Math 338 Linear Algebra Spring 13 Final a

Name: Solutions

- Do any 8 of the following 10 questions.
- You may use a calculator without symbolic algebra capabilities, but no notes.

1	10	
2	10	
3	10	
4	10	
5	10	
6	10	
7	10	
8	10	
9	10	
10	10	
	80	

Final	
Overall	

(1) Find all solutions to the following system of linear equations.

$$x_1 - x_2 + x_3 + 2x_4 = 0$$
$$3x_1 - x_2 - x_3 + 9x_4 = 0$$
$$2x_1 - x_2 + 6x_4 = 0$$

$$\begin{bmatrix} 1 & -1 & 1 & 2 \\ 3 & -1 & -1 & 9 \\ 2 & -1 & 0 & 6 \end{bmatrix} \rightsquigarrow \begin{bmatrix} 1 & -1 & 1 & 2 \\ 0 & 2 & -4 & 3 \\ 0 & 1 & -2 & 2 \end{bmatrix} \rightsquigarrow \begin{bmatrix} \boxed{1} & -1 & 1 & 2 \\ 0 & \boxed{2} & -4 & 3 \\ 0 & 0 & 0 & \boxed{1/2} \end{bmatrix}$$

$x_4 = 0$, $x_3 = t$, $2x_2 - 4x_3 + 3x_4 = 0$, $x_2 = 2t$

$x_1 - x_2 + x_3 + 2x_4 = 0$, $x_1 = 2t - t = t$

solutions: $t \begin{bmatrix} 1 \\ 2 \\ 1 \\ 0 \end{bmatrix}$

(2) (a) Write down a matrix for a linear transformation of $\mathbb{R}^2$ which rotates by $\pi/2$ anticlockwise about the origin.

(b) Write down a matrix for a linear transformation of $\mathbb{R}^2$ which doubles lengths in the x-direction, and halves lengths in the y-direction.

(c) Use your answers above to write down a matrix for the linear transformation of $\mathbb{R}^{\cancel{3}\,2}$ obtained by first doubling lengths in the x-direction and halving lengths in the y-direction, and then rotating by $\pi/2$.

a) $\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$ $\quad \theta = \frac{\pi}{2} \quad$ $\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$

b) $\begin{bmatrix} 2 & 0 \\ 0 & 1/2 \end{bmatrix}$

c) $\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 2 & 0 \\ 0 & 1/2 \end{bmatrix} = \begin{bmatrix} 0 & -1/2 \\ 2 & 0 \end{bmatrix}$

(3) (a) Write down a spanning set in $\mathbb{R}^4$ which is not a basis.
(b) Write down a basis for $\mathbb{R}^4$ which is orthogonal, but not orthonormal.
(c) Write down a set of four distinct vectors which span a three-dimensional subspace of $\mathbb{R}^4$.

a) $\left\{ \begin{bmatrix}1\\0\\0\\0\end{bmatrix}, \begin{bmatrix}0\\1\\0\\0\end{bmatrix}, \begin{bmatrix}0\\0\\1\\0\end{bmatrix}, \begin{bmatrix}0\\0\\0\\1\end{bmatrix}, \begin{bmatrix}1\\1\\1\\1\end{bmatrix} \right\}$

b) $\left\{ \begin{bmatrix}2\\0\\0\\0\end{bmatrix}, \begin{bmatrix}0\\2\\0\\0\end{bmatrix}, \begin{bmatrix}0\\0\\2\\0\end{bmatrix}, \begin{bmatrix}0\\0\\0\\2\end{bmatrix} \right\}$

c) $\left\{ \begin{bmatrix}1\\0\\0\\0\end{bmatrix}, \begin{bmatrix}0\\1\\0\\0\end{bmatrix}, \begin{bmatrix}0\\0\\1\\0\end{bmatrix}, \begin{bmatrix}1\\1\\1\\0\end{bmatrix} \right\}$

(4) Apply the Gram-Schmidt process to the following three vectors.

$$\mathbf{v}_1 = \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix}, \mathbf{v}_2 = \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix}, \mathbf{v}_3 = \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}.$$

What do you notice about $\mathbf{v}_3$? What does this tell you about the original three vectors?

$$\underline{q}_1 = \frac{1}{\sqrt{6}} \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix}$$

$$\underline{q}_2' = \underline{v}_2 - (\underline{v}_2 \cdot \underline{q}_1)\underline{q}_1 = \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix} - \overbrace{\frac{1}{\sqrt{6}}(3)\frac{1}{\sqrt{6}}}^{1/2} \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix} = \begin{bmatrix} 3/2 \\ 3/2 \\ 0 \end{bmatrix}.$$

$$\underline{q}_2 = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$$

$$\underline{q}_3' = v_3 - (v_3 \cdot q_1)q_1 - (v_3 \cdot q_2)q_2$$

$$= \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix} - \underbrace{\frac{1}{\sqrt{6}}(-3)\frac{1}{\sqrt{6}}}_{-1/2} \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix} - \underbrace{\frac{1}{\sqrt{2}}(3)\frac{1}{\sqrt{2}}}_{3/2} \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$q_1, q_2 \neq 0$

$\underline{q}_3 = \underline{0} \Rightarrow \underline{v}_1, \underline{v}_2$ independent but $\underline{v}_3 \in \text{span}\{\underline{v}_1, \underline{v}_2\}$.

(5)

$$A = \begin{bmatrix} 7 & 4 \\ -8 & -5 \end{bmatrix}$$

(a) Find the eigenvalues of A.
(b) Find the eigenvectors for A.

a) $\begin{vmatrix} 7-\lambda & 4 \\ -8 & -5-\lambda \end{vmatrix} = (7-\lambda)(-5-\lambda) + 32 = \lambda^2 - 2\lambda - 3.$

$= (\lambda-3)(\lambda+1)$

$\lambda = 3, -1.$

b) $\lambda_1 = 3$: $\begin{bmatrix} 4 & 4 \\ -8 & -3 \end{bmatrix} \Rightarrow \underline{v}_1 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}.$

$\lambda_2 = -1$: $\begin{bmatrix} 8 & 4 \\ -8 & -4 \end{bmatrix} \Rightarrow \underline{v}_2 = \begin{bmatrix} 1 \\ -2 \end{bmatrix}.$

(7) If A is a non-singular $n \times n$ matrix such that $A^{-1} = -A$, what can you say about the determinant of A? (Hint: there are two cases depending on whether n is odd or even.)

$A^{-1} = -A$

$\det(A^{-1}) = \dfrac{1}{\det(A)}$

$\det(-A) = \det(A)$ n even

$-\det(A)$ n odd.

n even : $\dfrac{1}{d} = d \Rightarrow d^2 = 1 \Rightarrow d = \pm 1$

n odd : $\dfrac{1}{d} = -d \Rightarrow d^2 = -1 \Rightarrow d = \pm i$

(8) Let A be a 4×5 matrix such that there are two different vectors x_1 and x_2 such that $Ax_1 = Ax_2$.

(a) What can you say about the kernel of A?

(b) What can you say about the column rank of A?

a) $A(x_1 - x_2) = 0$ so $\underset{\neq 0}{x_1 - x_2} \in \ker(A) \Rightarrow 5 \geq \dim(\ker(A)) \geq 1$

b) $0 \leq \dim(C(A)) \leq 4$

(9) Let B be the basis for $\mathbb{R}^2$ given by

$$B = \left\{ \begin{bmatrix} 1 \\ -1 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \end{bmatrix} \right\}.$$

(a) Find a matrix which converts vectors written in the standard basis to vectors written with respect to the basis B.

(b) Use your answer to (a) to write $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ (in the standard basis) as a linear combination of vectors in B.

(c) Use your answer to (a) to write the matrix $\begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$ with respect to the ~~matrix~~ basis B.

a) $\begin{bmatrix} 1 & 2 \\ -1 & 1 \end{bmatrix}^{-1} = \frac{1}{3}\begin{bmatrix} 1 & -2 \\ 1 & 1 \end{bmatrix}$

b) $\frac{1}{3}\begin{bmatrix} 1 & -2 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \frac{1}{3}\begin{bmatrix} -1 \\ 2 \end{bmatrix}$ check $-\frac{1}{3}\begin{bmatrix} 1 \\ -1 \end{bmatrix} + \frac{2}{3}\begin{bmatrix} 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$.

c)
$$\begin{tikzcd}
\mathbb{R}^2_E \arrow[r, "\begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}"] & \mathbb{R}^2_E \\
\mathbb{R}^2_B \arrow[u] \arrow[r] & \mathbb{R}^2_B \arrow[u]
\end{tikzcd}$$

$\frac{1}{3}\begin{bmatrix} 1 & -2 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 1 & 2 \\ -1 & 1 \end{bmatrix}$

$\frac{1}{3}\begin{bmatrix} 1 & -2 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 1 & 2 \\ 0 & 3 \end{bmatrix} = \frac{1}{3}\begin{bmatrix} 1 & -4 \\ 1 & 5 \end{bmatrix}$.

(10) Let $A = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 0 & 2 \\ 0 & -2 & 4 \end{bmatrix}$.

(a) Find the eigenvalues and eigenvectors for A.

(b) Can you diagonalize A? Explain.

a) $\begin{vmatrix} 2-\lambda & 0 & 0 \\ 0 & -\lambda & 2 \\ 0 & -2 & 4-\lambda \end{vmatrix} = (2-\lambda)\left[-\lambda(4-\lambda)+4\right] = (2-\lambda)(\lambda^2-4\lambda+4).$

$= -(\lambda-2)^3.$

$\lambda = 2$ with multiplicity 3.

b) $\begin{bmatrix} 0 & 0 & 0 \\ 0 & -2 & 2 \\ 0 & -2 & 2 \end{bmatrix} \rightsquigarrow \begin{bmatrix} 0 & 0 & 0 \\ 0 & -2 & 2 \\ 0 & 0 & 0 \end{bmatrix} \rightsquigarrow \begin{bmatrix} 0 & \boxed{-2} & 2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

$\begin{matrix} s & & t \end{matrix}$

$x_3 = t \quad -2x_2 + 2x_3 = 0 \quad x_2 = t \quad x_1 = s.$

eigenvectors $\begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$

No, cannot diagonalize A, not enough eigenvectors.