Math 330 ODEs Fall 15 Linear Algebra questions

(1) Are the following statement true or false? Explain.
   (a) Every vector space V contains a subspace W such that $W \neq V$.
   (b) If S is a linearly dependent set then each element of S may be written as a linear combination of the other elements.
   (c) Every subset of a linearly dependent set is linearly dependent.
   (d) Every subset of a linearly independent set is linearly independent.
   (e) The intersection of any two subspaces of V is a subspace of V.
   (f) The union of any two subspaces of V is a subspace of V.

(2) Let P_n be the vector space of degree n polynomials.
   (a) Write down a basis for P_n. What is the dimension of P_n?
   (b) Write down an explicit matrix giving the map $P_3 \to P_2$ defined by $p(x) \mapsto p'(x)$.
   (c) Is the evaluation map $P_n \to \mathbb{R}$ defined by $p(x) \mapsto p(4)$ a linear map?
   (d) Define an inner product on P_n by $p.q = \int_{-1}^{1} p(t)q(t)\ dt$. Use Gram-Schmidt to find an orthonormal basis for P_2.

(3) Let $M_{n,m}$ be the vector space of $n \times m$ matrices, and let A be an $n \times m$ matrix.
   (a) Is the map $M_{m,p} \to M_{n,p}$ given by $B \mapsto AB$ a linear map?
   (b) Is the map $A \mapsto A^T$ a linear map?
   (c) Is the map $M_{n,n} \to \mathbb{R}$ given by $A \mapsto \det(A)$ a linear map?

(4) A matrix is symmetric if $A = A^T$.
   (a) Show directly that the set of symmetric matrices forms a vector space.
   (b) Show that the map $A \mapsto A - A^T$ is linear, and that its kernel is equal to the symmetric matrices.

(5) (a) Find a 2×2 matrix A such that $A \neq 0$ but $A^2 = 0$. What does this do geometrically as a map $A \colon \mathbb{R} \to \mathbb{R}$?
   (b) Can you find a 2×2 matrix A such that $A \neq 0, A^2 \neq 0$ but $A^3 = 0$?
   (c) Find a 3×3 matrix A such that $A \neq 0, A^2 \neq 0$ but $A^3 = 0$. What does this do geometrically as a map $A \colon \mathbb{R} \to \mathbb{R}$?
   (d) Can you generalize this to $n \times n$ matrices?

(6) (a) Show that the complex numbers $\mathbb{C}$ form a 2-dimensional vector space over $\mathbb{R}$, with the usual complex addition and multiplication by real numbers.
(b) Show that multiplication by i is a linear map, and represent it as a matrix with respect to your favourite basis for $\mathbb{C}$.
(c) Find an explicit identification of $\mathbb{C}$ with a subset of 2×2 matrices which takes complex multiplication to matrix multiplication.

(7) Let U and V be subspaces of W, and define $U+V$ to be the sum of all vectors in U and V.
(a) Show that $U + V$ a subspace of W.
(b) Is the dimension of $U + V$ equal to $\dim(U) + \dim(V)$?
(c) Can you express $\dim(U+V)$ in terms of $\dim(U)$, $\dim(V)$ and $\dim(U \cap V)$?

(8) (a) Let A be a matrix such that $A^m = 0$ for some $m > 0$. Show that every eigenvalue of A is zero.
(b) Let A be a matric such that $A^m = I$ for some $m > 0$. What can you say about the eigenvalues of A?
(c) Give explicit examples of 2×2 matrices with $A^m = I$ but $A \neq I$ for all $m > 0$. What do these do geometrically? What are their eigenvalues?

(9) (a) Consider the matrix $D = \begin{bmatrix} -2 & 0 \\ 0 & 1 \end{bmatrix}$. Find an explicit formula for D^n.
(b) Consider the matrix $A = \begin{bmatrix} 1 & 0 \\ 3 & -2 \end{bmatrix}$. Find the eigenvalues and eigenvectors for A.
(c) Find a matrix T such that $A = TDT^{-1}$. Find an explicit formula for A^n.
(d) Define e^{At} to be $I + A + \frac{1}{2!}A^2 + \frac{1}{3!}A^3 + \cdots$. Find e^{Dt}, then use this to find e^{At}.

(10) The Fibonacci sequence is the sequence $1, 1, 2, 3, 5, 8, \ldots$, i.e. $\{a_n\}$ where $a_1 = a_2 = 1$ and $a_{n+2} = a_{n+1} + a_n$ for $n \geq 2$.
(a) Consider the sequence of vectors $v_n = \begin{bmatrix} a_{n+1} \\ a_n \end{bmatrix}$, and show there is a matrix A such that $v_{n+1} = Av_n$.
(b) Find the eigenvalues and eigenvectors of A, and find an explicit formula for $v_n = A^{n-1}v_1$, and hence for a_n.