Math 338 Linear Algebra Spring 13 Midterm 2b

Name: Solutions

- Do any 8 of the following 10 questions.
- You may use a calculator without symbolic algebra capabilities, but no notes.

1	20	
2	10	
3	10	
4	10	
5	10	
6	10	
7	10	
8	10	
9	10	
10	10	
	80	

Midterm 2	
Overall	

(1) Let S be the following set of vectors.

$$S = \{ \begin{bmatrix} -1 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}, \begin{bmatrix} -2 \\ 0 \\ 4 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \}.$$

(a) Find a subset of S which is a basis for the span of S.
(b) Do the vectors in S span $\mathbb{R}^3$?

$$\begin{bmatrix} -1 & 1 & -2 & 1 \\ 2 & 2 & 0 & 1 \\ 3 & -1 & 4 & 1 \end{bmatrix} \rightsquigarrow \begin{bmatrix} -1 & 1 & -2 & 1 \\ 0 & 4 & -4 & 3 \\ 0 & 2 & -2 & 4 \end{bmatrix} \rightsquigarrow \begin{bmatrix} \boxed{-1} & 1 & -2 & 1 \\ 0 & \boxed{4} & -4 & 3 \\ 0 & 0 & 0 & \boxed{5/2} \end{bmatrix}$$

a) $\{ \begin{bmatrix} -1 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \}$

b) yes.

(2) (a) Write down a spanning set for $\mathbb{R}^4$ which is not a basis.
(b) Write down a linearly independent set in $\mathbb{R}^4$ which is not a basis.

a) $\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \right\}$

b) $\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \right\}.$

(3) If $\overset{A}{\cancel{S}}$ is a 4×5 matrix, and the rows of A add up to the zero vector, what can you say about the rank and nullity of A?

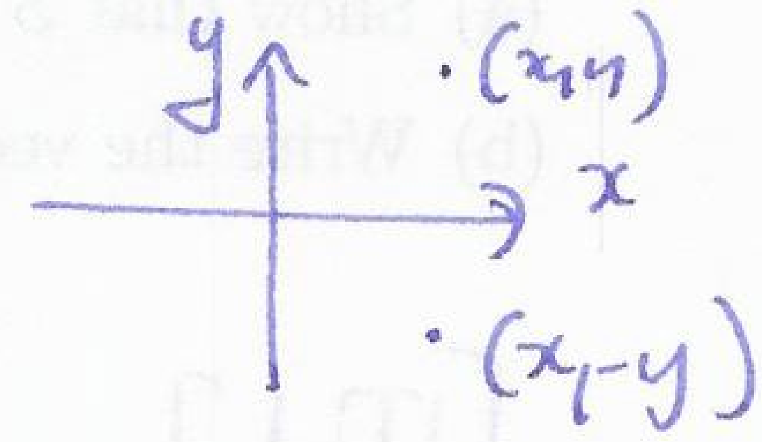
$0 \leq \text{rank}(A) \leq 3$

$2 \leq \text{nullity}(A) \leq 5$

If the rows of A add up to zero, then they are dependent, so when you row reduce, the bottom row will be zero, so there are at most 3 pivots, so $\text{rank}(A) \leq 3$

(4) (a) Write down a 2×2 matrix giving reflection in the x-axis.
(b) Write down a 2×2 matrix corresponding to a rotation by $\pi/2$.
(c) Use your answers to (a) and (b) to write down a product of matrices which gives reflection in the y-axis.

a) $\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$



b) $\begin{bmatrix} \cos(\pi/2) & -\sin(\pi/2) \\ \sin(\pi/2) & \cos(\pi/2) \end{bmatrix} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$

c) $\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}^{-1}$

(5) Let S be the set of vectors

$$S = \{\begin{bmatrix}1\\1\end{bmatrix}, \begin{bmatrix}1\\2\end{bmatrix}\}$$

(a) Show that S is a basis for $\mathbb{R}^2$.

(b) Write the vector $\begin{bmatrix}0\\1\end{bmatrix}$ as a linear combination of elements of S.

a) $\begin{bmatrix}1 & 1\\1 & 2\end{bmatrix} \rightsquigarrow \begin{bmatrix}\boxed{1} & 1\\0 & \boxed{1}\end{bmatrix}$ two pivots in $\mathbb{R}^2 \Rightarrow$ spans, contains 2 vectors, so is a basis. indep.

b) $c_1\begin{bmatrix}1\\1\end{bmatrix} + c_2\begin{bmatrix}1\\2\end{bmatrix} = \begin{bmatrix}0\\1\end{bmatrix}$.

$\left[\begin{array}{cc|c}1 & 1 & 0\\1 & 2 & 1\end{array}\right] \rightsquigarrow \left[\begin{array}{cc|c}1 & 1 & 0\\0 & 1 & 1\end{array}\right]$ $\quad c_2 = 1$

$c_1 + c_2 = 0 \quad c_1 = -1$

so $-1\begin{bmatrix}1\\1\end{bmatrix} + 1\begin{bmatrix}1\\2\end{bmatrix} = \begin{bmatrix}0\\1\end{bmatrix}$ ✓.

(6) Find a basis for the column space of the following matrix.

$$A = \begin{bmatrix} 1 & 0 & 2 \\ -1 & 1 & -3 \\ 3 & -1 & 7 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 2 \\ -1 & 1 & -3 \\ 3 & -1 & 7 \end{bmatrix} \rightsquigarrow \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & -1 \\ 0 & -1 & 1 \end{bmatrix} \rightsquigarrow \begin{bmatrix} \boxed{1} & 0 & 2 \\ 0 & \boxed{1} & -1 \\ 0 & 0 & 0 \end{bmatrix}$$

$\left\{ \begin{bmatrix} 1 \\ -1 \\ 3 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix} \right\}$ is a basis for col(A)

(7) Find a basis for the subspace of vectors in $\mathbb{R}^4$ perpendicular to the following vectors.

$$\left\{ \begin{bmatrix} 1 \\ 2 \\ 0 \\ 2 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ -1 \\ 1 \end{bmatrix} \right\}$$

solve $v_i \cdot x = 0$

$$\begin{bmatrix} 1 & 2 & 0 & 2 \\ 1 & 1 & -1 & 1 \end{bmatrix} \rightsquigarrow \begin{array}{cccc} x_1 & x_2 & x_3 & x_4 \\ \multicolumn{4}{c}{\begin{bmatrix} \boxed{1} & 2 & 0 & 2 \\ 0 & \boxed{-1} & -1 & -1 \end{bmatrix}} \\ & & s & t \end{array}$$

$x_2 = -s - t$

$x_1 = +2s + 2\cancel{t} - 2t$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = s \begin{bmatrix} +2 \\ -1 \\ 1 \\ 0 \end{bmatrix} + t \begin{bmatrix} \overset{0}{\cancel{4}} \\ -1 \\ 0 \\ 1 \end{bmatrix}$$

basis $\left\{ \begin{bmatrix} +2 \\ -1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} \overset{0}{\cancel{4}} \\ -1 \\ 0 \\ 1 \end{bmatrix} \right\}$

(8) This 3×4 matrix depends on c:

$$A = \begin{bmatrix} 1 & 2 & -1 & 2 \\ 2 & 5 & 0 & 3 \\ 0 & 1 & c & 0 \end{bmatrix}$$

For each c find a basis for the column space of A.

$$\rightsquigarrow \begin{bmatrix} 1 & 2 & -1 & 2 \\ 0 & 1 & 2 & -1 \\ 0 & 1 & c & 0 \end{bmatrix} \rightsquigarrow \begin{bmatrix} \boxed{1} & 2 & -1 & 2 \\ 0 & \boxed{1} & 2 & -1 \\ 0 & 0 & \boxed{c-2} & 1 \end{bmatrix} \qquad \begin{bmatrix} \boxed{1} & 2 & -1 & 2 \\ 0 & \boxed{1} & 2 & -1 \\ 0 & 0 & 0 & \boxed{1} \end{bmatrix}$$

$c \neq 2$ $\qquad\qquad$ $c = 2$

basis $\quad \left\{ \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 5 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ c \end{bmatrix} \right\} \qquad \left\{ \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 5 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 3 \\ 0 \end{bmatrix} \right\}.$

(9) This 3×4 matrix depends on c:

$$A = \begin{bmatrix} 1 & 2 & -1 & 2 \\ 2 & 5 & 0 & 3 \\ 0 & 1 & c & 0 \end{bmatrix}$$

For each c find a basis for the nullspace of A.

$c \neq 2$

$$\begin{array}{cccc} x_1 & x_2 & x_3 & x_4 \end{array}$$
$$\begin{bmatrix} \boxed{1} & 2 & -1 & 2 \\ 0 & \boxed{1} & 2 & -1 \\ 0 & 0 & \boxed{c-2} & 1 \end{bmatrix}$$
$$\qquad\qquad\qquad\qquad t$$

$x_4 = t \qquad (c-2)x_3 + t = 0$

$x_3 = \frac{-t}{c-2}$

$x_2 + 2\left(\frac{-t}{c-2}\right) - t = 0 \Rightarrow x_2 = t\left(1 + \frac{2}{c-2}\right)$

$x_1 + 2t\left(1 + \frac{2}{c-2}\right) + \frac{t}{c-2} + 2t = 0 \Rightarrow x_1 = t\left[-2 - \frac{1}{c-2} - 2 - \frac{4}{c-2}\right] = t\left(-4 - \frac{5}{c-2}\right)$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = t \begin{bmatrix} -4 - \frac{5}{c-2} \\ 1 + 2/c-2 \\ -1/c-2 \\ 1 \end{bmatrix}$$

so basis is $\left\{ \begin{bmatrix} -4 - 5/c-2 \\ 1 + 2/c-2 \\ -1/c-2 \\ 1 \end{bmatrix} \right\}$

$c = 2$

$$\begin{bmatrix} \boxed{1} & 2 & -1 & 2 \\ 0 & \boxed{1} & 2 & -1 \\ 0 & 0 & 0 & \boxed{1} \end{bmatrix}$$

$x_4 = 0$

$x_3 = t$

$x_2 = -2t$

$x_1 = 4t + t$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = t \begin{bmatrix} 5 \\ -2 \\ 1 \\ 6 \end{bmatrix}$$

basis $\left\{ \begin{bmatrix} -5 \\ -2 \\ 1 \\ 6 \end{bmatrix} \right\}$

(10) This 3×4 matrix depends on c:

$$A = \begin{bmatrix} 1 & 2 & -1 & 2 \\ 2 & 5 & 0 & 3 \\ 0 & 1 & c & 0 \end{bmatrix}$$

For each c find the complete solution x to $Ax = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$. (Hint: use your answer to the previous question.)

Already know kernel, so just need to find 1 solution.

$\underline{c \neq 2}$

$$\left[\begin{array}{cccc|c} 1 & 2 & -1 & 2 & 1 \\ 2 & 5 & 0 & 3 & -1 \\ 0 & 1 & c & 0 & 1 \end{array}\right] \rightsquigarrow \left[\begin{array}{cccc|c} 1 & 2 & -1 & 2 & 1 \\ 0 & 1 & 2 & -1 & -3 \\ 0 & 1 & c & 0 & 1 \end{array}\right] \rightsquigarrow \left[\begin{array}{cccc|c} \boxed{1} & 2 & -1 & 2 & 1 \\ 0 & \boxed{1} & 2 & -1 & -3 \\ 0 & 0 & \boxed{c-2} & \underset{t}{1} & 4 \end{array}\right]$$

can choose $x_4 = 0$, $x_3 = \frac{4}{c-2}$, $x_2 + \frac{8}{c-2} = -3 \Rightarrow x_2 = -3 - \frac{8}{c-2}$

$x_1 + 2\left(-3 - \frac{8}{c-2}\right) - \frac{4}{c-2} = 1 \Rightarrow x_1 = 7 + \frac{20}{c-2}$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 7 + \frac{20}{c-2} \\ -3 - 8/c-2 \\ 4/c-2 \\ 0 \end{bmatrix} + t \begin{bmatrix} -4 - 5/c-2 \\ 1 + 2/c-2 \\ -1/c-2 \\ 1 \end{bmatrix}$$

$\underline{c = 2}$

$$\left[\begin{array}{cccc|c} 1 & 2 & -1 & 2 & 1 \\ 2 & 5 & 0 & 3 & -1 \\ 0 & 1 & c & 0 & 1 \end{array}\right] \rightsquigarrow \left[\begin{array}{cccc|c} \boxed{1} & 2 & -1 & 2 & 1 \\ 0 & \boxed{1} & 2 & -1 & -3 \\ 0 & 0 & \underset{t}{0} & \boxed{1} & 4 \end{array}\right]$$

$x_4 = 4$

can choose $x_3 = 0$

$x_2 - 4 = -3 \Rightarrow x_2 = 1$

$x_1 + 2 + 8 = 1 \Rightarrow x_1 = -9$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} -9 \\ 1 \\ 0 \\ 4 \end{bmatrix} + t \begin{bmatrix} -5 \\ -2 \\ 1 \\ 0 \end{bmatrix}.$$