

Math 338 Linear Algebra Spring 13 Midterm 2a

Name: Solutions

- Do any 8 of the following 10 questions.
- You may use a calculator without symbolic algebra capabilities, but no notes.

1	20	
2	10	
3	10	
4	10	
5	10	
6	10	
7	10	
8	10	
9	10	
10	10	
	80	

Midterm 2	
Overall	

(1) Let S be the following set of vectors.

$$S = \left\{ \begin{bmatrix} -1 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}, \begin{bmatrix} -2 \\ 1 \\ 4 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \right\}.$$

- (a) Find a subset of S which is a basis for the span of S .
 (b) Do the vectors in S span \mathbb{R}^3 ?

$$\begin{bmatrix} -1 & 1 & -2 & 1 \\ 2 & 1 & 1 & 1 \\ 3 & -1 & 4 & 1 \end{bmatrix} \rightarrow \begin{bmatrix} -1 & 1 & -2 & 1 \\ 0 & 3 & -3 & 4 \\ 0 & 2 & -2 & 4 \end{bmatrix} \rightsquigarrow \begin{bmatrix} -1 & 1 & -2 & 1 \\ 0 & 3 & -3 & 4 \\ 0 & 0 & 0 & 4/3 \end{bmatrix}$$

a) $\left\{ \begin{bmatrix} -1 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \right\}$

b) yes

	Span of S
	\mathbb{R}^3

- (2) (a) Write down a spanning set for \mathbb{R}^4 which is not a basis.
 (b) Write down a linearly independent set in \mathbb{R}^4 which is not a basis.

a) $\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \right\}$

b) $\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} \right\}$

$$\det(A) \geq 0$$

$$\det(A) \geq 0$$

If the rows of A add up to zero then the rows are linearly dependent, so when you do row reduction, the bottom row will be zero, so there are at most 3 pivots.

4

 A

(3) If A is a 4×5 matrix, and the rows of A add up to the zero vector, what can you say about the rank and nullity of A ?

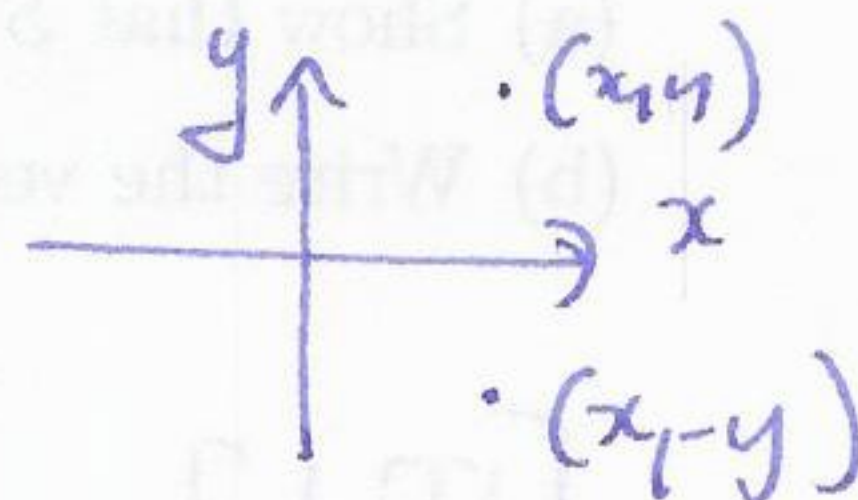
$$0 \leq \text{rank}(A) \leq 3$$

$$2 \leq \text{nullity}(A) \leq 5$$

If the rows of A add up to zero, then they are dependent, so when you row reduce, the bottom row will be zero, so there are at most 3 pivots,
so $\text{rank}(A) \leq 3$

- (4) (a) Write down a 2×2 matrix giving reflection in the x -axis.
 (b) Write down a 2×2 matrix corresponding to a rotation by $\pi/2$.
 (c) Use your answers to (a) and (b) to write down a product of matrices which gives reflection in the y -axis.

a)
$$\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$



b)
$$\begin{bmatrix} \cos(\pi/2) & -\sin(\pi/2) \\ \sin(\pi/2) & \cos(\pi/2) \end{bmatrix} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$$

c)
$$\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}^{-1}$$

(5) Let S be the set of vectors

$$S = \left\{ \begin{bmatrix} 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right\}$$

(a) Show that S is a basis for \mathbb{R}^2 .

(b) Write the vector $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ as a linear combination of elements of S .

a) $\begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix} \rightsquigarrow \begin{bmatrix} \boxed{2} & 1 \\ 0 & \boxed{1/2} \end{bmatrix}$ 2 pivots \Rightarrow rank = 2, so columns linearly independent, and span has dim 2.
 $\dim(\mathbb{R}^2) = 2 \Rightarrow$ basis.

b) $x_1 \begin{bmatrix} 2 \\ 1 \end{bmatrix} + x_2 \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

$$\left[\begin{array}{cc|c} 2 & 1 & 1 \\ 1 & 1 & 0 \end{array} \right] \rightsquigarrow \left[\begin{array}{cc|c} 2 & 1 & 1 \\ 0 & 1/2 & -1/2 \end{array} \right]$$

$$x_2 = -1$$

$$2x_1 - 1 = 1 \Rightarrow x_1 = 1$$

$$\begin{bmatrix} 2 \\ 1 \end{bmatrix} - \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

(6) Find a basis for the column space of the following matrix.

$$A = \begin{bmatrix} 1 & 0 & 2 \\ -1 & 1 & -3 \\ 3 & -1 & 7 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 2 \\ -1 & 1 & -3 \\ 3 & -1 & 7 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & -1 \\ 0 & -1 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix}$$

$\left\{ \begin{bmatrix} 1 \\ -1 \\ 3 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix} \right\}$ is a basis for $\text{col}(A)$

- (7) Find a basis for the subspace of vectors in \mathbb{R}^4 perpendicular to the following vectors.

solve $v_i \cdot x = 0$

$$\left\{ \begin{bmatrix} 1 \\ 2 \\ 0 \\ 2 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ -1 \\ 1 \end{bmatrix} \right\}$$

$$\begin{bmatrix} 1 & 2 & 0 & 2 \\ 1 & 1 & -1 & 1 \end{bmatrix} \rightsquigarrow \begin{bmatrix} x_1 & x_2 & x_3 & x_4 \\ \boxed{1} & 2 & 0 & 2 \\ 0 & \boxed{-1} & -1 & -1 \\ 0 & 0 & 0 & s & t \end{bmatrix} \rightsquigarrow \begin{bmatrix} s & 0 & 1 \\ 1 & -1 & 0 \\ 1 & 1 & 0 \end{bmatrix} \rightsquigarrow \begin{bmatrix} s & 0 & 1 \\ 2 & -1 & 1 \\ 1 & 1 & 0 \end{bmatrix}$$

$$x_2 = -s - t$$

$$x_1 = +2s + 2t - 2t$$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = s \begin{bmatrix} +2 \\ -1 \\ 1 \\ 0 \end{bmatrix} + t \begin{bmatrix} 0 \\ -1 \\ 0 \\ 1 \end{bmatrix}$$

basis $\left\{ \begin{bmatrix} +2 \\ -1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ -1 \\ 0 \\ 1 \end{bmatrix} \right\}$

(8) This 3×4 matrix depends on c :

$$A = \begin{bmatrix} 1 & 2 & -1 & 2 \\ 2 & 5 & 0 & 3 \\ 0 & 1 & c & 0 \end{bmatrix}$$

For each c find a basis for the column space of A .

$$\leadsto \begin{bmatrix} 1 & 2 & -1 & 2 \\ 0 & 1 & 2 & -1 \\ 0 & 1 & c & 0 \end{bmatrix} \leadsto \begin{bmatrix} 1 & 2 & -1 & 2 \\ 0 & 1 & 2 & -1 \\ 0 & 0 & c-2 & 1 \end{bmatrix} \quad c \neq 2$$

$$\begin{bmatrix} 1 & 2 & -1 & 2 \\ 0 & 1 & 2 & -1 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad c = 2$$

basis $\left\{ \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 5 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ c \end{bmatrix} \right\}$

$$\left\{ \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 5 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 3 \\ 0 \end{bmatrix} \right\}$$

$$\left\{ \begin{bmatrix} 2 \\ 5 \\ 1 \\ 0 \end{bmatrix} \right\}$$

$$\begin{aligned} x_1 &= 0 \\ x_2 &= t \\ x_3 &= -5t \\ x_4 &= t+5t \end{aligned}$$

$$\begin{bmatrix} 1 & 2 & -1 & 2 \\ 0 & 1 & 2 & -1 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad c=2$$

(9) This 3×4 matrix depends on c :

$$A = \begin{bmatrix} 1 & 2 & -1 & 2 \\ 2 & 5 & 0 & 3 \\ 0 & 1 & c & 0 \end{bmatrix}$$

For each c find a basis for the nullspace of A .

$$\underline{c \neq 2} \quad \begin{array}{cccc} x_1 & x_2 & x_3 & x_4 \\ \begin{bmatrix} 1 & 2 & -1 & 2 \\ 0 & 1 & 2 & -1 \\ 0 & 0 & c-2 & 1 \end{bmatrix} & & & \\ & & & t \end{array}$$

$$x_4 = t \quad (c-2)x_3 + t = 0$$

$$x_3 = \frac{-t}{c-2}$$

$$x_2 + 2\left(\frac{-t}{c-2}\right) - t = 0 \Rightarrow x_2 = t\left(1 + \frac{2}{c-2}\right)$$

$$x_1 + 2t\left(1 + \frac{2}{c-2}\right) + \frac{t}{c-2} + 2t = 0 \Rightarrow x_1 = t\left[-2 - \frac{1}{c-2} - 2 - \frac{4}{c-2}\right] = t\left(-4 - \frac{5}{c-2}\right)$$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = t \begin{bmatrix} -4 - \frac{5}{c-2} \\ 1 + \frac{2}{c-2} \\ -\frac{1}{c-2} \\ 1 \end{bmatrix}$$

so basis is $\left\{ \begin{bmatrix} -4 - \frac{5}{c-2} \\ 1 + \frac{2}{c-2} \\ -\frac{1}{c-2} \\ 1 \end{bmatrix} \right\}$

$$\underline{c = 2} \quad \begin{bmatrix} 1 & 2 & -1 & 2 \\ 0 & 1 & 2 & -1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$x_4 = 0$$

$$x_3 = t$$

$$x_2 = -2t$$

$$x_1 = 4t + t$$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = t \begin{bmatrix} 5 \\ -2 \\ 1 \\ 0 \end{bmatrix}$$

basis $\left\{ \begin{bmatrix} 5 \\ -2 \\ 1 \\ 0 \end{bmatrix} \right\}$

(10) This 3×4 matrix depends on c :

$$A = \begin{bmatrix} 1 & 2 & -1 & 2 \\ 2 & 5 & 0 & 3 \\ 0 & 1 & c & 0 \end{bmatrix}$$

For each c find the complete solution x to $Ax = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$. (Hint: use your answer to the previous question.)

Already knows kernel, so just need to find 1 solution.

$$\underline{c \neq 2} \quad \left[\begin{array}{cccc|c} 1 & 2 & -1 & 2 & 1 \\ 2 & 5 & 0 & 3 & -1 \\ 0 & 1 & c & 0 & 1 \end{array} \right] \rightsquigarrow \left[\begin{array}{cccc|c} 1 & 2 & -1 & 2 & 1 \\ 0 & 1 & 2 & -1 & -3 \\ 0 & 1 & c & 0 & 1 \end{array} \right] \rightsquigarrow \left[\begin{array}{cccc|c} 1 & 2 & -1 & 2 & 1 \\ 0 & 1 & 2 & -1 & -3 \\ 0 & 0 & c-2 & 1 & 4 \end{array} \right]$$

can choose $x_4 = 0$, $x_3 = \frac{4}{c-2}$, $x_2 + \frac{8}{c-2} = -3 \Rightarrow x_2 = -3 - \frac{8}{c-2}$

$$x_1 + 2\left(-3 - \frac{8}{c-2}\right) - \frac{4}{c-2} = 1 \Rightarrow x_1 = 7 + \frac{20}{c-2}$$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 7 + \frac{20}{c-2} \\ -3 - \frac{8}{c-2} \\ \frac{4}{c-2} \\ 0 \end{bmatrix} + t \begin{bmatrix} -4 - \frac{5}{c-2} \\ 1 + \frac{2}{c-2} \\ -1/c-2 \\ 1 \end{bmatrix}$$

$$\underline{c=2} \quad \left[\begin{array}{cccc|c} 1 & 2 & -1 & 2 & 1 \\ 2 & 5 & 0 & 3 & -1 \\ 0 & 1 & c & 0 & 1 \end{array} \right] \rightsquigarrow \left[\begin{array}{cccc|c} 1 & 2 & -1 & 2 & 1 \\ 0 & 1 & 2 & -1 & -3 \\ 0 & 0 & 0 & 1 & 4 \end{array} \right]$$

$$x_4 = 4$$

can choose $x_3 = 0$

$$x_2 - 4 = -3 \Rightarrow x_2 = 1$$

$$x_1 + 2 + 8 = 1 \Rightarrow x_1 = -9$$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} -9 \\ 1 \\ 0 \\ 4 \end{bmatrix} + t \begin{bmatrix} -5 \\ -2 \\ 1 \\ 0 \end{bmatrix}$$