Math 330 Differential Equations Fall 15 Final b

Name: Solutions

- I will count your best 8 of the following 10 questions.
- You may use yor textbooks and notes, but no electronic devices.

1	10	
2	10	
3	10	
4	10	
5	10	
6	10	
7	10	
8	10	
9	10	
10	10	
	80	

Final	
Overall	

(1) (10 points) Find the general solution to the following differential equation.

$$y' = \frac{x+3y}{x}$$

$y = ux$

$y' = u'x + u$

$$u'x + u = \frac{x + 3ux}{x} = 1 + 3u$$

$$u'x = 1 + 2u$$

$$\int \frac{du}{1+2u} = \int \frac{1}{x}\,dx$$

$$\frac{1}{2} \ln|1+2u| = \ln|x| + c$$

$$1 + 2u = Ax^2$$

$$\frac{y}{x} = u = \frac{Ax^2 - 1}{2}$$

$$y = \frac{1}{2}Ax^3 - \frac{1}{2}x$$

(2) (10 points) Find the solution to

$$y'' - 4y = e^{-2x}$$

with $y(0) = 1$ which stays bounded as $x \to \infty$.

homogeneous : $y''-4y=0$ try $y = e^{\lambda x}$: $e^{\lambda x}(\lambda^2-4)=0$

$(\lambda-2)(\lambda+2)$

$\lambda = \pm 2$

$y_h(x) = c_1 e^{2x} + c_2 e^{-2x}$

particular solution: try $y = Axe^{-2x}$

$y' = Ae^{-2x} - 2Axe^{-2x}$

$y'' = -2Ae^{-2x} - 2Ae^{-2x} + 4Axe^{-2x}$

plug in : $-4Ae^{-2x} + \cancel{4Axe^{-2x}} - \cancel{4Axe^{-2x}} = e^{-2x}$ $\Rightarrow$ $-4A = 1$ $A = -1/4$

general solution $c_1 e^{2x} + c_2 e^{-2x} - \frac{1}{4} xe^{-2x}$

bounded $\Rightarrow c_1 = 0$

$y(0) = 1 = c_2$

solution : $e^{-2x} - \frac{1}{4} xe^{-2x}$

(3) (10 points) Find the eigenvalues and eigenvectors for the following matrix.

$$\begin{bmatrix} 3 & -1 \\ 2 & 0 \end{bmatrix}$$

$$\begin{vmatrix} 3-\lambda & -1 \\ 2 & -\lambda \end{vmatrix} = \lambda(\lambda-3)+2 = \lambda^2 - 3\lambda + 2$$

$$(\lambda-2)(\lambda-1) \quad \lambda = 1, 2$$

$\lambda = 1$: $\begin{bmatrix} 2 & -1 \\ 2 & -1 \end{bmatrix} \sim \begin{bmatrix} 2 & -1 \\ 0 & 0 \end{bmatrix}$ $\quad \underline{v} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$

$\lambda = 2$: $\begin{bmatrix} 1 & -1 \\ 2 & -2 \end{bmatrix} \sim \begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix}$ $\quad \underline{v} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

(4) (10 points) Consider the following differential equation.

$$X' = AX, \text{ where } A = \begin{bmatrix} 3 & -1 \\ 2 & 0 \end{bmatrix}$$

(a) Find the general solution in the form $\Omega(t)C$, where $\Omega(t)$ is the fundamental matrix solution and $C = \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$.

(b) Find Ω^{-1}.

You may use your solution to the previous question.

a) $$c_1 \begin{bmatrix} 1 \\ 2 \end{bmatrix} e^{t} + c_2 \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{2t} = \underbrace{\begin{bmatrix} e^{t} & e^{2t} \\ 2e^{t} & e^{2t} \end{bmatrix}}_{\Omega} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$$

b) $$\Omega^{-1} = \frac{1}{e^{3t} - 2e^{3t}} \begin{bmatrix} e^{2t} & -e^{2t} \\ -2e^{t} & e^{t} \end{bmatrix} = - \begin{bmatrix} e^{-t} & -e^{-t} \\ -2e^{-2t} & e^{-2t} \end{bmatrix} = \begin{bmatrix} -e^{-t} & e^{-t} \\ 2e^{-2t} & -e^{-2t} \end{bmatrix}$$

(5) (10 points) Find the general solution to the following differential equation, by looking for a solution of the form $X(t) = \Omega(t)U(t)$.

$$X' = AX + F(t), \text{ where } A = \begin{bmatrix} 3 & -1 \\ 2 & 0 \end{bmatrix}, F(t) = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

You may use your solution to the previous question.

plug in: $\cancel{\Omega' U} + \Omega U' = \cancel{A\Omega U} + F$ $\qquad U' = \Omega^{-1} F$

$$U = \int \Omega^{-1} F \, dt$$

$$\Omega^{-1} F = \begin{bmatrix} -e^{-t} & e^{-t} \\ 2e^{-2t} & -e^{-2t} \end{bmatrix} \begin{bmatrix} 2 \\ 1 \end{bmatrix} = \begin{bmatrix} -e^{-t} \\ 3e^{-2t} \end{bmatrix}$$

$$\int \Omega^{-1} F \, dt = \begin{bmatrix} e^{-t} \\ -\frac{3}{2} e^{-2t} \end{bmatrix}$$

general solution $\Omega C + \Omega \int \Omega^{-1} F \, dt$

$$X(t) = \begin{bmatrix} e^{t} & e^{2t} \\ 2e^{t} & e^{2t} \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} + \begin{bmatrix} e^{t} & e^{2t} \\ 2e^{t} & e^{2t} \end{bmatrix} \begin{bmatrix} e^{-t} \\ -\frac{3}{2} e^{-2t} \end{bmatrix}$$

$$= \begin{bmatrix} e^{t} & e^{2t} \\ 2e^{t} & e^{2t} \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} + \begin{bmatrix} -1/2 \\ 1/2 \end{bmatrix}$$

(6) (10 points) Find the equilibrium solutions and investigate their stability for the following differential equation.

$$x'' = ex - xe^x - (x')^2$$

$x' = y$

$y' = ex - xe^x - y^2$

$X' = F(X)$ solve $F(X) = 0$: $y = 0$

$x(e - e^x) = 0$ $x = 0, 1$

$(0,0)$ and $(1,0)$

$$DF = \begin{bmatrix} 0 & 1 \\ e - e^x - xe^x & -2y \end{bmatrix}$$

$DF\begin{bmatrix} 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ e-1 & 0 \end{bmatrix}$ eigenvalues: $\begin{vmatrix} -\lambda & 1 \\ e-1 & -\lambda \end{vmatrix} = \lambda^2 - (e-1) = 0$

$\lambda = \pm\sqrt{e-1}$

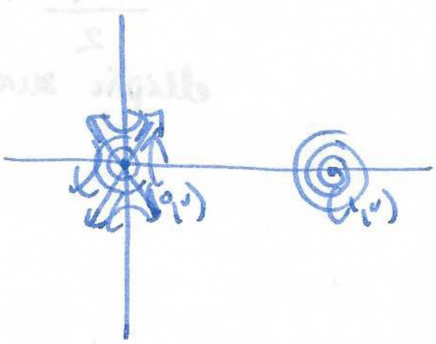
saddle

$DF\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ $\begin{bmatrix} 0 & 1 \\ e-e-e & 0 \end{bmatrix}$ eigenvalues: $\begin{vmatrix} -\lambda & 1 \\ -e & -\lambda \end{vmatrix} = \lambda^2 + e = 0$

$= -e$

$\lambda = \pm\sqrt{e}\, i$

elliptic



(7) (10 points) Find the equilibrium solutions and investigate their stability for the following system of differential equations.

$$x' = xy - x$$
$$y' = x + y$$

$X' = F(X)$

solve $F(X) = 0$: $x(y-1) = 0$ $\quad x = 0,\ y = 1 \quad (0,0) \quad (-1,1)$

$x = -y$

$$DF = \begin{bmatrix} y-1 & x \\ 1 & 1 \end{bmatrix}$$

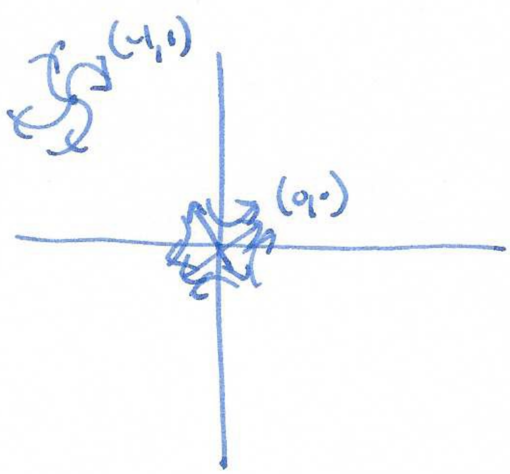
$$DF\begin{bmatrix} 0 \\ 0 \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 1 & 1 \end{bmatrix}$$ eigenvalues ± 1 saddle

$$DF\begin{bmatrix} -1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 & -1 \\ 1 & 1 \end{bmatrix}$$ eigenvalues: $\begin{vmatrix} -\lambda & -1 \\ 1 & 1-\lambda \end{vmatrix} = \lambda(\lambda-1)+1$

$\lambda^2 - \lambda + 1$

$\lambda = \dfrac{1 \pm \sqrt{1-4}}{2}$

elliptic source



(8) (10 points) Find the inverse Laplace transform for the following function.

$$F(s) = \frac{4e^{-2s}}{s^2 - 2s + 2} = 4e^{-2s} \frac{1}{(s-1)^2+1}$$

$$\sin(at) \overset{L}{\longmapsto} \frac{a}{s^2+a^2} \Rightarrow \sin(t) \longmapsto \frac{1}{s^2+1}$$

$$e^{at} f(t) \overset{L}{\longmapsto} F(s-a) \Rightarrow e^{t}\sin(t) \longmapsto \frac{1}{(s-1)^2+1}$$

$$H(t-a) f(t-a) \overset{L}{\longmapsto} e^{-as} F(s)$$

$$\Rightarrow L^{-1}\left(4e^{-2s} \frac{1}{(s-1)^2+1}\right) = 4H(t-2)\, e^{t(t-2)} \sin(t-2)$$

(9) (10 points) Find the Laplace transform of the function.

$$f(t) = \begin{cases} 0 & t < 2 \\ e^{-3t} & t \geq 2 \end{cases}$$

$$f(t) = H(t-2)e^{-3t} = H(t-2)e^{-3(t-2+2)} = e^{-6} H(t-2) \underbrace{e^{-3(t-2)}}_{f(t) = e^{-3t}}$$

$$H(t-a)f(t-a) \xrightarrow{L} e^{-as}F(s), \quad e^{at} \xrightarrow{L} \frac{1}{s-a}$$

$$L(f(t)) = e^{-6} \cdot e^{-2s} \frac{1}{s+3}$$

(10) (10 points) Use the Laplace transform to solve the following IVP.

$$y'' + 5y' + 6y = \begin{cases} 0 & t < 2 \\ e^{-3t} & t \geq 2 \end{cases} \quad y(0) = 0,\ y(0) = 0$$

Hint: you may use your answer to the previous question.

$$s^2Y - s\underbrace{y(0)}_{0} - \underbrace{y'(0)}_{0} + 5(sY - \underbrace{y(0)}_{0}) + 6Y = e^{-6}e^{-2s}\frac{1}{s+3}.$$

$$Y(s^2+5y+6) = Y(s+2)(s+3) = \frac{e^{-6}e^{-2s}}{s+3}.$$

$$\frac{1}{(s+2)(s+3)^2} = \frac{A}{s+2} + \frac{B}{s+3} + \frac{C}{(s+3)^2}$$

$$1 = A(s+3)^2 + B(s+2)(s+3) + C(s+2)$$

$s=-3$: $1 = -C$

$s=-2$ $1 = A$

$s=0$ $1 = \underbrace{9A}_{9} + 6B + \underbrace{2C}_{-2}$ $B = -1$

$$Y = e^{-6}e^{-2s}\left[\frac{1}{s+2} - \frac{1}{s+3} - \frac{1}{(s+3)^2}\right]$$

$$\frac{1}{s+2} \to e^{-2t} \qquad \frac{1}{s+3} \to -e^{-3t} \qquad \frac{1}{(s+3)^2} \to -te^{-3t}$$

$$y(t) = e^{-6}H(t-2)\left[e^{-2(t-2)} - e^{-3(t-2)} - (t-2)e^{-3(t-2)}\right].$$