Sample final solutions

Q1

$$\begin{bmatrix} 1 & 1 & 2 & -1 \\ 2 & 1 & 3 & 1 \\ 1 & 0 & 1 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & 2 & -1 \\ 0 & -1 & -1 & 3 \\ 0 & -1 & -1 & 2 \end{bmatrix} \sim \begin{bmatrix} \boxed{1} & 1 & 2 & -1 \\ 0 & \boxed{-1} & -1 & 3 \\ 0 & 0 & 0 & \boxed{-1} \end{bmatrix}$$

$x_4 = 0 \quad x_3 = t \quad -x_2 - t = 0 \Rightarrow x_2 = -t \qquad x_1 + x_2 + 2x_3 = 0$

$x_1 = -t.$

sol^{n}: $t\begin{bmatrix} -1 \\ -1 \\ 1 \\ 0 \end{bmatrix}$

Q2 a) $\begin{bmatrix} \cos(\pi/2) & -\sin(\pi/2) & 0 \\ \sin(\pi/2) & \cos(\pi/2) & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1/\sqrt{2} & -1/\sqrt{2} & 0 \\ 1/\sqrt{2} & 1/\sqrt{2} & 0 \\ 0 & 0 & 1 \end{bmatrix} = A.$

b) $B = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ c) $C = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$ d) CAB

$$CAB = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}\begin{bmatrix} 1/\sqrt{2} & -1/\sqrt{2} & 0 \\ 1/\sqrt{2} & 1/\sqrt{2} & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}\begin{bmatrix} 2/\sqrt{2} & -1/\sqrt{2} & 0 \\ 2/\sqrt{2} & 1/\sqrt{2} & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 2/\sqrt{2} & -1/\sqrt{2} & 0 \\ 2/\sqrt{2} & 1/\sqrt{2} & 0 \\ 0 & 0 & -1 \end{bmatrix}.$$

Q3 a) $\left\{\begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}\right\}$

b) $\left\{\begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}\right\}.$

c) $\left\{\begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix}\right\}.$

Q4 $\underline{q}_1 = \frac{1}{\sqrt{3}}\begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}$

$$\underline{q}_2' = \underline{v}_2 - (\underline{v}_2 \cdot \underline{q}_1)\underline{q}_1 = \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix} - \overbrace{\frac{1}{\sqrt{3}}(-2)\frac{1}{\sqrt{3}}}^{-2/3}\begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 5/3 \\ -1/3 \\ 4/3 \end{bmatrix}.$$

$$\underline{q}_2 = \frac{1}{\sqrt{42}}\begin{bmatrix} 5 \\ -1 \\ 4 \end{bmatrix}$$

$$\underline{q_3}' = \underline{v_3} - (\underline{v_3}\cdot\underline{q_1})\underline{q_1} - (\underline{v_3}\cdot\underline{q_2})\underline{q_2}$$

$$\begin{bmatrix}1\\1\\1\end{bmatrix} - \underbrace{\frac{1}{\sqrt{3}}(1)\frac{1}{\sqrt{3}}}_{1/3.}\begin{bmatrix}1\\1\\-1\end{bmatrix} - \underbrace{\frac{1}{\sqrt{42}}(8)\frac{1}{\sqrt{42}}}_{4/21}\begin{bmatrix}5\\-1\\4\end{bmatrix} = \frac{1}{21}\begin{bmatrix}-6\\18\\12\end{bmatrix}$$

$$\underline{q_3} = \frac{1}{\sqrt{14}}\begin{bmatrix}-1\\3\\2\end{bmatrix}$$

Q5

a) $\begin{bmatrix}-1 & 1 & -3\\2 & 3 & 1\\1 & -1 & 3\end{bmatrix} \rightsquigarrow \begin{bmatrix}\boxed{-1} & 1 & -3\\0 & \boxed{5} & -5\\0 & 0 & 0\end{bmatrix} \Rightarrow \left\{\begin{bmatrix}-1\\2\\1\end{bmatrix}, \begin{bmatrix}1\\3\\-1\end{bmatrix}\right\}$ is a basis for V

b) solve $v_i \cdot x = 0$: $\begin{bmatrix}-1 & 2 & 1\\1 & 3 & -1\\-3 & 1 & 3\end{bmatrix} \rightsquigarrow \begin{bmatrix}-1 & 2 & 1\\0 & 5 & 0\\0 & -5 & 0\end{bmatrix} \rightsquigarrow \begin{bmatrix}\boxed{-1} & 2 & 1\\0 & \boxed{5} & 0\\0 & 0 & 0\end{bmatrix}$

$x_3 = t \quad x_2 = 0 \quad -x_1 + t = 0$ soln: $t\begin{bmatrix}+1\\0\\1\end{bmatrix}$ so basis for $V^\perp$ is $\left\{\begin{bmatrix}+1\\0\\1\end{bmatrix}\right\}$

Q6a) $\det(A - \lambda I) = \begin{vmatrix}7-\lambda & 5\\-10 & -8-\lambda\end{vmatrix} = -(7-\lambda)(8+\lambda) + 56 = \lambda^2 + \lambda - 6$

$= (\lambda+3)(\lambda-2) \qquad \lambda = 2, -3.$

b) $\lambda = 2$: $\begin{bmatrix}5 & 5\\-10 & -10\end{bmatrix} \Rightarrow \underline{v} = \begin{bmatrix}1\\-1\end{bmatrix}$

$\lambda = -3$ $\begin{bmatrix}10 & 5\\-10 & -5\end{bmatrix} \Rightarrow \underline{v} = \begin{bmatrix}1\\-2\end{bmatrix}$

c) $D = \begin{bmatrix}2 & 0\\0 & -3\end{bmatrix}$ $P = \begin{bmatrix}1 & 1\\-1 & -2\end{bmatrix}$

$$P^{-1}AP = -\begin{bmatrix}-2 & -1\\1 & 1\end{bmatrix}\begin{bmatrix}7 & 5\\-10 & -8\end{bmatrix}\begin{bmatrix}1 & 1\\-1 & -2\end{bmatrix} = \begin{bmatrix}2 & 1\\-1 & -1\end{bmatrix}\begin{bmatrix}2 & -3\\-2 & 6\end{bmatrix} = \begin{bmatrix}2 & 0\\0 & -3\end{bmatrix}$$

d) $A^k = PD^kP^{-1} \qquad e^{At} = Pe^{Dt}P^{-1} = P\begin{bmatrix}e^{2t} & 0\\0 & e^{-3t}\end{bmatrix}P^{-1}$

Q7 a) $L_T = \begin{bmatrix} 1 & -3 \\ 2 & 1 \end{bmatrix}$

$$\begin{tikzcd} \mathbb{R}^2 \arrow[r, "L_T"] & \mathbb{R}^2 \\ \mathbb{R}^2 \arrow[u, "P"] \arrow[r, "L_S"'] & \mathbb{R}^2 \arrow[u, "P"'] \end{tikzcd}$$

b) $\begin{bmatrix} 2 & -1 \\ 3 & -1 \end{bmatrix} = P$

c) $L_S = P^{-1} L_T P = \begin{bmatrix} -1 & 1 \\ -3 & 2 \end{bmatrix} \begin{bmatrix} 1 & -3 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} 2 & -1 \\ 3 & -1 \end{bmatrix} = \begin{bmatrix} -1 & 1 \\ -3 & 2 \end{bmatrix} \begin{bmatrix} -7 & 2 \\ 7 & -3 \end{bmatrix}.$

$= \begin{bmatrix} 14 & -5 \\ 35 & -12 \end{bmatrix}$

Q8 $\det(A^{-1}) = \dfrac{1}{\det(A)}$ $\quad \det(A^T) = \det(A)$

$A^{-1} = A^T \Rightarrow \dfrac{1}{\det(A)} = \det(A) \Rightarrow \det(A)^2 = 1 \Rightarrow \det(A) = \pm 1$

Q9 a) $0 \leq \text{col rank} \leq 2$

b) $3 < \text{dim kernel} \leq 5$