Math 341 Advanced Calculus Spring 13 Final

Name: ______________________________

- Do any 7 of the following 10 questions.
- You may use two letter size pages of notes, you may write on both sides. You do not need a calculator.

(1) Give the definition of what it means for a function $f\colon \mathbb{R} \to \mathbb{R}$ to be continuous at a point $x_0 \in \mathbb{R}$.

(2) Use the definition of convergence to show that the sequence given by $s_n = \dfrac{3}{2n+1}$ converges.

(3) Use the definition of convergence to show that the sequence given by $s_n = \dfrac{1}{n^2-2}$ converges.

(4) Is it possible to have an unbounded sequence (s_n) such that $\displaystyle\lim_{n\to\infty} \frac{s_n}{n} = 0$? Explain.

(5) Let (a_n) and (b_n) be sequences of positive terms such that $\frac{a_n}{b_n}$ diverges to $+\infty$. Prove that if (a_n) is bounded then $b_n \to 0$.

(6) Consider the sequence given by $a_1 = 2$ and $a_{n+1} = \frac{1}{3}(a_n + 2)$. Show this sequence is convergent, for example by using monotone convergence, or by any other method.

(7) Let $E \subset \mathbb{R}$ be bounded. Show that the boundary of E is non-empty and bounded.

(8) Let $f\colon (0,\infty) \to \mathbb{R}$ be given by $f(x) = x/(x+1)$. Find the limit $\displaystyle\lim_{x\to 1} f(x)$, and use the $(\epsilon\text{-}\delta)$-definition of the limit to prove your answer is correct.

(9) Use the $(\epsilon\text{-}\delta)$-definition of continuity to show that the function $f\colon (0,\infty) \to \mathbb{R}$ given by $f(x) = \sqrt{x}$ is continuous at $x = 2$.

(10) Use the definition of the derivative to find the derivative of the function $f\colon (0,\infty) \to \mathbb{R}$ given by $f(x) = 1/x$ at $x = 1$.

Final	
Overall	

Solutions

Q1 $f:\mathbb{R}\to\mathbb{R}$ is continuous at $x_0\in\mathbb{R}$ if for all $\epsilon>0$ there is a $\delta>0$ s.t. if $|x-x_0|<\delta$ then $|f(x)-f(x_0)|<\epsilon$.

Q2 show $s_n=\frac{3}{2n+1}\to 0$ as $n\to\infty$.

[rough work: want $\left|\frac{3}{2n+1}-0\right|<\epsilon$ $\quad \frac{3}{2n+1}<\frac{3}{2n}<\frac{3}{n}=\epsilon$, so can choose $n=\frac{3}{\epsilon}$.]

given $\epsilon>0$, choose $N=\frac{3}{\epsilon}$, then $\left|\frac{3}{2n+1}-0\right|<\frac{3}{2n}<\frac{3}{N}=\epsilon$ for all $n\geq N$, as required.

Q3 show $s_n=\frac{1}{n^2-2}\to 0$ as $n\to\infty$.

[rough work: $\left|\frac{1}{n^2-2}-0\right|=\left|\frac{1}{(n-\sqrt{2})(n+\sqrt{2})}\right|\leq\frac{1}{n+\sqrt{2}}<\frac{1}{n}=\epsilon$, so choose $N=\max\{3,\frac{1}{\epsilon}\}$.] $\quad n-\sqrt{2}>1$ for all $n\geq 3$.

given $\epsilon>0$, choose $N=\max\{3,\frac{1}{\epsilon}\}$, then $\left|\frac{1}{n^2-2}\right|=\frac{1}{|n-\sqrt{2}|\,|n+\sqrt{2}|}\leq\frac{1}{n-\sqrt{2}}$ for $n\geq 3$

$\frac{1}{n}\leq\frac{1}{N}=\epsilon$, ~~as requ~~ for all $n\geq N$, as required.

Q4 Yes. Example: $s_n=\sqrt{n}$, $s_n\to\infty$ as $n\to\infty$, but $\frac{s_n}{n}=\frac{\sqrt{n}}{n}=\frac{1}{\sqrt{n}}\to 0$ as $n\to\infty$.

Q5 (a_n) bounded means there is an $M_1>0$ s.t. $|a_n|\leq M_1$ for all n.

$\left(\frac{a_n}{b_n}\right)$ diverges to $+\infty$ means for all $M>0$ there is an N s.t. $\frac{a_n}{b_n}\geq M$ for all $n\geq N$.

so for all $M>0$, there is an N s.t. for all $n\geq N$ $\quad b_n<Ma_n\leq M.M_1$ ($=\epsilon$ want).

so ~~choose~~ given $\epsilon>0$, choose $M=M_1/\epsilon$, then there is an N s.t. for all $n\geq N$,

$\frac{a_n}{b_n}\geq\frac{M_1}{\epsilon}\Rightarrow b_n\leq\frac{\epsilon a_n}{M_1}\leq\epsilon$ as required.

Q6 Show (a_n) decreasing: induction, base case $a_1=2$, $a_2=\frac{4}{3}$, so $a_1>a_2$. induction step: spse $a_n>a_{n+1}$, then $a_n+2>a_{n+1}+2$, so $\frac{1}{3}(a_n+2)>\frac{1}{3}(a_{n+1}+2)\Rightarrow a_{n+1}>a_{n+2}$.

All terms positive, so bounded below by 0, so (a_n) converges by the monotone convergence theorem.

Q7 $E \subset \mathbb{R}$ bounded means there is an M s.t. $|e| \le M$ for all $e \in E$.
set $x = \sup(E)$, finite number as M is an upper bound. claim: $x \in$ boundary.
consider $(x-c, x+c)$, $x+c \notin E$ as every $e \in E$ satisfies $e \le x$. If no point of $(x-c, x+c)$ lies in E then $x-c$ is an upper bound # x is least upper bound, so $(x-c, x+c)$ contains points of E and $\mathbb{R} \setminus E$ for all $c > 0$, so x is in boundary(E), so not empty. If $x \in$ boundary(E) then for all $c>0$ $(x-c, x+c) \cap E$ is not empty, in particular, for $c = 1$, so there is an e s.t. $|x-e| < 1$, as $|e| \le M$ triangle inequality $|x| - |e| \le 1 \Rightarrow |x| \le 1 + |e| < 1 + M$, so boundary$(E)$ is bounded.

Q8 $\lim_{x \to 1} \frac{x}{x+1} = \frac{1}{2}$. Given $\epsilon > 0$ want to find $\delta > 0$ s.t. $|x-1| < \delta \Rightarrow |f(x) - \frac{1}{2}| < \epsilon$.

[rough work: $\left|\frac{x}{x+1} - \frac{1}{2}\right| = \left|\frac{2x-x-1}{2(x+1)}\right| = \left|\frac{x-1}{2(x+1)}\right|$ know $|x-1| < \delta$ need lower bound on $2|x+1|$.
$-\delta < x-1 < \delta \Rightarrow -\delta + 2 < x+1 < \delta + 2$ so if $|\delta| < 1$ $|x+1| > 1$.]

given $\epsilon > 0$ choose $\delta = \min\{\epsilon, 1\}$, then if $|x-1| < \delta$,

$$\left|f(x) - \tfrac{1}{2}\right| = \left|\frac{x}{x+1} - \frac{1}{2}\right| = \frac{|x-1|}{2|x+1|} \quad \left.\begin{matrix} |x-1| < \delta \\ |x+1| > 1 \text{ as } |\delta| < 1 \end{matrix}\right\} \Rightarrow \frac{|x-1|}{2|x+1|} < \frac{\delta}{2} < \delta = \epsilon,$$

as required.

Q9 show $f : (0, \infty) \to \mathbb{R}$, $f(x) = \sqrt{x}$ cts at $x = 2$.

[rough work: given $|x-2| < \delta$, bound $|f(x) - \sqrt{2}| = \frac{|\sqrt{x} - \sqrt{2}| \cdot |\sqrt{x} + \sqrt{2}|}{|\sqrt{x} + \sqrt{2}|} = \frac{|x-2|}{|\sqrt{x} + \sqrt{2}|}$.
if $\delta \le 1$ then $-\delta \le x-2 \le \delta \Rightarrow 1 \le x \le 3$
$\Rightarrow x \ge 1 \Rightarrow \sqrt{x} \ge 1$ so $\sqrt{x} + \sqrt{2} \ge 1 + \sqrt{2} \ge \sqrt{2}$.]

given $\epsilon > 0$ choose $\delta = \epsilon$, then if $|x-2| < \delta$, consider $|f(x) - \sqrt{2}| = \frac{|x-2|}{|\sqrt{x} + \sqrt{2}|} \le \frac{\delta}{\sqrt{2}}$
$\le \delta = \epsilon$, as required.

Q10 $f'(x) = \lim_{x \to x_0} \frac{f(x_0) - f(x)}{x_0 - x} = \frac{\frac{1}{x_0} - \frac{1}{x}}{x_0 - x} \overset{x_0 = 1}{=} \frac{1 - \frac{1}{x}}{1 - x} = \frac{x-1}{x(1-x)} = -\frac{1}{x}$

so $\lim_{x \to 1} \frac{f(1) - f(x)}{1 - x} = \lim_{x \to 1} -\frac{1}{x} = -1$.