MTH 330 Sample final solutions

Q1 a) $y'\tan(y) = 1 + \frac{x^2}{1+x^2} = 1 + 1 + \frac{-1}{1+x^2} = 2 - \frac{1}{1+x^2}$

$\int \tan(y)\,dy = \int 2 - \frac{1}{1+x^2}\,dx \qquad \ln|\sec(y)| = 2x - \tan^{-1}(x) + c$

$y = \cos^{-1}\left(Ae^{\tan^{-1}(x) - 2x}\right)$

b) $y'' - 2y' + 2y = e^t \sin t$

homogeneous system: $y'' - 2y' + 2y$ try $y = e^{\lambda t}$: $e^{\lambda t}(\lambda^2 - 2\lambda + 2) = 0$

$\lambda = \frac{2 \pm \sqrt{4-8}}{2} = 1 \pm i$: gen homogeneous solution $h(t) = c_1 e^t \cos t + c_2 e^t \sin t$

particular solution: try

$y = th(t)$

$y' = h + th'$

$y'' = h' + h' + th'' = 2h' + th''$

plug in: $2h' + th'' - 2h - 2th' + 2th = e^t \sin t$

$t\underbrace{(h'' - 2h' + 2h)}_{=0} + 2h' - 2h = e^t \sin t$

$h' = c_1 e^t \cos t \cancel{+} - c_1 e^t \sin t + c_2 e^t \sin t + c_2 e^t \cos t$

$e^t \cos t\left[2c_1 + 2c_2 - 2c_1\right] + e^t \sin t\left[-2c_1 + 2c_2 - 2c_2\right] = e^t \sin t$.

$c_2 = 0$, $\qquad -2c_1 = 1 \qquad c_1 = -\frac{1}{2}$.

$y_p(t) = -\frac{1}{2}te^t \cos t$

general solution: $c_1 e^t \cos t + c_2 e^t \sin t - \frac{1}{2}te^t \cos t$.

c) $y' = \frac{3t}{y + t^2 y} = \frac{1}{y}\,\frac{3t}{1+t^2}$

$\int y\,dy = \int \frac{3t}{1+t^2}\,dt$

$\frac{1}{2}y^2 = \frac{3}{2}\ln|1+t^2| + c \qquad y = \sqrt{3\ln|1+t^2|+2c}$

$y(0) = 1 \Rightarrow c = \frac{1}{2}$ IVP solution: $y(t) = \sqrt{3\ln|1+t^2|+1}$

d) $\underbrace{(x-1)y' + y}_{\text{exact } = ((x-1)y)'} = x^2 - 2, \quad y(2) = 1$

$$(x-1)y = \int x^2 - 2\,dx = \frac{1}{3}x^3 - 2x + c$$

$y(2) = 1$: $\quad 1 = \frac{1}{3}8 - 4 + c \qquad c = 1 + \frac{4}{3} = \frac{7}{3}$.

IVP solution: $\quad y = \dfrac{\frac{1}{3}x^3 - 2x + 7/3}{x-1}$.

Q2 a) $f(t) = (t^2 - 2t + 1)(e^{-t} - 1) = t^2e^{-t} - 2te^{-t} + e^{-t} - t^2 + 2t - 1$.

$$F(s) = \frac{2}{(s+1)^3} - \frac{2}{(s+1)^2} + \frac{1}{s+1} - \frac{2}{s^3} + \frac{2}{s^2} - \frac{1}{s}$$

b) $f(t) = te^{-t}\cos(3t) = e^{-t}g(t)$ where $g(t) = t\cos(3t)$

so $G(s) = \dfrac{s^2-9}{(s^2+9)^2}$

$$F(s) = G(s+1) = \frac{(s+1)^2 - 9}{((s+1)^2+9)^2}$$

c) $f(t) = \begin{cases} -t & t<4 \\ t^2+1 & t \geq 4 \end{cases}$

$f(t) = -t + H(t-4)(t^2+t+1)$

$= -t + H(t-4)\left[(t-4+4)^2 + t-4+4+1\right]$

$(t-4)^2 + 8(t-4) + 16 + (t-4) + 5$

$(t-4)^2 + 9(t-4) + 21$

$= -t + H(t-4)g(t-4)$ where $g(t) = t^2 + 9t + 2$

$$F(s) = \frac{-1}{s^2} + e^{-4s}G(s) = -\frac{1}{s^2} + e^{-4s}\left[\frac{2}{s^3} + \frac{9}{s^2} + \frac{2}{s}\right].$$

d) $F(s) = \dfrac{3e^{-2s}}{s^2-9} = e^{-2s}\dfrac{3}{s^2-9} = e^{-2s}G(s)$.

$\dfrac{3}{s^2-9} = \dfrac{3}{(s-3)(s+3)} = \dfrac{A}{s-3} + \dfrac{B}{s+3}$.

$3 = A(s+3) + B(s-3)$

$s = 3: \quad 3 = 6A \quad A = 1/2$

$s = -3: \quad 3 = -6B \quad B = -1/2$

$f(t) = H(t-2)g(t-2)$

$G(s) = \frac{1/2}{s-3} - \frac{1/2}{s+3} \qquad g(t) = \frac{1}{2}e^{3t} - \frac{1}{2}e^{-3t}$

$f(t) = H(t-2)\frac{1}{2}\left(e^{3(t-2)} - e^{-3(t-2)}\right).$

e) $F(s) = \frac{-2s+1}{s^2+4s+13} = \frac{-2s+1}{(s+2)^2+9} = \frac{-2(s+2-2)+1}{(s+2)^2+9} = \frac{-2(s+2)}{(s+2)^2+9} + \frac{5}{(s+2)^2+9}$

$f(t) = -2e^{-2t}\cos 3t + \frac{5}{3}e^{-2t}\sin(3t)$

<u>Q3</u> $y'' + 3y' + 2y = f(t) = \begin{cases} e^t & t<2 \\ 0 & t \geq 2 \end{cases} = H(2-t)e^t = e^t - H(t-2)e^t$ $\quad y(0)=0,\ y'(0)=0.$

$= e^t - H(t-2)e^{(t-2)+2}$

Laplace transform:

$s^2Y - \underbrace{sy(0)}_{0} - \underbrace{y'(0)}_{0} + 3sY - \underbrace{y(0)}_{0} + 2Y = \frac{1}{s-1} - e^{-2s}\frac{1}{s-1}e^2$

$Y\underbrace{(s^2+3s+2)}_{(s+2)(s+1)} = \frac{1-e^{2-2s}}{s-1} \qquad Y = \frac{1-e^{2-2s}}{(s-1)(s+1)(s+2)}$

$\frac{1}{(s-1)(s+1)(s+2)} = \frac{A}{s-1} + \frac{B}{s+1} + \frac{C}{s+2} \qquad 1 = A(s+1)(s+2) + B(s-1)(s+2) + C(s-1)(s+1)$

$s=1: \ 1 = 6A \qquad s=-1: \ 1 = -2B \qquad s=-2: \ 1 = C.(-3)(-1) = 3C.$

$Y = \frac{1/6}{s-1} - \frac{1/2}{s+1} + \frac{1/3}{s+2} + e^{-2s}\left[\frac{1/6}{s-1} - \frac{1/2}{s+1} + \frac{1/3}{s+2}\right](-e^2).$

$y(t) = \frac{1}{6}e^t - \frac{1}{2}e^{-t} + \frac{1}{3}e^{-2t} + H(t-2)(-e^2)\left[\frac{1}{6}e^{t-2} - \frac{1}{2}e^{-(t-2)} + \frac{1}{3}e^{-2(t-2)}\right].$

<u>Q4</u> $y'' + 4y' + 4y = 2\delta(t+1), \ y(0)=0, \ y'(0) = -1$

$s^2Y - \underbrace{sy(0)}_{0} - \underbrace{y'(0)}_{-1} + 4sY - \underbrace{4y(0)}_{0} + 4Y = 2e^{s}$

$Y(s^2+4s+4) = 2e^{s} - 1$

$Y = \frac{2e^{s}-1}{s^2+4s+4} = \frac{2e^{s}-1}{(s+2)^2} = \frac{2e^{s}}{(s+2)^2} - \frac{1}{(s+2)^2}$

$$y(t) = 2H(t+1)e^{-2t}\cdot t - e^{-2t}\cdot t$$

Q5 $X' = AX \quad A = \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} \quad X(0) = \begin{bmatrix} -2 \\ 3 \end{bmatrix}$

eigenvalues: $\begin{vmatrix} 1-\lambda & 1 \\ -1 & 1-\lambda \end{vmatrix} = (1-\lambda)^2 + 1 = \lambda^2 - 2\lambda + 2 \qquad \lambda = \dfrac{2 \pm \sqrt{4-8}}{2} = 1 \pm i$

eigenvectors: $\lambda = 1+i$: $\begin{bmatrix} -i & 1 \\ -1 & -i \end{bmatrix} \rightsquigarrow \begin{bmatrix} 1 & i \\ 0 & 0 \end{bmatrix} \quad \underline{v} = \begin{bmatrix} i \\ -1 \end{bmatrix} = \begin{bmatrix} 0 \\ -1 \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \end{bmatrix} i$

general solution:
$$X(t) = c_1 e^t\left(\begin{bmatrix} 0 \\ -1 \end{bmatrix}\cos t - \begin{bmatrix} 1 \\ 0 \end{bmatrix}\sin t\right) + c_2 e^t\left(\begin{bmatrix} 0 \\ -1 \end{bmatrix}\sin t + \begin{bmatrix} 1 \\ 0 \end{bmatrix}\cos t\right).$$

$X(0) = \begin{bmatrix} -2 \\ 3 \end{bmatrix} = c_1\begin{bmatrix} 0 \\ -1 \end{bmatrix} + c_2\begin{bmatrix} 1 \\ 0 \end{bmatrix} \qquad c_1 = -3,\ c_2 = -2$

IVP solution: $X(t) = -3e^t\left(\begin{bmatrix} 0 \\ -1 \end{bmatrix}\cos t - \begin{bmatrix} 1 \\ 0 \end{bmatrix}\sin t\right) - 2e^t\left(\begin{bmatrix} 0 \\ -1 \end{bmatrix}\sin t + \begin{bmatrix} 1 \\ 0 \end{bmatrix}\cos t\right).$

Q6 $X' = AX + F(t) \quad A = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix} \quad F(t) = \begin{bmatrix} 3t \\ 2 \end{bmatrix}$

solve homogeneous: $X' = AX$ eigenvalues: $\begin{vmatrix} 1-\lambda & 2 \\ 2 & 1-\lambda \end{vmatrix} = (1-\lambda)^2 - 4 = \lambda^2 - 2\lambda - 3 = (\lambda-3)(\lambda+1) = 0$

$\lambda = 3, -1.$

$\lambda = 3$: $\begin{bmatrix} -2 & 2 \\ 2 & -2 \end{bmatrix} \rightsquigarrow \begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix} \quad \underline{v} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \qquad \lambda = -1$: $\begin{bmatrix} 2 & 2 \\ 2 & 2 \end{bmatrix} \rightsquigarrow \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} \quad \underline{v} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}.$

homogeneous ~~general~~ sol^n $\Omega . C = \begin{bmatrix} \underline{v}_1 e^{\lambda_1 t} & \underline{v}_2 e^{\lambda_2 t} \end{bmatrix}\begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} e^{3t} & e^{-t} \\ e^{3t} & -e^{-t} \end{bmatrix}\begin{bmatrix} c_1 \\ c_2 \end{bmatrix}.$

look for solution $\Omega . Y$ plug in: $\underbrace{\Omega' Y + \Omega Y' = A\Omega Y}_{\text{equal}} + F \qquad Y' = \Omega^{-1}F.$

$Y = \int \Omega^{-1}F\,dt \qquad \Omega^{-1} = \begin{bmatrix} -e^{-t} & -e^{-t} \\ -e^{3t} & e^{3t} \end{bmatrix}\dfrac{1}{-e^{2t} - e^{2t}} = \dfrac{1}{2}\begin{bmatrix} e^{-3t} & e^{-3t} \\ e^{t} & -e^{t} \end{bmatrix}$

$\Omega^{-1}F = \dfrac{1}{2}\begin{bmatrix} e^{-3t} & e^{-3t} \\ e^{t} & -e^{t} \end{bmatrix}\begin{bmatrix} 3t \\ 2 \end{bmatrix} = \dfrac{1}{2}\begin{bmatrix} 3te^{-3t} + 2e^{-3t} \\ 3te^{t} - 2e^{t} \end{bmatrix}.$

$Y = \dfrac{1}{2}\begin{bmatrix} -\frac{1}{3}e^{-3t}\cdot 3t + \frac{1}{3}\int 3\cancel{t}e^{-3t}\,dt - \frac{2}{3}e^{-3t} + c_1 \\ 3te^{t} - \int 3e^{t}\,dt - 2e^{t}. \end{bmatrix} = \dfrac{1}{2}\begin{bmatrix} -te^{-3t} - \frac{1}{3}e^{-3t} - \frac{2}{3}e^{-3t} \\ 3te^{t} - 3e^{t} - 2e^{t} \end{bmatrix}.$

$$Y = \frac{1}{2}\begin{bmatrix} -te^{-3t} - e^{-3t} \\ 3te^{t} - 5e^{t} \end{bmatrix} \qquad \text{solution } \Omega Y = \begin{bmatrix} e^{3t} & e^{-t} \\ e^{3t} & -e^{-t} \end{bmatrix}\begin{bmatrix} -te^{-3t} - e^{-3t} \\ 3te^{t} - 5e^{t} \end{bmatrix}.$$

Q7

$$\begin{tikzcd} \mathbb{R}^3_{e_i} \arrow[r, "A"] & \mathbb{R}^3_{e_i} \\ \mathbb{R}^3_{\underline{v_i}} \arrow[r, "D"'] \arrow[u, "T"] & \mathbb{R}^3_{\underline{v_i}} \arrow[u, "T"'] \end{tikzcd}$$

$$D = \begin{bmatrix} 3 & 0 & 0 \\ 0 & \cos\frac{\pi}{4} & -\sin\frac{\pi}{4} \\ 0 & \sin\frac{\pi}{4} & \cos\frac{\pi}{4} \end{bmatrix} = \begin{bmatrix} 3 & 0 & 0 \\ 0 & \sqrt{2}/2 & -\sqrt{2}/2 \\ 0 & \sqrt{2}/2 & \sqrt{2}/2 \end{bmatrix}.$$

$$\underline{v_1} = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} \quad \underline{v_2} = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} \quad \underline{v_3} = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$$

so $A = TDT^{-1}$ where $T = \begin{bmatrix} 1 & 1 & 1 \\ -1 & 0 & 2 \\ 1 & -1 & 1 \end{bmatrix}$.

Q8 $U+V$ has basis $\left\{ \begin{bmatrix} 0 \\ 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix} \right\}$ 3-dimensional

$U \cap V$ is 1-dimensional,

$$\begin{array}{cccc} v_1 & v_2 & w_1 & w_2 \\ \hline \end{array}$$
$$\begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$
$$c_1 \quad c_2 \quad c_3 \quad \underbrace{c_4}_{t}$$

$$c_3 = t \qquad c_2 = -t \qquad c_1 = -t \qquad \begin{bmatrix} -1 \\ -1 \\ 1 \\ 1 \end{bmatrix} \Rightarrow \underline{v_1} + \underline{v_2} = \underline{w_1} + \underline{w_2}$$

so basis is $\begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$ for $U \cap V$.

Q9 $x''' = x - x^2 - x'x''$

$$x' = y \qquad y' = z \qquad z' = x - x^2 - yz$$

critical points $X' = F(X) = 0$: $y = 0$, $z = 0$, $x - x^2 = 0$, $x(1-x)$, $x = 0, 1$.

$$DF = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1-2x & -z & -y \end{bmatrix} \quad \text{at } (0,0,0)\ DF = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix} \text{ eigenvalues } \begin{vmatrix} -\lambda & 1 & 0 \\ 0 & -\lambda & 1 \\ 1 & 0 & -\lambda \end{vmatrix}$$

$$= -\lambda \begin{vmatrix} -\lambda & 1 \\ 0 & -\lambda \end{vmatrix} - \begin{vmatrix} 0 & 1 \\ 1 & -\lambda \end{vmatrix} = -\lambda^3 + 1 = 0 \quad \lambda^3 = 1$$

$\Rightarrow$ unstable

$\lambda = 1, \omega, \bar{\omega}$

$\omega = \cos\frac{2\pi}{3} + i\sin\frac{2\pi}{3}$ real$(\omega) < 0$

unstable.

$$DF\left(\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -1 & 0 & 0 \end{bmatrix} \text{ eigenvalues } \begin{vmatrix} -\lambda & 1 & 0 \\ 0 & -\lambda & 1 \\ -1 & 0 & -\lambda \end{vmatrix}$$

$$= -\lambda \begin{vmatrix} -\lambda & 1 \\ 0 & -\lambda \end{vmatrix} - \begin{vmatrix} 0 & 1 \\ -1 & -\lambda \end{vmatrix} = -\lambda^3 - 1 = 0 \quad \lambda^3 = -1$$

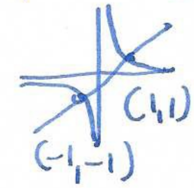
$\lambda = -1, a, \bar{a}$

real$(a) \geq 0 \Rightarrow$ unstable.

Q10 $\quad x' = xy - 1 \qquad \dot{x}' = F(x)$ equilibrium solutions $F(x) = 0$: $\quad xy = 1$

$\qquad y' = y - x \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad y = x$

$\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad x = \pm 1$



$$DF = \begin{bmatrix} y & x \\ -1 & 1 \end{bmatrix} \qquad \text{at } (1,1): DF\left(\begin{bmatrix}1\\1\end{bmatrix}\right) = \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}$$

eigenvalues $\begin{vmatrix} 1-\lambda & 1 \\ -1 & 1-\lambda \end{vmatrix} = (1-\lambda)^2 + 1 = 0$

$\lambda^2 - 2\lambda + 2 = 0$

$\lambda = 1 \pm i$

unstable

$$DF\left(\begin{bmatrix}-1\\-1\end{bmatrix}\right) = \begin{bmatrix} -1 & -1 \\ -1 & 1 \end{bmatrix}$$

eigenvalues $\begin{vmatrix} -1-\lambda & -1 \\ -1 & 1-\lambda \end{vmatrix} = (\lambda+1)(\lambda-1) - 1$

$= \lambda^2 - 2 = 0$

$\lambda = \pm\sqrt{2}$

unstable.