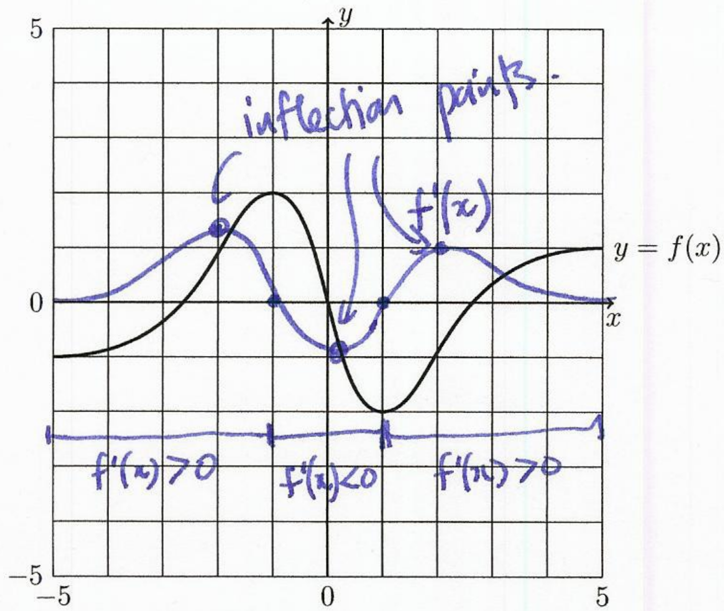
Solutions



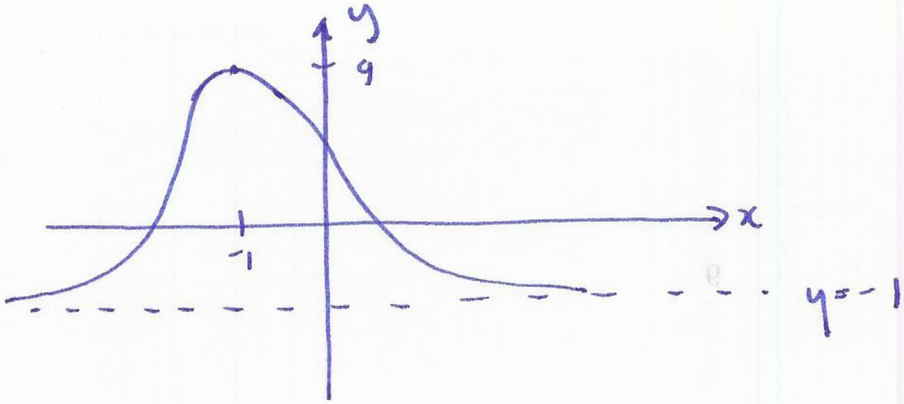
**Math 231 Calculus 1 Fall 13 Sample Midterm 3**

(1) Consider the function $f(x)$ defined by the following graph.




(a) Label all regions where $f'(x) < 0$.
(b) Label all regions where $f'(x) > 0$.
(c) What is $\lim_{x\to\infty} f'(x)$? 0
(d) What is $\lim_{x\to-\infty} f''(x)$? 0
(e) Sketch a graph of $f'(x)$ on the figure.
(f) Label the approximate locations of all points of inflection.

**Q2**




**Q3** a) vertical asymptotes $\Leftrightarrow \quad 27-x^3=0$

$\Leftrightarrow \quad x=3$

horizontal asymptotes $\Leftrightarrow \lim\limits_{x\to\pm\infty} \dfrac{x}{27-x^3} = 0$

b) $$f'(x) = \frac{(27-x^3).1 - x.-3x^2}{(27-x^3)^2} = \frac{27+2x^3}{(27-x^3)^2}$$

critical points: $f'(x)=0 \Leftrightarrow x = -\dfrac{3}{\sqrt[3]{2}}$

$f'(x)$ undefined at $x=3$

c) $f'(x) > 0 \quad x > -\dfrac{3}{\sqrt[3]{2}}$

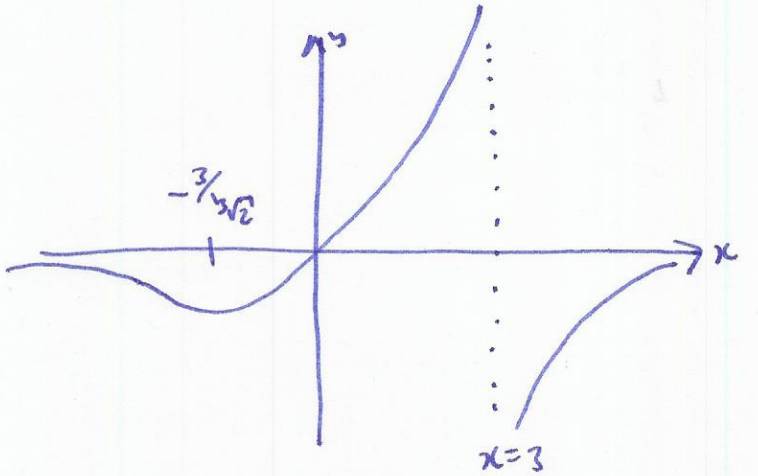
$f'(x) < 0 \quad x < -\dfrac{3}{\sqrt[3]{2}}$

d) $$f''(x) = \frac{(27-x^3)^2.6x^2 - 2(27-x^3).(-3x^2).(27+2x^3)}{(27-x^3)^4}$$

$$= \frac{6x^2(27-x^3)^2 + 6x^2(27-x^3)(27+2x^3)}{(27-x^3)^4}$$

$$f''\left(-\frac{3}{\sqrt[3]{2}}\right) = \frac{\text{square} + 0}{(-)^4} > 0 \Rightarrow \text{local min.}$$

e)




**Q4** $g(x) = (x^2-2x)e^x$

a) $g'(x) = (2x-2)e^x + (x^2-2x)e^x$

$= (x^2-2)e^x$

$g'(x)=0 \quad x = \pm\sqrt{2}$

first derivative test

$$\begin{array}{ccccc} + & & - & & + \\ \hline & -\sqrt{2} & & +\sqrt{2} & \end{array}$$

local max     local min.

b) $f'(x) > 0$ on $(-\infty, -\sqrt{2})$ and $(\sqrt{2}, \infty)$

$f'(x) < 0 \quad (-\sqrt{2}, \sqrt{2})$

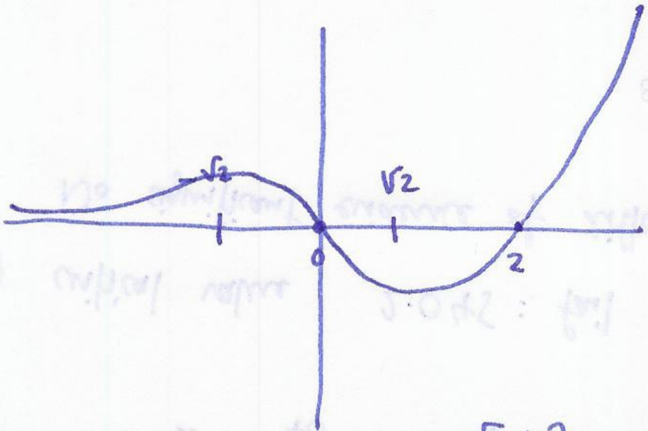
c) $g''(x) = 2xe^x + (x^2-2)e^x = (x^2+2x-2)e^x$

solve $g''(x)=0$: $x = \dfrac{-2 \pm \sqrt{4+8}}{2} = -1 \pm \sqrt{3}$ inflection points.

d) $g''(x) > 0 \quad (-\infty, -1-\sqrt{3}), (-1+\sqrt{3}, \infty)$ concave up

$g''(x) < 0 \quad (-1-\sqrt{3}, -1+\sqrt{3})$ concave down.

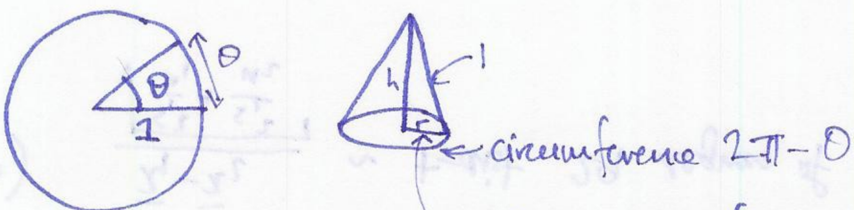
e)




in $[1,4]$

**Q5** $f'(x) > 0$ for all $x$, so increasing on $[1,4]$ so max value at $x=4$.

**Q6**




radius $= \frac{\text{circumference}}{2\pi} = \frac{2\pi - \theta}{2\pi} = 1 - \frac{\theta}{2\pi}$

height $h$: $\quad h^2 + r^2 = 1 \qquad h^2 = 1 - r^2 = 1 - \left(1 - \frac{\theta}{2\pi}\right)^2 = \frac{\theta}{\pi} - \frac{\theta^2}{4\pi^2}$

$$h = \sqrt{\frac{\theta}{\pi} - \frac{\theta^2}{4\pi^2}}.$$

volume of cone: $V = \frac{1}{3}\pi r^2 h$

$$= \frac{1}{3}\pi\left(1 - \frac{\theta}{2\pi}\right)^2 \sqrt{\frac{\theta}{\pi} - \frac{\theta^2}{4\pi^2}} = \frac{1}{3}\pi r^2 \sqrt{1 - r^2} = \frac{1}{3}\pi\sqrt{r^4 - r^6}$$

$$\frac{dV}{d\theta} = \frac{dV}{dr}\frac{dr}{d\theta} = \frac{1}{3}\pi \, \frac{1}{2}\left(r^4 - r^6\right)^{-1/2}\left(4r^3 - 6r^5\right) \cdot -\frac{1}{2\pi} = \frac{1}{3} \, \frac{4r^3 - 6r^5}{\sqrt{r^4 - r^6}}$$

$= 0$ when $\quad 4r^3 - 6r^5 = 0$

$2r^3(2 - 3r^2) = 0 \qquad r = \sqrt{\frac{2}{3}}, \quad = 1 - \frac{\theta}{2\pi}$

$$\theta = \left(1 - \sqrt{2/3}\right)2\pi.$$

**Q7** a) $$\lim_{x\to+\infty} \frac{6x+2}{\sqrt{2x-4}} = \frac{6\sqrt{x}+2/\sqrt{x}}{\sqrt{2-4/x}} \to \infty \text{ as } x\to\infty.$$

b) $$\lim_{x\to0+} \sqrt{x}\ln(x) = \lim_{x\to0+} \frac{\ln(x)}{x^{-1/2}} \underset{l'H}{=} \lim_{x\to0+} \frac{1/x}{-\frac{1}{2}x^{-3/2}} = \lim_{x\to0+} -2\sqrt{x} = 0$$

c) $$\lim_{x\to0}\left(\frac{e^{2x}}{e^{2x}-1} - \frac{1}{2x}\right) = \lim_{x\to0} \frac{2xe^{2x} - e^{2x}+1}{(e^{2x}-1)2x}$$

$$\underset{l'H}{=} \lim_{x\to0} \frac{2e^{2x} + 2x\,2e^{2x} - 2e^{2x}}{2e^{2x} + 2xe^{2x} - 2} = \lim_{x\to0} \frac{4xe^{2x}}{\cancel{4xe^{2x}-2}\ 2e^{2x}+2xe^{2x}-2}$$

$$\underset{l'H}{=} \lim_{x\to0} \frac{4e^{2x} + 4x.2e^{2x}}{4e^{2x} + 2e^{2x} + 2x.2e^{2x}} = \frac{4}{6} = \frac{2}{3}$$

d) $$\lim_{x\to0} \frac{3\sin x - \sin 3x}{\sin x - x\cos 3x} \underset{l'H}{=} \lim_{x\to0} \frac{3\cos x - 3\cos 3x}{\cos x - \cos 3x + x\,3\sin 3x}$$

$$\underset{l'H}{=} \lim_{x\to0} \frac{-3\sin x + 9\sin 3x}{-\sin x + \underbrace{3\sin 3x + 3\sin 3x}_{6\sin 3x} + x\,9\cos 3x}$$

$$\underset{l'H}{=} \lim_{x\to0} \frac{-3\cos x + 27\cos 3x}{-\cos x + 18\cos 3x + 9\cos 3x - x.27\sin 3x} = \frac{24}{26} = \frac{12}{13}$$

**Q8** a) $$\int \frac{x^2-2x+1}{x}\,dx = \int x - 2 + \frac{1}{x}\,dx = \frac{1}{2}x^2 - 2x + \ln|x| + c$$

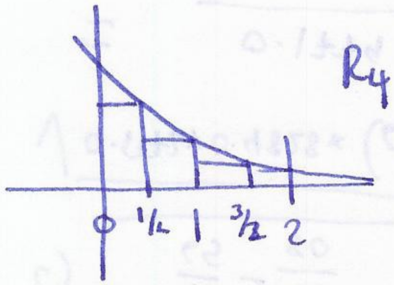
b) $$\int 2e^x - 4\cos x\,dx = 2e^x - 4\sin x + c$$

c) $\displaystyle\int_1^2 3\sqrt[3]{x}\,dx = \left[3\cdot\frac{3}{4}x^{4/3}\right]_1^2 = \frac{9}{4}\left(2^{4/3}-1\right)$

d) $\displaystyle\int_0^t \frac{1}{x+1}\,dx = \Big[\ln|x+1|\Big]_0^t = \ln|t+1| - \ln|1|$

$= \ln|t+1|$ .

**Q9**




$R_4 = \frac{1}{2}\left(e^{-1/2}+e^{-1}+e^{-3/2}+e^{-2}\right) \approx 0.6664$

under estimate.

**Q10** $v = \frac{dx}{dt} = \frac{1}{(t+1)^2}$ $\quad x(t) = -\frac{1}{t+1} + c$

$x(0) = 0 \Rightarrow -\frac{1}{0+1} + c = 0 \Rightarrow c = 1$

$x(t) = 1 - \frac{1}{1+t}$ $\quad \lim_{t\to\infty} 1 - \frac{1}{1+t} = 1$ , but particle will not reach $x = 1$ in finite time.