Math 338 Linear Algebra Spring 13 Midterm 1b

Name: Solutions

- You must do Q1. Do any 7 of the following 9 questions.
- You may use a calculator without symbolic algebra capabilities, but no notes.

1	20	
2	10	
3	10	
4	10	
5	10	
6	10	
7	10	
8	10	
9	10	
10	10	
	90	

Midterm 1	
Overall	

(1) Find all solutions to the following set of linear equations.

$$\begin{aligned} x_1 - x_2 + 2x_3 + 3x_4 &= 4 \\ 2x_1 + x_2 + x_3 + 6x_4 &= 12 \\ -x_1 + 3x_2 - 8x_3 - 5x_4 &= 6 \end{aligned}$$

$$\left[\begin{array}{cccc|c} 1 & -1 & 2 & 3 & 4 \\ 2 & 1 & 1 & 6 & 12 \\ -1 & 3 & -8 & -5 & 6 \end{array}\right]$$

$$\left[\begin{array}{cccc|c} 1 & -1 & 2 & 3 & 4 \\ 0 & 3 & -3 & 0 & 4 \\ 0 & 2 & -6 & -2 & 10 \end{array}\right]$$

$$\left[\begin{array}{cccc|c} \boxed{1} & -1 & 2 & 3 & 4 \\ 0 & \boxed{3} & -3 & 0 & 4 \\ 0 & 0 & \boxed{-4} & -2 & 10 - 8/3 \end{array}\right]$$

$$x_4 = t$$

$$-4x_3 = \frac{22}{3} + 2x_4$$

$$x_3 = \frac{-11}{6} - \frac{1}{2}t$$

$$3x_2 = 4 + 3x_3$$

$$= 4 - \frac{11}{2} - \frac{3}{2}t$$

$$x_2 = -\frac{1}{2} - \frac{1}{2}t$$

$$x_1 = 4 + x_2 - 2x_3 - 3x_4$$

$$= 4 - 1/2 + 11/3$$

$$-1/2\,t + t - 3t$$

$$= \frac{24-3+22}{6} - \frac{5}{2}t = \frac{43}{6} - \frac{5}{2}t$$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 43/6 \\ -1/2 \\ -11/6 \\ 0 \end{bmatrix} + t \begin{bmatrix} -5/2 \\ -1/2 \\ -1/2 \\ 1 \end{bmatrix}$$

(2) Suppose A and B are invertible matrices.

(a) Is $A^{-1}BA$ invertible? If so, write down the inverse.

(b) Is $A + B$ invertible? Explain why or give a counterexample.

a) $(A^{-1}BA)^{-1} = A^{-1}B^{-1}A$

b) no. $A = I$, $B = -I$ $A+B = I - I = 0$ not invertible.

(3) Give an example of a system of three equations in two unknowns which is inconsistent.

$x+y=0$

$x+y=1$

$x+y=2$

(4) (a) If a system has three equations and four unknowns can it be inconsistent? Give an example or justify your answer.
(b) If a system has three equations and four unknowns can there be a unique solution? Justify your answer.

a) yes

$$x_1 + x_2 + x_3 + x_4 = 0$$
$$x_1 + x_2 + x_3 + x_4 = 1$$
$$x_1 + x_2 + x_3 + x_4 = 2$$

b) no. if there are solutions, there are at most 3 pivots, so there is a free variable, so there are infinitely many solutions.

(5) Consider the matrix

$$L = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ -2 & 0 & 1 \end{bmatrix}$$

(a) Describe in words the row operations corresponding to left multiplication by L.

(b) Write down L^{-1}.

a) ① row ① stays the same

② + 2① row ② has 2 copies of ① added to it

③ − 2① row ③ has 2 copies of ① subtracted from it.

b) $L^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 2 & 0 & 1 \end{bmatrix}$

(6) Use row operations to find the row echelon form the following matrix, writing out clearly what row operations you used.

$$A = \begin{bmatrix} 1 & -1 & 2 \\ 2 & 0 & 5 \\ -1 & 3 & 0 \end{bmatrix}$$

$$\begin{array}{c} ① \\ ② - 2① \\ ③ + ① \end{array} \begin{bmatrix} 1 & -1 & 2 \\ 0 & 2 & 1 \\ 0 & 2 & 2 \end{bmatrix}$$

$$\begin{array}{c} ① \\ ② \\ ③ - ② \end{array} \begin{bmatrix} 1 & -1 & 2 \\ 0 & 2 & 1 \\ 0 & 0 & +1 \end{bmatrix}$$

(7) Find the LU factorization for the matrix A in the previous question. Hint: use your answer to the previous question.

$$L_2 L_1 A = U \quad \Rightarrow \quad A = \underbrace{L_1^{-1} L_2^{-1}}_{L} U$$

where

$$L_1 = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} \qquad L_1^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix}$$

$$L_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix} \qquad L_2^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & +1 & 1 \end{bmatrix}$$

$$L = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & +1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ -1 & +1 & 1 \end{bmatrix}$$

(9) Find a 3×3 matrix A such that $A \neq I$ but $A^2 = I$.

$$\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

(8) Are the following vectors linearly independent?

$$\{[1,-1,2],[2,0,5],[-1,3,0]\}$$

Hint: use your solution to Q6 or Q7.

$$\begin{bmatrix} 1 & -1 & 2 \\ 2 & 0 & 5 \\ -1 & 3 & 0 \end{bmatrix} \underset{\text{ref}}{\rightsquigarrow} \begin{bmatrix} 1 & -1 & 2 \\ 0 & 2 & 1 \\ 0 & 0 & -1 \end{bmatrix}$$

3 pivots $\Rightarrow$ linearly independent

(10) Use row operations to find the inverse of the following matrix.

$$A = \begin{bmatrix} 1 & 2 & -1 \\ 1 & 4 & -2 \\ 1 & 2 & -2 \end{bmatrix}$$

Check your answer.

$$\left[\begin{array}{ccc|ccc} 1 & 2 & -1 & 1 & 0 & 0 \\ 1 & 4 & -2 & 0 & 1 & 0 \\ 1 & 2 & -2 & 0 & 0 & 1 \end{array}\right]$$

$$\left[\begin{array}{ccc|ccc} 1 & 2 & -1 & 1 & 0 & 0 \\ 0 & 2 & -1 & -1 & 1 & 0 \\ 0 & 0 & -1 & -1 & 0 & 1 \end{array}\right]$$

$$\left[\begin{array}{ccc|ccc} 1 & 2 & 0 & 2 & 0 & -1 \\ 0 & 2 & 0 & 0 & 1 & -1 \\ 0 & 0 & -1 & -1 & 0 & 1 \end{array}\right]$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 2 & -1 & 0 \\ 0 & 2 & 0 & 0 & 1 & -1 \\ 0 & 0 & -1 & -1 & 0 & 1 \end{array}\right]$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 2 & -1 & 0 \\ 0 & 1 & 0 & 0 & 1/2 & -1/2 \\ 0 & 0 & 1 & 1 & 0 & -1 \end{array}\right]$$

Check

$$\begin{bmatrix} 2 & -1 & 0 \\ 0 & 1/2 & -1/2 \\ 1 & 0 & -1 \end{bmatrix} \begin{bmatrix} 1 & 2 & -1 \\ 1 & 4 & -2 \\ 1 & 2 & -2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$