}{x+1} = \frac{-x-1}{x+1} = -1$$

$xv'' - v' = 0$ set $z = v'$

$xz' - z = 0$

$$\int\frac{dz}{z} = \int\frac{dx}{x}$$

$\ln|z| = \ln|x| + C$

$z = Ax$

so $v' = Ax$

$v = \frac{1}{2}Ax^2 + B$ choose simplest x^2

so other solution is $\boxed{y = \frac{x^2}{1+x}}$

b) $xy'' - y' - 4x^3 y = 0$ $\quad u(x) = e^{x^2}$ is a solution $\quad$ try $y = uv$

$u' = 2xe^{x^2}$ $\quad y' = u'v + uv'$

gives: $\quad y'' = u''v + 2u'v' + uv''$

$$v\underbrace{\left[xu'' - u' - 4x^3 u\right]}_{0} + x(2u'v' + uv'') - uv' = 0$$

$$4x^2 e^{x^2} v' + xe^{x^2} v'' - e^{x^2} v' = 0$$

$$xv'' + (4x^2 - 1)v' = 0 \quad \text{set } z = v'$$

$$z' + \left(4x - \tfrac{1}{x}\right)z = 0 \quad \text{use integrating factor } e^{\int 4x - \frac{1}{x}dx} = e^{2x^2 - \ln|x|}$$

$$= \cancel{xe^{2x^2}}$$

$$\left(z e^{2x^2 - \ln(x)}\right)' = 0$$

$$z e^{2x^2 - \ln(x)} = c$$

$$v' = z = ce^{-2x^2 + \ln(x)} = cxe^{-2x^2} \quad \text{(simplify choose } c=1)$$

$$v = \int xe^{-2x^2}dx = -\tfrac{1}{2}e^{-2x^2} + c \quad \text{(simplify choose } c = 0)$$

(don't care about mult. const)

$$y = uv = e^{x^2}.e^{-2x^2} = e^{-x^2} \quad \text{second solution: } \boxed{y = e^{-x^2}}.$$

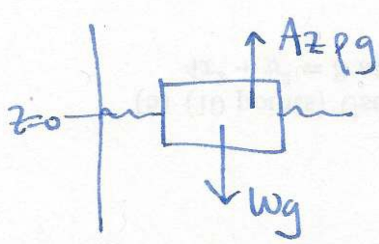
Q8 $\quad y'' - y' - 2y$ $\quad$ try $y = e^{\lambda x}$

$$e^{\lambda x}(\lambda^2 - \lambda - 2) = 0 \quad \text{general solution} \quad c_1 e^{+2x} + c_2 e^{-x}$$

$$(\lambda - 2)(\lambda + 1)$$

bounded $\Rightarrow c_1 = 0$ $\quad y(0) = 1 \Rightarrow c_2 = 1$ $\quad$ solution: $\boxed{y = e^{-x}}$

Q9



$$F = m\ddot{z} = Wg - A\rho g(z + z_0) \quad \text{where } z_0 \text{ is equilibrium}$$

$$Wz'' = -A\rho gz + Wg - A\rho g\frac{W}{A\rho} \qquad Wg = A\rho g z_0$$

$$z_0 = \frac{W}{A\rho}.$$

$$Wz'' + A\rho g z = 0 \Rightarrow z'' + \frac{Ag\rho}{W}z = 0.$$

$z'' + \frac{g\rho A}{W} z = m \sin(\omega t) \qquad \omega^2 = \frac{g\rho A}{W}$

solve:

$z'' + \omega^2 z = 0$ general solution $z = c_1 \cos(\omega t) + c_2 \sin(\omega t)$

particular solution will be of form $At\cos\omega t + Bt\sin\omega t$ for some A, B

so $z(t)$ becomes arbitrarily large as $t \to \infty$.

$z'' + kz' + \omega^2 z = 0$ try $e^{\lambda x}$: $e^{\lambda x}(\lambda^2 + k\lambda + \omega^2 \lambda)$

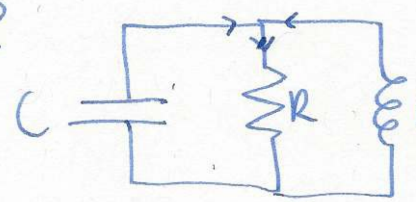
$\lambda = \frac{-k \pm \sqrt{k^2 - 4\omega^2}}{2}$ gives oscillating / exponential decay $e^{-k/2}(A\cos\omega' t + B\sin\omega' t)$ if $k^2 - 4\omega^2 < 0$ (large damping / underdamped).

critical damping $k^2 - 4\omega^2 = 0$

solution $Ae^{-k/2} + Bte^{-k/2}$ also exp. decay

over-damped $k^2 - 4\omega^2 > 0$, exponential decay $Ae^{-\lambda_1} + Be^{-\lambda_2}$ $\lambda_1, \lambda_2 > 0$.

Q10



$L = 2R^2C$

Voltage drops

$iR \qquad q/C \qquad Li'$

Kirchhoff's laws: ① sum of currents at a junction is zero

$i_c + i_L = i_R \Rightarrow q_c' + q_L' = q_R'$ where $i = q'$

differentiate $\Rightarrow q_c'' + q_L'' = q_R''$

② voltage around a loop is zero so $V_C = V_R = V_L$

$V_C = \frac{q_c}{C}, \quad V_R = i_R R, \quad V_L = Li_L' = Lq_L''$

$i_R = \frac{q_c}{CR}$

$i_R' = \frac{q_c'}{CR}$

$q_L'' = \frac{q_c}{CL}$

so $q_c'' + \underbrace{q_L''}_{\frac{q_c}{CL}} = \underbrace{q_R''}_{\frac{q_c'}{CR}}$

just write q for q_c now: $q'' - \frac{q'}{CR} + \frac{q}{CL} = 0 \qquad L = 2R^2C$

$q'' - \frac{q'}{CR} + \frac{q}{2R^2C^2} = 0$ try $q = e^{\lambda t}$: $e^{\lambda t}\left(\lambda^2 - \frac{\lambda}{CR} + \frac{1}{2R^2C^2}\right)$.

$$\lambda = \frac{\frac{1}{CR} \pm \sqrt{\frac{1}{C^2R^2} - \frac{4}{2R^2C^2}}}{2} = \frac{1}{2CR} \pm \sqrt{\frac{-1}{2CR}} = k \pm ik$$

so general solution is $q(t) = Ae^{-kt}\cos kt + Be^{-kt}\sin kt$

initial conditions $\left.\begin{matrix} q(0) = Q_0 \\ \cancel{q'(0)=0} \end{matrix}\right\} \Rightarrow A = Q_0.$ $\cancel{q(t)=}$

$i_L(0) = 0 \Rightarrow i_C(0) = i_R(0) = \frac{q_C(0)}{CR} = 0$

$q'(t) = -kQ_0e^{-kt}\cos kt - kQ_0e^{-kt}\sin kt - kBe^{-kt}\sin kt + kBe^{-kt}\cos kt$

$q'(0) = 0 \Rightarrow -kQ_0 + kB = 0 \Rightarrow B = -Q.$

solution: $\boxed{q(t) = Q_0e^{-kt}(\cos kt - \sin kt)}$.