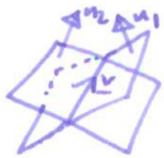
MTH 233 Sample final solutions 

**Q1**



$\underline{v} = \underline{n}_1 \times \underline{n}_2$ $\quad \underline{n}_1 = \langle 2,-4,-1\rangle$ $\quad \underline{n}_2 = \langle 1,-1,-2\rangle$

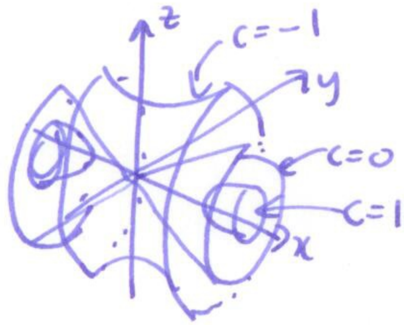
$$\underline{v} = \begin{vmatrix} i & j & k \\ 2 & -4 & -1 \\ 1 & -1 & -2 \end{vmatrix} = \langle 7,3,2\rangle$$

find a point on the line, try $z=0$: $\left.\begin{array}{l} 2x-4y=-2 \\ x-y=4 \end{array}\right\}$ $-2y=-10$ $\quad y=5$, $x=9$

line is $\underline{r}(t) = \langle 9,5,0\rangle + t\langle 7,3,2\rangle$

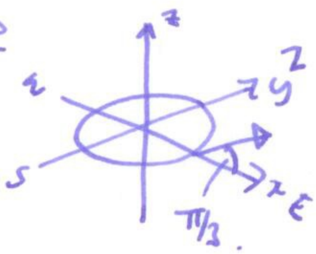
**Q2** $f(x,y,z) = x^2-y^2-z^2 = c$ $\quad \nabla f = \langle 2x,-2y,-2z\rangle$

$\nabla f(2,1,1) = \langle 4,-2,-2\rangle$




tangent plane $4x-2y-2z=4$

**Q3**



$\underline{c}(t) = \langle 10\cos t, 10\sin t, 0\rangle$

$\underline{c}'(t) = \langle -10\sin t, 10\cos t, 0\rangle$ check $\|c'(t)\| = 10$

$\underline{x}(0) = \langle 10,0,0\rangle$

$\underline{x}'(0) = \langle 0,10,0\rangle + \langle 20\cos\underbrace{\tfrac{\pi}{3}}_{\frac{1}{2}}, 0, 20\sin\underbrace{\tfrac{\pi}{3}}_{\frac{\sqrt3}{2}}\rangle$

$\underline{x}''(t) = \langle 0,0,-g\rangle$

$\underline{x}'(t) = \langle 0,0,-gt\rangle + \underline{c}$ $\qquad \underline{c} = \langle 10,10,10\sqrt3\rangle$

$\underline{x}(t) = \langle 0,0,-\frac{1}{2}gt^2\rangle + \langle 10,10,10\sqrt3\rangle t + \underline{c}$ $\qquad \underline{c} = \langle 10,0,0\rangle$

lands when $z=0$: $\quad -\frac{1}{2}gt^2 + 10\sqrt3 t = 0$ $\qquad (g \approx 10)$

$t(10\sqrt3 - 5t) = 0$ $\quad t = 2\sqrt3$.

$\underline{x}(2\sqrt3) = \langle 0,0,60\rangle + \langle 10,10,10\sqrt3\rangle 2\sqrt3 + \langle 10,0,0\rangle$.

$= \langle 10+20\sqrt3, 10, \cancel{60}+120\rangle$.

**Q4** $f(x,y) = x^2+2y^2-2y+4$

a) $\frac{\partial f}{\partial x} = 2x = 0 \Rightarrow x=0$

$\frac{\partial f}{\partial y} = 4y-2 = 0 \Rightarrow y=\frac{1}{2}$ critical point $(0,\frac{1}{2})$

$f_{xx} = 2$

$f_{xy} = 0$ $D(0,\frac{1}{2}) = f_{xx}f_{yy}-(f_{xy})^2 = 8 > 0$

$f_{yy} = 4$ $f_{xx}=2>0$ $\Big\} \Rightarrow$ local minimum.

b) $f(x,y) = x^2+2y^2-2y+4$ $\nabla f = \langle 2x, 4y-2\rangle$

$g(x,y) = x^2+y^2 = 4$ $\nabla g = \langle 2x, 2y\rangle$.

$\nabla f = \lambda \nabla g$ $2x = \lambda 2x \Rightarrow \lambda = 1$ or $x=0 \Rightarrow y=\pm 2$

$g=4$ $4y-2 = \lambda 2y \Rightarrow y=1$

$x^2+y^2 = 4 \Rightarrow x=\pm\sqrt{3}$.

c) $f(0,\frac{1}{2}) = \frac{1}{2}-1+4 = \frac{9}{2}$ min

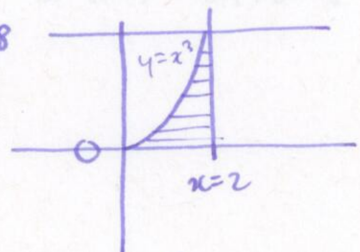
$f(\sqrt{3},1) = 3+2-2+4 = 7$

$f(-\sqrt{3},1) = 3+2-2+4 = 7$

$f(0,2) = 8-4+4 = 8$

$f(0,-2) = 8+4+4 = 16$ max.

**Q5** $\int_0^8 \int_{\sqrt[3]{y}}^2 \cos(x^4)\,dx\,dy$

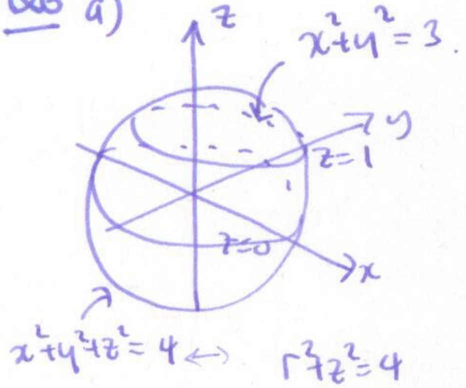



$x=\sqrt[3]{y}$ $x^3=y$

$\int_0^2 \int_0^{x^3} \cos(x^4)\,dy\,dx$ $\Big[y\cos(x^4)\Big]_0^{x^3} = x^3\cos(x^4)$.

$\int_0^2 x^3\cos(x^4)\,dx = \Big[\frac{1}{4}\sin(x^4)\Big]_0^2 = \frac{1}{4}\sin(16)$

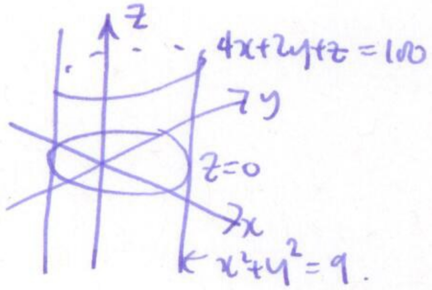
**Q6** a)




$$\int_0^{2\pi}\int_0^{\sqrt{3}}\int_0^1 f(r,\theta,z)\,dz\,dr\,d\theta$$

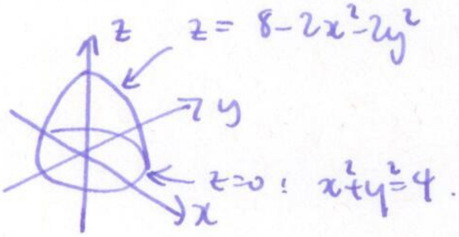
$$+\int_0^{2\pi}\int_{\sqrt{3}}^{2}\int_0^{\sqrt{4-r^2}} f(r,\theta,z)\,dz\,dr\,\theta$$

b)



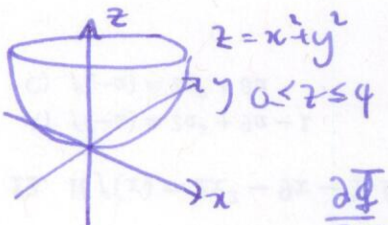

$$\int_0^{2\pi}\int_0^{3}\int_0^{100-4x-2y} f(r,\theta,z)\,dz\,dr\,d\theta$$

c)




$$\int_0^{2\pi}\int_0^{2}\int_0^{8-2r^2} r\,1\,dz\,dr\,d\theta$$

**Q7**




$\cancel{\Phi:(u,v)\mapsto(u,v,u^2+v^2)}$

$\Phi(r,\theta)\mapsto(r\cos\theta, r\sin\theta, r^2)$

$$\frac{\partial\Phi}{\partial r} = (\cos\theta, \sin\theta, 2r)$$

$$\frac{\partial\Phi}{\partial\theta} = (-r\sin\theta, r\cos\theta, 0)$$

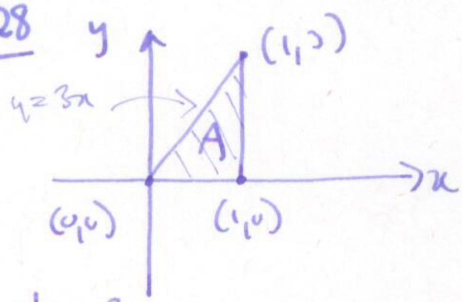
$$\underline{n} = \frac{\partial\Phi}{\partial r}\times\frac{\partial\Phi}{\partial\theta} = \begin{vmatrix} i & j & k \\ \cos\theta & \sin\theta & 2r \\ -r\sin\theta & r\cos\theta & 0 \end{vmatrix} = \langle -2r^2\cos\theta, -2r^2\sin\theta, r^2\rangle$$

$$\underline{F} = \langle -r\sin\theta, r\cos\theta, r^4\rangle.$$

$$\int\int \underline{F}\cdot\underline{dS} = \int_0^{2\pi}\int_0^2 \cancel{2r^3\cos\theta\sin\theta} - \cancel{2r^3\sin\theta\cos\theta} + r^6 \,dr\,d\theta$$

$$[r^6]_0^2 = \left[\frac{1}{7}r^7\right]_0^2 = \frac{128}{7} \qquad \int_0^{2\pi}\frac{128}{7}\,d\theta = \frac{256\pi}{7}$$

**Q8**

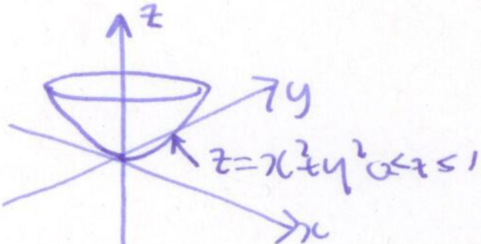



$$\underline{F} = \langle \sqrt{1+x^3}, 2xy \rangle$$

$$\int_C \underline{F}.\underline{ds} = \iint_A \frac{\partial f_2}{\partial x_1} - \frac{\partial f_1}{\partial x_2}\, dx\,dy$$

$$\int_0^1 \int_0^{3x} 2y - 0 \; dy\,dx \qquad \left[y^2\right]_0^{3x} = 9x^2 \qquad \int_0^1 9x^2 dx = \left[3x^3\right]_0^1 = 3.$$

**Q9**




$$\iint_S \underline{\nabla \times F}.\underline{dS} = \int_{\partial S} \underline{F}.\underline{ds} \qquad \underline{F} = \langle y^2, x, z^2 \rangle$$

$$\partial S: \quad \underline{c}(\theta) = \langle \cos\theta, \sin\theta, 1 \rangle$$
$$\underline{c}'(\theta) = \langle -\sin\theta, \cos\theta, 0 \rangle.$$

$\cos 2\theta = \cos^2\theta - \sin^2\theta$
$= 2\cos^2\theta - 1$

$$\int_{\partial S} \underline{F}.\underline{ds} = \int_0^{2\pi} \langle \sin^2\theta, \cos\theta, 1 \rangle \langle -\sin\theta, \cos\theta, 0 \rangle\, d\theta = \int_0^{2\pi} \underbrace{-\sin^3\theta}_{+\sin\theta(\cos^2\theta - 1)} + \overbrace{\cos^2\theta}^{\frac{1}{2}+\frac{1}{2}\cos 2\theta}\, d\theta$$

$$= \left[-\frac{1}{3}\cos^3\theta + \cos\theta + \frac{1}{2}\theta + \frac{1}{4}\sin 2\theta\right]_0^{2\pi} = \pi.$$

parameterization

$$(r,\theta) \xmapsto{\Phi} (r\cos\theta, r\sin\theta, r^2)$$

$$\nabla \times \underline{F} = \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y^2 & x & z^2 \end{vmatrix} = \langle 0, 0, 1 - 2y \rangle$$

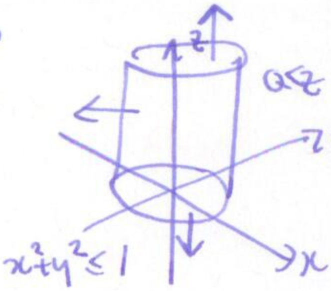
$$\frac{\partial \Phi}{\partial r} = (\cos\theta, \sin\theta, 2r)$$
$$\frac{\partial \Phi}{\partial \theta} = (-r\sin\theta, r\cos\theta, 0).$$

$$\underline{n} = \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ \cos\theta & \sin\theta & 2r \\ -r\sin\theta & r\cos\theta & 0 \end{vmatrix} = \langle -2r^2\cos\theta, -2r^2\sin\theta, r \rangle$$

$$\iint_S \underline{\nabla \times F}.\underline{dS} = \int_0^1 \int_0^{2\pi} (1 - 2r\sin\theta)\, r \; d\theta\, dr \qquad \left[r\theta + r^2\cos\theta\right]_0^{2\pi} = 2\pi r$$

$$\int_0^1 2\pi r\, dr = \left[\pi r^2\right]_0^1 = \pi.$$

Q10




$\underline{F} = \langle x, y, -z \rangle \qquad \nabla \cdot F = 1+1-1 = 1$ ⑤

$$\iint_{\partial W} \underline{F} \cdot d\underline{S} = \iiint_W \operatorname{div}(F)\, dV = \operatorname{vol}(W)$$

$$\iiint_W 1\, dV = \int_0^3 \int_0^{2\pi} \int_0^1 1\ r\, dr\, d\theta\, dz$$

$\left[\frac{1}{2}r^2\right]_0^1 = \frac{1}{2}$.

$\left[\frac{1}{2}\theta\right]_0^{2\pi} = \pi$.

$\left[\pi z\right]_0^3 = 3\pi$.

disc $z=0$: $\iint_D \underline{F} \cdot d\underline{S}$ $\quad \Phi: (r,\theta) \mapsto (r\cos\theta, r\sin\theta, 0)$

$\Phi_r = (\cos\theta, \sin\theta, 0)$

$\Phi_\theta = (-r\sin\theta, r\cos\theta, 0)$.

$\underline{n} = \Phi_r \times \Phi_\theta = \langle 0, 0, r \rangle$.

$$-\iint \langle r\cos\theta, r\sin\theta, 0 \rangle \cdot \langle 0, 0, r \rangle\, dA = 0.$$

disc $z=3$: $\Phi(r,\theta) = (r\cos\theta, r\sin\theta, 3) \quad \underline{n} = \langle 0,0,r \rangle$

$$\int_0^{2\pi} \int_0^1 \langle r\cos\theta, r\sin\theta, -3 \rangle \cdot \langle 0, 0, r \rangle\, dr\, d\theta = \int_0^{2\pi} \int_0^1 -3r\, dr\, d\theta$$

$\left[-\frac{3}{2}r^2\right]_0^1 = -\frac{3}{2} \qquad \left[-\frac{3}{2}\theta\right]_0^{2\pi} = -3\pi$.

curved surface of cylinder: $\Phi(\theta, z) = (\cos\theta, \sin\theta, z)$

$\Phi_\theta = (-\sin\theta, \cos\theta, 0) \quad \Phi_z = (0,0,1) \quad \underline{n} = \Phi_\theta \times \Phi_z = \begin{vmatrix} i & j & k \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{vmatrix} = \langle \cos\theta, \sin\theta, 0 \rangle$

$$\int_0^3 \int_0^{2\pi} \langle \cos\theta, \sin\theta, -z \rangle \cdot \langle \cos\theta, \sin\theta, 0 \rangle\, d\theta\, dz = \int_0^3 \int_0^{2\pi} 1\, d\theta\, dz = [\theta]_0^{2\pi} = 2\pi$$

$[2\pi z]_0^3 = 6\pi$ so $\iint_{\partial W} \underline{F} \cdot d\underline{S} = 6\pi - 3\pi = 3\pi$.