Sample midterm 2   Sample solutions

Q1 a) $\underset{m\times n}{A} : \underset{n\times 1}{\mathbb{R}^n} \longrightarrow \underset{m\times 1}{\mathbb{R}^m}$ $m=3 \Rightarrow 0 \le \text{rank}(A) \le 3$

$Ax = \begin{bmatrix}1\\1\\1\end{bmatrix}$ no solutions $\Rightarrow$ $\text{rank}(A) < 3$.

$Ax = \begin{bmatrix}0\\1\\0\end{bmatrix}$ exactly one solution $\Rightarrow$ nullity $N(A) = 0$ and $\text{rank}(A) > 0$
i.e. $\text{rank}(A) = n$

so $1 \le \text{rank}(A) \le 2$ and so $n = 1$ or 2.

b) as $N(A) = 0$ $Ax = 0$ has unique solution $x = \begin{bmatrix}0\\0\\0\end{bmatrix}$.

c) $A = \begin{bmatrix}0\\1\\0\end{bmatrix}$.

Q2 a) $A = \begin{array}{c} \begin{array}{cccc} x_1 & x_2 & x_3 & x_4 \end{array} \\ \begin{bmatrix} 1 & 1 & 2 & 4 \\ 3 & c & 2 & 8 \\ 0 & 0 & 2 & 2 \end{bmatrix} \end{array} \rightsquigarrow \begin{bmatrix} \boxed{1} & 1 & 2 & 4 \\ 0 & \boxed{c-3} & -4 & -4 \\ 0 & 0 & \boxed{2} & 2 \end{bmatrix} \underset{c\neq 3.}{} \overset{c=3.}{\rightsquigarrow} \begin{bmatrix} \boxed{1} & 1 & 2 & 4 \\ 0 & c-3 & \boxed{-4} & -4 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

if $c \neq 3$ then the first 3 cols are a basis for $C(A)$
if $c = 3$ then cols 1 and 3 are a basis for $C(A)$.

b) $c \neq 3$: $x_4 = t$ $2x_3 = -2t$, $x_3 = -t$ $(c-3)x_2 = 0$, $x_2 = 0$ $x_1 = 2t - 4t = -2t$.

$\begin{bmatrix}x_1\\x_2\\x_3\\x_4\end{bmatrix} = t\begin{bmatrix}-2\\0\\-1\\1\end{bmatrix}$ basis for $N(A)$ $\left\{\begin{bmatrix}-2\\0\\-1\\1\end{bmatrix}\right\}$.

$c = 3$: $x_4 = t$ $x_3 = -t$ $x_2 = s$ $x_1 = -s - 2t$

$\begin{bmatrix}x_1\\x_2\\x_3\\x_4\end{bmatrix} = s\begin{bmatrix}-1\\1\\0\\0\end{bmatrix} + t\begin{bmatrix}-2\\0\\-1\\1\end{bmatrix}$ basis for $N(A)$: $\left\{\begin{bmatrix}-1\\1\\0\\0\end{bmatrix}, \begin{bmatrix}-2\\0\\-1\\1\end{bmatrix}\right\}$.

c) $c \neq 3$ a solution is $\begin{bmatrix}0\\1\\0\\0\end{bmatrix}$ so general sol^n is $\begin{bmatrix}0\\1\\0\\0\end{bmatrix} + t\begin{bmatrix}-2\\0\\-1\\1\end{bmatrix}$.

$c = 3$ a solution is $\begin{bmatrix}1\\0\\0\\0\end{bmatrix}$ so general sol^n is $\begin{bmatrix}1\\0\\0\\0\end{bmatrix} + s\begin{bmatrix}-1\\1\\0\\0\end{bmatrix} + t\begin{bmatrix}-2\\0\\-1\\1\end{bmatrix}$.

Q3 A 3×5 $m\times n$ $\text{rank}(A) \leq 3$ $2 \leq \underset{n-r}{\underset{\shortparallel}{\dim N(A)}} \leq 5$

Q4 cols 1 ~~and~~, 4, 3 form a basis for $C(A)$

col 2 is a multiple of col 1
col 3 is zero.

Q5 space of upper triangular matrices.

Q6 a) $\{v_1, \dots, v_d\}$ $d \geq n+1$

b) $\{v_1, \dots, v_d\}$ $d \leq n-1$

Q7
$$\begin{bmatrix} 2 & -1 & 0 & 0 \\ 1 & 2 & 5 & 3 \\ -3 & 2 & 1 & 3 \end{bmatrix} \rightsquigarrow \begin{bmatrix} 2 & -1 & 0 & 0 \\ 0 & 5/2 & 5 & 3 \\ 0 & 1/2 & 1 & 3 \end{bmatrix} \rightsquigarrow \begin{bmatrix} \boxed{2} & -1 & 0 & 0 \\ 0 & \boxed{5/2} & 5 & 3 \\ 0 & 0 & 0 & \boxed{12/5} \end{bmatrix}.$$

a) $\left\{ \begin{bmatrix} 2 \\ 1 \\ -3 \end{bmatrix}, \begin{bmatrix} -1 \\ 2 \\ 2 \end{bmatrix}, \begin{bmatrix} 0 \\ 3 \\ 3 \end{bmatrix} \right\}$ is a basis.   b) yes.

Q8 let $W = \{ x \in \mathbb{R}^n \mid x \cdot v = 0 \}$. check W is a vector space.

sums: if $x, y \in W$ then $x \cdot v = 0$ $y \cdot v = 0$ so $(x+y) \cdot v = x \cdot v + y \cdot v = 0 + 0 = 0$. ✓.

scalar mult: $(cx) \cdot v = c(x \cdot v) = c0 = 0$ ✓.

so W is a vector space.

Q9 a)
$$\begin{bmatrix} 1 & 1 & 0 & -2 & 3 \\ 0 & -1 & 1 & 3 & 1 \\ 2 & -1 & 3 & 5 & 9 \end{bmatrix} \rightsquigarrow \begin{bmatrix} 1 & 1 & 0 & -2 & 3 \\ 0 & -1 & 1 & 3 & 1 \\ 0 & -3 & 3 & 9 & 3 \end{bmatrix} \rightsquigarrow \begin{array}{c} \begin{matrix} x_1 & x_2 & x_3 & x_4 & x_5 \end{matrix} \\ \begin{bmatrix} \boxed{1} & 1 & 0 & -2 & 3 \\ 0 & \boxed{-1} & 1 & 3 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \end{array}.$$

b) rank $= 2$ nullity $= 3$.

c) $\left\{ \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} \right\}$.

d) $x_5 = a$ $x_4 = b$ $x_3 = c$ $x_2 = -c - 3b - a$ $x_1 = c + 3b + a + 2b - 3a$
$= c + 5b + 2a$.

so $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = c \begin{bmatrix} 1 \\ -1 \\ 1 \\ 0 \\ 0 \end{bmatrix} + b \begin{bmatrix} 5 \\ -3 \\ 0 \\ 1 \\ 0 \end{bmatrix} + a \begin{bmatrix} 2 \\ -1 \\ 0 \\ 0 \\ 1 \end{bmatrix}$ basis $N(A)$: $\left\{ \begin{bmatrix} 1 \\ -1 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 5 \\ -3 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ -1 \\ 0 \\ 0 \\ 1 \end{bmatrix} \right\}$.

Q10 a) rotation matrix: $\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$ $\theta = \pi/6$: $\begin{bmatrix} \cos\pi/6 & -\sin\pi/6 \\ \sin\pi/6 & \cos\pi/6 \end{bmatrix}$.

$$= \begin{bmatrix} \sqrt{3}/2 & -1/2 \\ 1/2 & \sqrt{3}/2 \end{bmatrix}.$$

b) $\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$

c) $\begin{bmatrix} \sqrt{3}/2 & -1/2 \\ 1/2 & \sqrt{3}/2 \end{bmatrix} \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \sqrt{3}/2 & 1/2 \\ -1/2 & \sqrt{3}/2 \end{bmatrix}$

$$\begin{bmatrix} -\sqrt{3}/2 & -1/2 \\ -1/2 & \sqrt{3}/2 \end{bmatrix} \begin{bmatrix} \sqrt{3}/2 & 1/2 \\ -1/2 & \sqrt{3}/2 \end{bmatrix} = \begin{bmatrix} -\frac{3}{4} + \frac{1}{4} & -\frac{\sqrt{3}}{4} - \frac{\sqrt{3}}{4} \\ -\frac{\sqrt{3}}{4} - \frac{\sqrt{3}}{4} & -\frac{1}{4} + \frac{3}{4} \end{bmatrix} = \begin{bmatrix} 1/2 & -\sqrt{3}/2 \\ -\frac{\sqrt{3}}{2} & 1/2 \end{bmatrix}.$$