Sample midterm 3 solutions

Q1 a) $\det(A-\lambda I) = (3-\lambda)(-3-\lambda)+5 = \lambda^2-4=0 = (\lambda+2)(\lambda-2)=0$

$\lambda_1 = 2 \quad \lambda_2 = -2$.

b) find eigenvectors: $\lambda=2$: $\begin{bmatrix} 1 & 1 \\ -5 & -5 \end{bmatrix}$ $\quad v_1 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$. $\quad D = \begin{bmatrix} 2 & 0 \\ 0 & -2 \end{bmatrix}$

$\lambda_2=-2$: $\begin{bmatrix} 5 & 1 \\ -5 & -1 \end{bmatrix}$ $\quad v_2 = \begin{bmatrix} 1 \\ -5 \end{bmatrix}$ $\quad P = \begin{bmatrix} 1 & 1 \\ -1 & -5 \end{bmatrix}$

c) $A^k = (PDP^{-1})^k = PD^kP^{-1} = \begin{bmatrix} 1 & 1 \\ -1 & -5 \end{bmatrix}\begin{bmatrix} 2^k & 0 \\ 0 & (-2)^k \end{bmatrix}\begin{bmatrix} -5 & -1 \\ 1 & 1 \end{bmatrix}\cdot\frac{1}{-4}$

d) $2^k, (-2)^k$

Q2 a) LT.

$$\begin{bmatrix} x-2y \\ x+2y \end{bmatrix} = \begin{bmatrix} 1 & -2 \\ 1 & 2 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}$$

b)
$$\begin{tikzcd}
\mathbb{R}^2_T \arrow[r, "LT"] & \mathbb{R}^2_T \\
\mathbb{R}^2_S \arrow[r, "L_S"] \arrow[u, leftrightarrow] & \mathbb{R}^2_S \arrow[u, leftrightarrow]
\end{tikzcd}$$

$L_S = P^{-1}LP \qquad P = \begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix}$.

$$L_S = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 1 & -2 \\ 1 & 2 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & -2 \\ 2 & 0 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix} = \begin{bmatrix} 3 & -2 \\ 2 & 0 \end{bmatrix}$$

c) $\text{rank}(L) = 2$
$\text{nullity}(L) = 0$

Q3 a) ± 1 for orthogonal matrices $Q^{-1} = Q^T \Rightarrow \det(Q) = \pm 1$, as $\det(Q^T) = \det(Q) = \frac{1}{\det(Q^T)}$

b) $\det[q_1+q_2, q_2+q_3, q_3+q_1] = \det[q_1, q_2+q_3, q_3+q_1] + \det[q_2, q_2+q_3, q_3+q_1]$

$= \det[q_1, q_2+q_3, q_3] + \det[q_2, q_3, q_3+q_1] = \det[q_1, q_2, q_3] + \det[q_2, q_3, q_1]$

$= \pm 2$

c) -2 (two dets have opposite sign)

Q4 a) $\deg \leq 2$ (each permutation can contain at most 2 x's).

b)
$$\det\begin{bmatrix} x & x & x & x \\ x & 1 & 0 & 0 \\ x & 0 & 1 & 0 \\ x & 0 & 0 & 1 \end{bmatrix} = \det\begin{bmatrix} x-3x^2 & 0 & 0 & 0 \\ x & 1 & 0 & 0 \\ x & 0 & 1 & 0 \\ x & 0 & 0 & 1 \end{bmatrix}$$

$x(1-3x)=0$

$x = 0, \tfrac{1}{3}$.

Q5 a) $\det(A) = 0\cdot1\cdot2\cdot3 = 0$

b) $S^{-1} = S^T$

c) $A = SDS^T \qquad D = \begin{bmatrix} 0 & & & 0 \\ & 1 & & \\ & & 2 & \\ 0 & & & 3 \end{bmatrix}$.

d) $A + I = SDS^T + I = SDS^T + SIS^T = S(D+I)S^T$.

Q6 a) $\lambda_1 = -1$, $\lambda_2 = 0$, $\lambda_3 = 0$

$$\lambda_1 = -1: \quad \begin{array}{ccc} x_1 & x_2 & x_3 \end{array} \quad \begin{bmatrix} 0 & 2 & 4 \\ 0 & \boxed{1} & 5 \\ 0 & 0 & \boxed{2} \end{bmatrix} \qquad x_3 = 0,\ x_2 = 0 \qquad v_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

$$\lambda_2 = 0: \quad \begin{bmatrix} \boxed{-1} & 2 & 4 \\ 0 & 0 & 5 \\ 0 & 0 & 1 \end{bmatrix} \rightsquigarrow \begin{array}{ccc} x_1 & x_2 & x_3 \end{array} \begin{bmatrix} \boxed{-1} & 2 & 4 \\ 0 & 0 & \boxed{5} \\ 0 & 0 & 0 \end{bmatrix}$$

$x_3 = 0$, $x_2 = t$, $-x_1 + 2t = 0$ $\qquad v_2 = \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}$

$$\lambda_2 = 1: \quad \begin{bmatrix} \boxed{-2} & 2 & 4 \\ 0 & \boxed{-1} & 5 \\ 0 & 0 & 0 \end{bmatrix}$$

$x_3 = t$

$-x_2 + 5t = 0 \quad x_2 = 5t$

$-2x_1 + 2x_2 + 4x_3 = 0 \quad x_1 = 5t + 2t = 7t$ $\qquad v_3 = \begin{bmatrix} 7 \\ 5 \\ 1 \end{bmatrix}$

b) $D = \begin{bmatrix} -1 & & 0 \\ & 0 & \\ 0 & & 1 \end{bmatrix} \qquad A = SDS^{-1} \qquad A^{101} = SD^{101}S^{-1}$ but $D^{101} = \begin{bmatrix} -1^{101} & & 0 \\ & 0^{101} & \\ 0 & & 1^{101} \end{bmatrix} = D$.

so $A^{101} = SDS^{-1} = A$. $\qquad A^{100} = SD^{100}S^{-1}$

where $D^{100} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ so $A^{100} \neq I$, as D^{100} has rank 2

c) $A^k = \begin{bmatrix} -1^k & * & * \\ 0 & 0^k & * \\ 0 & 0 & 1^k \end{bmatrix}$. $\quad e^{At} = I + At + \frac{A^2t^2}{2!} + \frac{A^3t^3}{3!} + \cdots$

$$= \begin{bmatrix} e^{-1} & * & * \\ 0 & e^0 & * \\ 0 & 0 & e^1 \end{bmatrix}.$$

Q7 a) $Av_i = \lambda_i v_i \qquad v_i = \lambda_i A^{-1} v_i \qquad A^{-1} v_i = \frac{1}{\lambda_i} v_i$

so A^{-1} has same eigenvecs v_i with eigenvalues $\frac{1}{\lambda_i}$

b) $v = c_1 q_1 + c_2 q_2 + \cdots + c_n q_n$.

$v \cdot q_i = c_i q_i q_i = c_i$ so $c_i = v q_i$

c) $A^{-1} b = d_1 q_1 + \cdots + d_n q_n$

$b = A(d_1 q_1 + \cdots + d_n q_n) = \lambda_1 d_1 q_1 + \cdots + \lambda_n q_n d_n$.

$b \cdot q_i = \lambda_i d_i \cdot q_i q_i = \lambda_i d_i$ so $d_i = \frac{b q_i}{\lambda_i}$

Q8 a) $\begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix} \rightarrow \begin{bmatrix} \boxed{2} & 1 \\ 0 & \boxed{1/2} \end{bmatrix}$ 2 pivots $\Rightarrow$ basis.

b) $\begin{bmatrix} 1 \\ -1 \end{bmatrix} = c_1 \begin{bmatrix} 2 \\ 1 \end{bmatrix} + c_2 \begin{bmatrix} 1 \\ 1 \end{bmatrix} \qquad \left[\begin{array}{cc|c} 2 & 1 & 1 \\ 1 & 1 & -1 \end{array}\right] \rightarrow \left[\begin{array}{cc|c} 2 & 1 & 1 \\ 0 & 1/2 & -3/2 \end{array}\right]$

$c_2 = -3 \qquad c_1 = +2$

c) $\begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}^{-1} = \begin{bmatrix} 1 & -1 \\ -1 & 2 \end{bmatrix}$.

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Q9 $\quad \underline{q_1} = \frac{1}{\sqrt{6}}\begin{bmatrix}1\\-1\\2\end{bmatrix}.$

$\frac{3}{\sqrt{6}} \cdot = \frac{1}{2}.$

$$q_2 = \frac{v_2 - v_2 \cdot q_1 q_1}{\| \quad \|} = \frac{\begin{bmatrix}0\\-1\\1\end{bmatrix} - \frac{1}{\sqrt{6}} 3 \frac{1}{\sqrt{6}}\begin{bmatrix}1\\-1\\2\end{bmatrix}}{\| \quad \|} = \frac{\begin{bmatrix}-1/2\\-1/2\\0\end{bmatrix}}{\| - \|} = \frac{1}{\sqrt{2}}\begin{bmatrix}-1\\-1\\0\end{bmatrix}$$

$$q_3 = \frac{v_3 - (v_3 \cdot q_1) q_1 - (v_3 \cdot q_2) q_2}{\| \quad \|} = \begin{bmatrix}0\\1\\1\end{bmatrix} - \overbrace{\frac{1}{\sqrt{6}} 1 \frac{1}{\sqrt{6}}}^{1/6}\begin{bmatrix}1\\-1\\2\end{bmatrix} - \overbrace{\frac{1}{\sqrt{2}}(-1)\frac{1}{\sqrt{2}}}^{-1/2}\begin{bmatrix}-1\\-1\\0\end{bmatrix}$$

$$= \begin{bmatrix}-2/3\\2/3\\2/3\end{bmatrix} = \frac{1}{3}\begin{bmatrix}-2\\2\\2\end{bmatrix} = \frac{1}{\sqrt{3}}\begin{bmatrix}-1\\1\\1\end{bmatrix}$$

Q10 a) $\det(A - \lambda I)$ is a degree 3 polynomial so has a real root

b) every rotation preserves a direction

c) $\begin{bmatrix}0&1&0&0\\1&0&0&0\\0&0&0&1\\0&0&1&0\end{bmatrix}$

d) rotations in $\mathbb{R}^4$ don't need to preserve a direction.