College of Staten Island
Department of Mathematics

MTH 230/231 Calculus I Fall 2013 Common Final version 2

NAME: Solutions

Each part of each question is worth **4 points**.

SHOW YOUR WORK—OTHERWISE THERE IS NO CREDIT!

Question	Possible Points	Earned Points
1	4	
2a	4	
2b	4	
2c	4	
3a	4	
3b	4	
3c	4	
3d	4	
4	4	
5	4	
6a	4	
6b	4	
6c	4	
6d	4	
7a	4	
7b	4	
8	4	
9a	4	
9b	4	
9c	4	
9d	4	
10a	4	
10b	4	
10c	4	
11	4	

1. Let f(x) be a function. Carefully define f ′ (x), the derivative of f(x).

f ′ (x) = $\lim_{h \to 0} \frac{f(x+h)-f(x)}{h}$

Graphically (i.e. geometrically) what does the derivative of f(x) measure?

Answer: slope of tangent line to graph of f(x)

2. Let f(x) = $\frac{\sin(x)}{x}$. This function will be used in all parts of this question 2.

a. Use the "squeeze theorem" to compute $\lim_{x \to \infty} \frac{\sin(x)}{x} =$ 0

Hint: Look carefully at this limit.
Show your computation:

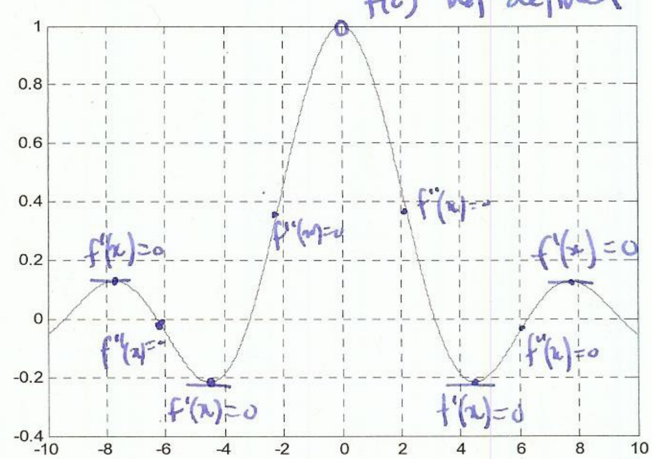
$|\sin(x)| \le 1$ so $\frac{-1}{x} \le \frac{\sin(x)}{x} \le \frac{1}{x}$ $\lim_{x \to \infty} \frac{-1}{x} = 0$ $\lim_{x \to \infty} \frac{1}{x} = 0$

b. Use L'Hopital's rule to compute: $\lim_{x \to 0} \frac{\sin(x)}{x} =$ 1

Show your computation: $\lim_{x \to 0} \frac{\cos(x)}{1} = 1$

b. In Matlab, f(x) = $\frac{\sin(x)}{x}$ on the interval [-10,10] looks like

f(0) not defined



Without calculation:

On the sketch above, label point(s) of discontinuity: Point(s) of discontinuity of f(x): x=0

On the sketch above, label the critical points of f(x): Critical points of f(x): -7, -5, 0, 5, 7

On the sketch above, label the inflection points of f(x): Inflection points of f(x): -6, -2, 2, 6

On the sketch above, label all regions where $f'(x) < 0$? Answer: (-7, 5) (0, 5) (7, 10)

3. Compute the first **derivative** for each of the following functions (show your work):

a $f(x) = x^4 + 4^x + 4^4 = x^4 + e^{x\ln(4)} + 4^4$

$f'(x) =$ $4x^3 + \ln(4)\, e^{x\ln(4)}$

b. $f(x) = \dfrac{\cos(x)}{x^5 + 3}$

$$\frac{(x^5+3)(-\sin(x)) - (5x^4)\cos(x)}{(x^5+3)^2}$$

$f'(x) =$ ____________________

Computation:

c. $f(x) = \sqrt{x^4 + \sin^2(x)}$

$\frac{1}{2}\left(x^4 + \sin^2(x)\right)^{-1/2} \left(4x^3 + 2\sin(x)\cos(x)\right)$

$f'(x) =$ ____________________

Computation

d. $f(x) = \ln(\cos(x)) + \sin^{-1}(3x)$

$f'(x) =$ $\frac{1}{\cos(x)} \cdot -\sin(x) + \frac{1}{\sqrt{1-9x^2}} \cdot 3$

Computation:

4. Find the <u>equation of the tangent line</u> to the curve $x^2 + y^4 + 2y^2 = 7$ at the point (2,1).
Hint: Use implicit differentiation.

$2x + 4y\frac{dy}{dx} + 4y\frac{dy}{dx} = 0$

$4 + 4\frac{dy}{dx} + 4\frac{dy}{dx} = 0$

$\frac{dy}{dx} = -\frac{1}{2}$

$y - 1 = -\frac{1}{2}(x-2)$

5. An oil tanker in the Atlantic Ocean has sprung a leak, creating a circular oil slick. If the area of the oil slick is increasing at a rate of 4m^2 per minute, how fast is the radius of the oil slick increasing when the radius is 20m.. ANSWER:______________________

$A = \pi r^2$

$\frac{dA}{dt} = 2\pi r \frac{dr}{dt}$

$4 = 2\pi \cdot 20 \frac{dr}{dt}$

$\frac{dr}{dt} = \frac{1}{\pi \cdot 10}$

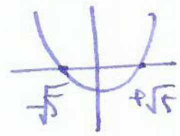
6. Let $f(x) = x^3 - 15x + 8$

a. Where is the function increasing? Answer: $(-\infty, -\sqrt{5})$ $(\sqrt{5}, \infty)$.

Show your work:

$f'(x) = 3x^2 - 15$

$f'(x) = 0 \quad : x = \pm\sqrt{5}$



b. What are the inflection point(s)? Answer $x = 0$.

Show your work.

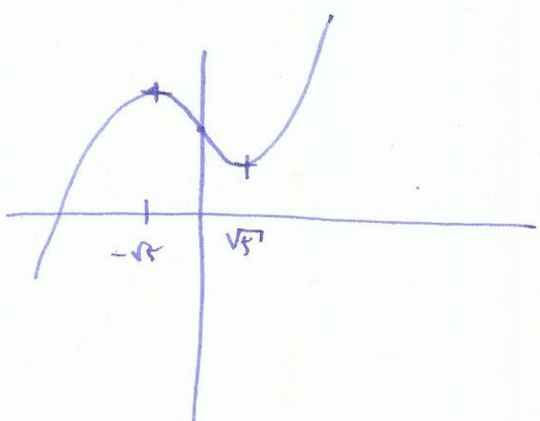
$f''(x) = 6x$

c. Where is the function concave down? Answer: $(-\infty, 0)$

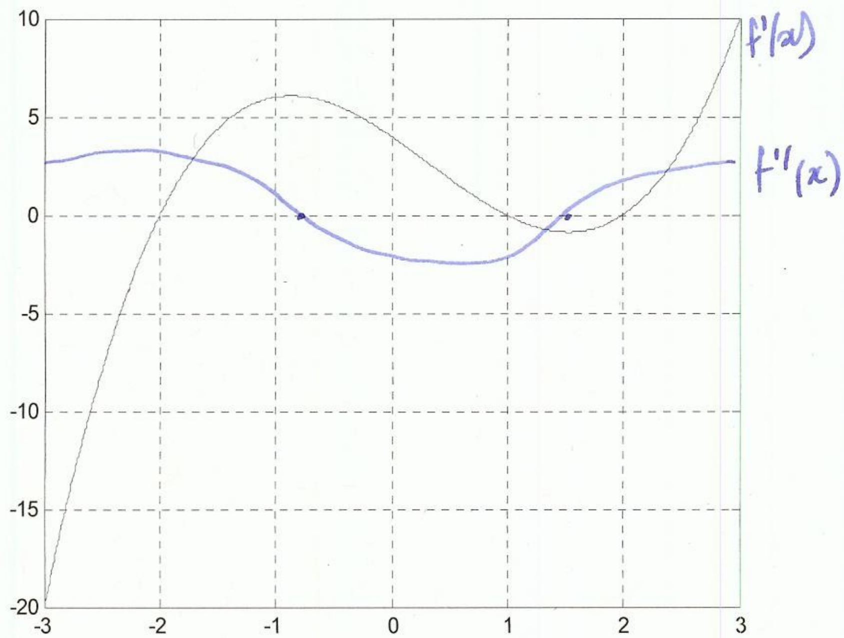
Explain:

$f''(x) < 0 \Leftrightarrow x < 0$

d. Draw a graph of the function.



7. The graph of the **derivative of f(x)** is



a. Where is the **function f(x)** increasing?. Answer: $(-2, 1)$ $(2, 3)$

Explain:

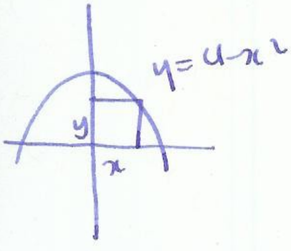
$f(x)$ increasing $\leftrightarrow f'(x) > 0$

b. Where is the **function f(x)** concave up? Answer: $(-3, -1)$ $(1.5, 3)$

Explain:

$f''(x) > 0$

8. Find the maximum area of a rectangle inscribed in the region bounded by the graph of $y = 4 - x^2$ and the axes, in the first quadrant. Hint: Sketch this function in the first quadrant. Assume the base of the rectangle is on the x-axis and starts at (0,0). Answer: ____________



$A = xy = x(4-x^2) = 4x - x^3$

$\frac{dA}{dx} = 4 - 3x^2$

$\frac{dA}{dx} = 0$: $x = \frac{2}{\sqrt{3}}$ $\quad y = 4 - \frac{4}{3} = \frac{8}{3}$ $\quad A = \frac{2}{\sqrt{3}} \cdot \frac{8}{3}$.

9 Compute the following antiderivatives (show your work):

a. $\int\left(\frac{5}{\sqrt{x}}+\frac{3}{x}+1\right)dx =$ $5x^{1/2}\cdot 2 + 3\ln|x| + x + C$

Computation:

b. $\int\left(\sin(3x)+\frac{4}{x\sqrt{x}}\right)dx =$ $-\cos(3x)\cdot\frac{1}{3} + \frac{4x^{-1/2}}{-1/2} + c$

Computation:

$-\frac{1}{3}\cos(3x) - \frac{8}{\sqrt{x}} + c$

c. $\int(6x+\sec^2(x))\,dx =$ $3x^2 + \tan(x) + c$

Computation:

d. $\int\left(x\left(x^2+7\right)^{23}\right)dx =$ $\frac{1}{48}\left(x^2+27\right)^{24} + c$

Computation:

$u = x^2+7$

$\frac{du}{dx} = 2x$

$\int x\,u^{23}\frac{1}{2x}\,du = \int \frac{1}{2}u^{23}\,du = \frac{u^{24}}{48} + c = \frac{1}{48}(x^2+7)^{24} + c$

10. a. Carefully state the First Fundamental Theorem of Calculus

$$\int_a^b f(x)\,dx = F(b) - F(a) \quad \text{where} \quad F'(x) = f(x)$$

b Calculate the area under the curve f(x) = cos(x) on the interval $\left[0, \frac{\pi}{2}\right]$

Answer: 1

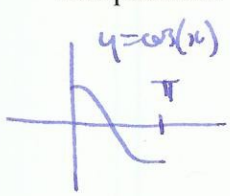
Computation:

$$\int_0^{\pi/2} \cos(x)\,dx = \left[\sin(x)\right]_0^{\pi/2} = \sin(\pi/2) - \sin(0) = 1$$

c. Calculate the area under the curve f(x) = |cos(x)| on the interval [0,π].

Answer: 2

Computation:



$$\int_0^{\pi/2} \cos(x)\,dx + \int_{\pi/2}^{\pi} -\cos(x)\,dx$$

11. If F(x) = $\int_0^x \sqrt{4 - t^2}\,dt$, then $F'(x) =$ $\sqrt{4-x^2}$