Math 338 Linear Algebra Spring 13 Final b

Name: solutions

- Do any 8 of the following 10 questions.
- You may use a calculator without symbolic algebra capabilities, but no notes.

1	10	
2	10	
3	10	
4	10	
5	10	
6	10	
7	10	
8	10	
9	10	
10	10	
	80	

Final	
Overall	

(1) Find all solutions to the following system of linear equations.

$$\begin{aligned} x_1 + x_2 - 2x_3 - x_4 &= 0 \\ 3x_1 + 5x_2 - 4x_3 + 2x_4 &= 0 \\ 2x_1 - 3x_2 - 3x_3 + x_4 &= 0 \end{aligned}$$

$$\begin{bmatrix} 1 & 1 & -2 & -1 \\ 3 & 5 & -4 & 2 \\ 2 & -3 & -3 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & -2 & -1 \\ 0 & 2 & 2 & 5 \\ 0 & -5 & 1 & 3 \end{bmatrix} \sim \begin{bmatrix} \boxed{1} & 1 & -2 & -1 \\ 0 & \boxed{2} & 2 & 5 \\ 0 & 0 & \boxed{6} & 31/2 \end{bmatrix}$$

$$\begin{array}{cccc} & & & t \end{array}$$

$x_4 = t$, $\quad 6x_3 + \frac{31}{2}t = 0 \qquad x_3 = -\frac{31}{12}t$

$2x_2 + 2\left(\frac{-31}{12}t\right) + 5t = 0 \qquad x_2 = \frac{t}{2}\left(\frac{31}{6} - 5\right) = \frac{1}{12}t$

$x_1 + \frac{1}{12}t - 2\left(-\frac{31}{12}t\right) - t = 0 \qquad x_1 = t\left(-\frac{1}{12} - \frac{31}{6} + 1\right) = -\frac{51}{12}t$

$$\underline{x} = t \begin{bmatrix} -51 \\ 1 \\ -31 \\ 12 \end{bmatrix}$$

(2) (a) Write down a matrix for a linear transformation of $\mathbb{R}^2$ which rotates by $\pi/2$ anticlockwise about the origin.

(b) Write down a matrix for a linear transformation of $\mathbb{R}^2$ which halves lengths in the x-direction, and doubles lengths in the y-direction.

(c) Use your answers above to write down a matrix for the linear transformation of $\mathbb{R}^{\cancel{3}\,2}$ obtained by first halving lengths in the x-direction and doubling lengths in the y-direction, and then rotating by $\pi/2$.

a) $\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$ $\quad \theta = \frac{\pi}{2} \quad$ $\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$

b) $\begin{bmatrix} 1/2 & 0 \\ 0 & 2 \end{bmatrix}$

c) $$\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 1/2 & 0 \\ 0 & 2 \end{bmatrix} = \begin{bmatrix} 0 & -2 \\ 1/2 & 0 \end{bmatrix}$$

(3) (a) Write down a spanning set in $\mathbb{R}^4$ which is not a basis.
(b) Write down a basis for $\mathbb{R}^4$ which is orthogonal, but not orthonormal.
(c) Write down a set of four distinct vectors which span a three-dimensional subspace of $\mathbb{R}^4$.

a) $\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} \right\}.$

b) $\left\{ \begin{bmatrix} 2 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 2 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 2 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 2 \end{bmatrix} \right\}$

c) $\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \\ 0 \end{bmatrix} \right\}$

(4) Apply the Gram-Schmidt process to the following three vectors.

$$\mathbf{v}_1 = \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix}, \mathbf{v}_2 = \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix}, \mathbf{v}_3 = \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}.$$

What do you notice about $\mathbf{v}_3$? What does this tell you about the original three vectors?

$$\underline{q_1} = \frac{1}{\sqrt{6}} \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix}$$

$$q_2' = \underline{v_2} - (\underline{v_2} \cdot \underline{q_1})\underline{v_1} = \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix} - \overbrace{\frac{1}{\sqrt{6}}(3)\frac{1}{\sqrt{6}}}^{1/2} \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ -3/2 \\ 3/2 \end{bmatrix}$$

$$q_2 = \frac{1}{\sqrt{2}} \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix}$$

$$q_3' = \underline{v_3} - (v_3 \cdot q_1)\underline{q_1} - (\underline{v_3} \cdot \underline{q_2})\underline{q_2} = \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix} - \overbrace{\frac{1}{\sqrt{6}}(3)\frac{1}{\sqrt{6}}}^{1/2} \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix} - \overbrace{\frac{1}{\sqrt{2}}(-3)\frac{1}{\sqrt{2}}}^{-3/2} \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix}$$

$$= \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$\Rightarrow$ $\underline{v_3}$ lives in span of $\{\underline{v_1}, \underline{v_2}\}$

(5)

$$A = \begin{bmatrix} -5 & -8 \\ 4 & 7 \end{bmatrix}$$

(a) Find the eigenvalues of A.
(b) Find the eigenvectors for A.

a) $\begin{vmatrix} -5-\lambda & -8 \\ 4 & 7-\lambda \end{vmatrix} = (-5-\lambda)(7-\lambda)+32 = \lambda^2-2\lambda-3$
$= (\lambda-3)(\lambda+1)$

$\lambda = 3, -1$

b) $\lambda_1 = 3$: $\begin{bmatrix} -8 & -8 \\ 4 & 4 \end{bmatrix}$ $\quad \underline{v}_1 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$

$\lambda_2 = -1$ $\begin{bmatrix} -4 & -8 \\ 4 & 8 \end{bmatrix}$ $\quad \underline{v}_2 = \begin{bmatrix} 2 \\ -1 \end{bmatrix}$

(7) If A is a non-singular $n \times n$ matrix such that $A^{-1} = -A$, what can you say about the determinant of A? (Hint: there are two cases depending on whether n is odd or even.)

$\det(A^{-1}) = \frac{1}{\det(A)}$

$\det(-A) = \det(A)$ n even
$\qquad\quad -\det(A)$ n odd

n even: $\frac{1}{d} = d \Rightarrow d^2 = 1 \quad d = \pm 1$

n odd: $\frac{1}{d} = -d \Rightarrow d^2 = -1 \quad d = \pm i$

(8) Let A be a 4×5 matrix such that there are two different vectors x_1 and x_2 such that $Ax_1 = Ax_2$.

(a) What can you say about the kernel of A?

(b) What can you say about the column rank of A?

a) $x_1 - x_2 \neq 0$ lies in $\ker(A)$ so $1 \leq \dim(\ker(A)) \leq 5$

b) $0 \leq C(A) \leq 4$

(9) Let B be the basis for $\mathbb{R}^2$ given by

$$B = \left\{ \begin{bmatrix} 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \end{bmatrix} \right\}.$$

(a) Find a matrix which converts vectors written in the standard basis to vectors written with respect to the basis B.

(b) Use your answer to (a) to write $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ (in the standard basis) as a linear combination of vectors in B.

(c) Use your answer to (a) to write the matrix $\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$ with respect to the ~~matrix~~ basis B.

a) $\begin{bmatrix} 2 & 1 \\ 1 & -1 \end{bmatrix}^{-1} = -\frac{1}{3}\begin{bmatrix} -1 & -1 \\ -1 & 2 \end{bmatrix} = \frac{1}{3}\begin{bmatrix} 1 & 1 \\ 1 & -2 \end{bmatrix}$

b) $\frac{1}{3}\begin{bmatrix} 1 & 1 \\ 1 & -2 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \frac{1}{3}\begin{bmatrix} 2 \\ -1 \end{bmatrix}$ check: $\frac{2}{3}\begin{bmatrix} 2 \\ 1 \end{bmatrix} + -\frac{1}{3}\begin{bmatrix} 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

c)
$$\begin{tikzcd} \mathbb{R}^2_{std} \arrow[r, "{\left[\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right]}"] & \mathbb{R}^2_{std} \\ \mathbb{R}_B \arrow[u] \arrow[r] & \mathbb{R}_B \arrow[u] \end{tikzcd}$$

$$\frac{1}{3}\begin{bmatrix} 1 & 1 \\ 1 & -2 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 2 & 1 \\ 1 & -1 \end{bmatrix} = \frac{1}{3}\begin{bmatrix} 1 & 1 \\ 1 & -2 \end{bmatrix}\begin{bmatrix} 3 & 0 \\ 1 & -1 \end{bmatrix}$$

$$= \frac{1}{3}\begin{bmatrix} 4 & -1 \\ 1 & 2 \end{bmatrix}$$

(10) Let $A = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 4 & 2 \\ 0 & -2 & 0 \end{bmatrix}$.

(a) Find the eigenvalues and eigenvectors for A.
(b) Can you diagonalize A? Explain.

a) $\begin{vmatrix} 2-\lambda & 0 & 0 \\ 0 & 4-\lambda & 2 \\ 0 & -2 & -\lambda \end{vmatrix} = (2-\lambda)\left[(4-\lambda)(-\lambda)+4\right]$

$= (2-\lambda)(\lambda^2-4\lambda+4) = -(\lambda-2)^3.$

$\lambda = 2$ with multiplicity 3

b) $\begin{bmatrix} 0 & 0 & 0 \\ 0 & 2 & 2 \\ 0 & -2 & -2 \end{bmatrix} \rightsquigarrow \begin{bmatrix} 0 & \boxed{2} & 2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ $\quad \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}$

$x_3 = t \qquad 2x_2 + 2t = 0 \quad x_2 = -t \qquad x_1 = s.$

No - only tho eigenvectas.