Linear Algebra Spring 13 Sample Final

(1) Find all solutions to the following system of linear equations.

$$\begin{aligned} x_1 + x_2 + 2x_3 - x_4 &= 0 \\ 2x_1 + x_2 + 3x_3 + x_4 &= 0 \\ x_1 + x_3 + x_4 &= 0 \end{aligned}$$

(2) (a) Write down a matrix for a linear transformation of $\mathbb{R}^3$ which rotates by $\pi/2$ anticlockwise about the z-axis.
(b) Write down a matrix for a linear transformation of $\mathbb{R}^3$ which doubles lengths in the x-direction.
(c) Write down a matrix for a linear transformation of $\mathbb{R}^3$ which reflects in the xy-plane.
(d) Use your answers above to write down a matrix for a linear transformation of $\mathbb{R}^3$ which doubles lengths in the x-direction, then rotates by $\pi/2$ anticlockwise about the z-axis, and then reflects in the xy-plane.

(3) (a) Write down a linearly independent set in $\mathbb{R}^4$ which is not a basis.
(b) Write down a basis for $\mathbb{R}^4$ which is not orthogonal.
(c) Write down a set of three vectors which span a two-dimensional subspace of $\mathbb{R}^4$.

(4) Let $S = \{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ be a basis for $\mathbb{R}^3$, where

$$\mathbf{v}_1 = \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}, \mathbf{v}_2 = \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix}, \mathbf{v}_3 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}.$$

Use the Gram-Schmidt process to find an orthogonal basis.

(5) Let $V = \mathrm{Span}\{(-1, 2, 1), (1, 3, -1), (-3, 1, 3)\}$ in $\mathbb{R}^3$.
(a) What is the dimension of V?
(b) Find a basis for $V^{\perp}$.

(6)

$$A = \begin{bmatrix} 7 & 5 \\ -10 & -8 \end{bmatrix}$$

(a) Find the eigenvalues of A.
(b) Find the eigenvectors for A.
(c) Diagonalize A, i.e. find matrices P and D such that $P^{-1}AP = D$.
(d) Write down products of matrices which give you A^k and e^{At}.

(7) Let $L\colon \mathbb{R}^2 \to \mathbb{R}^2$ be given by $L(x, y) = (x - 3y, 2x + y)$. Let

$$S = \left\{ \begin{bmatrix} 2 \\ 3 \end{bmatrix}, \begin{bmatrix} -1 \\ -1 \end{bmatrix} \right\}$$

be a basis for $\mathbb{R}^2$, and let T be the standard basis for $\mathbb{R}^2$.

(a) Find the matrix for L with respect to T.
(b) Find the matrix for the change of basis from S to T.
(c) Find the matrix for L with respect to S. Don't worry if its not diagonal.

(8) If A is a non-singular $n \times n$ matrix such that $A^{-1} = A^T$, what can you say about the determinant of A?

(9) Let A be a 3×5 matrix such that the sum of the rows add up to the zero vector.

(a) What can you say about the column rank of A?
(b) If A determines a linear map $L : \mathbb{R}^5 \to \mathbb{R}^3$ given by $L(\mathbf{x}) = A\mathbf{x}$, what can you say about the kernel of L?